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# ON RESIDUE FREE DIFFERENTIAL FORMS OF AN ALGEBRAIC SCHEME OVER A FIELD OF CHARACTERISTIC $p$

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**Abstract.** Let $V$ be an $n$-dimensional non-singular algebraic integral scheme over a perfect field $k$ of characteristic $p > 0$ and $K$ its algebraic function field. In this paper, we will prove the following:

**Theorem B.** *Let $\omega$ be a differential form in $Z_\infty(K/k)$. Then the following three conditions are equivalent:*
*(1) $\omega$ is residue free on $V$,*
*(2) there exists an integer $N$ such that $C_K^N(\omega) \in G_1(V)$,*
*(3) $\omega \in D(V)$.*

The above theorem is a generalization of the main theorem in Nakakoshi[5]. He proved the theorem in case of degree($\omega$) $= n$.

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## 1. Preliminaries

Throughout this paper, $k$ will denote a perfect field of characteristic $p > 0$. Let $V$ be an $n$-dimensional non-singular algebraic integral scheme over $k$, where a scheme $V$ is said to be algebraic over $k$ if $V$ is a separated scheme of finite type over $k$. We will denote by $K$ the function field of $V$.

Let $W$ be a prime divisor of $V$, $R$ the local ring at the generic point of $W$. We know that $R$ is a discrete valuation ring and call it the valuation ring of $W$. Let $\nu_R$ be its valuation. If $f \in R$, then we will denote by $\overline{f}$ its canonical image in the residue field $D$ of $R$. Since $k$ is perfect, we can choose a family $\{t_1, t_2, \ldots, t_n\}$ of elements in $R$ such that $t_1$ is a prime element of $R$ (i.e. $t_1 R$ is the maximal ideal of $R$.) and $\{\bar{t}_2, \ldots, \bar{t}_n\}$ is a separating transcendental basis of $D/k$. We will call such a family $\underline{t} = \{t_1, t_2, \ldots, t_n\}$ a parameter of $R$. Then we know that $\{d\bar{t}_2, \ldots, d\bar{t}_n\}$ forms a basis of the module of Kähler differentials $\Omega^1(D/k)$ of $D/k$ and $\{dt_1, \ldots, dt_n\}$ forms a free basis of $\Omega^1(R/k)$ as an $R$-module also a basis of $\Omega^1(K/k)$, (c.f. Kawahara-Uchibori[2]). We put $\Omega^r(K/k) = \bigwedge^r \Omega^1(K/k)$ and $\Omega(K/k) = \bigoplus_r \Omega^r(K/k)$. Then $\Omega(K/k)$ is a graded $K$-algebra. Similarly we define $\Omega(R/k)$ and $\Omega(D/k)$. If there is no confusion, we will omit a symbol "$/k$".

Let $\hat{R}$ be the completion of $R$. Then there exists a unique coefficient field $E$ of $\hat{R}$ such that $\hat{R} = E[[t_1]]$ and $E \supseteq k(t_2, \ldots, t_n)$ (c.f. Th. 28.3 in Matsumura[4]). The quotient field of $\hat{R}$ is the field $E((t_1))$ of formal power series and $K$ can be regarded as a subfield of $E((t_1))$.

Let $\omega$ be a differential form in $\Omega^r(K)$ $(r > 0)$. Then $\omega$ can be uniquely expressed in the form

$$\omega = \sum_{1<i_1<\ldots<i_r} g_{i_1,\ldots,i_r} dt_{i_1} \wedge \cdots \wedge dt_{i_r} + \sum_{1<i_2<\ldots<i_r} h_{i_2,\ldots,i_r} dt_1 \wedge dt_{i_2} \wedge \cdots \wedge dt_{i_r}, \quad (g_{i_1,\ldots,i_r}, h_{i_2,\ldots,i_r} \in K).$$

Elzein[1] defined the residue $\mathrm{res}_{R,\underline{t}}(\omega)$ of $\omega \in \Omega(K)$ as follows. The coefficient $h_{i_2,\ldots,i_r}$ can be uniquely expressed as an element of $E((t_1))$ in the following form:

$$h_{i_2,\ldots,i_r} = \sum_k h_{i_2,\ldots,i_r,k} t_1^k \quad (h_{i_2,\ldots,i_r,k} \in E).$$

Then the residue of $\omega$ is defined by

$$\mathrm{res}_{R,\underline{t}}(\omega) = \sum_{1<i_2<\ldots<i_r} \overline{h}_{i_2,\ldots,i_r,-1} d\bar{t}_{i_2} \wedge \cdots \wedge d\bar{t}_{i_r},$$

where $\overline{h}_{i_2,\ldots,i_r,-1}$ is the canonical image of $h_{i_2,\ldots,i_r,-1}$ in the residue field $D$ of $\hat{R}$. Moreover Elzein[1] proved the following property

$$\mathrm{res}_{R,\underline{t}} \circ d + d \circ \mathrm{res}_{R,\underline{t}} = 0.$$

It follows from this property that $\mathrm{res}_{R,\underline{t}}$ maps a closed differential to closed one and an exact differential to exact one. Since the map $\mathrm{res}_{R,\underline{t}} \colon \Omega^r(K) \to \Omega^{r-1}(D)$ is $k$-linear, we can define the map $\mathrm{res}_{R,\underline{t}} \colon \Omega(K) \to \Omega(D)$ by linearity.

We will denote by $Z_1(K)$ ( $= \ker d$) all of closed differential forms and by $B_1(K)$ ( $= \mathrm{im}\, d$) all of exact differential forms in $\Omega(K)$. We define the graded subalgebra $H(\underline{t})$ of $\Omega(K)$ as follows:

$$H(\underline{t}) := K^P[t_1^{p-1}dt_1, t_2^{p-1}dt_2, \ldots, t_n^{p-1}dt_n].$$

We have by Exercise(6) in §5 of E. Kunz[3] that

$$Z_1(K) = B_1(K) \oplus H(\underline{t}) \quad \text{(direct sum as } K^P\text{-modules)}.$$

The Cartier operator $C_K \colon Z_1(K) \to \Omega(K)$ is a surjective ring-homomorphism which is defined by the following equations:

$$C_K(\omega) = 0 \quad (\omega \in B_1(K)), \quad C_K(a^p) = a \quad (a^p \in K^P),$$
$$C_K(t_i^{p-1}dt_i) = dt_i \quad \text{(for each } i).$$

The Cartier operator $C_K$ is independent on the choice of a parameter $\underline{t}$ and also independent on the choice of $R$. We put, for every integer $m > 0$,

$$B_{m+1}(K) = C_K^{-1}(B_m(K)), \quad Z_{m+1}(K) = C_K^{-1}(Z_m(K)).$$

Moreover we also put $B_\infty(K) = \bigcup_{m=1}^\infty B_m(K)$ and $Z_\infty(K) = \bigcap_{m=1}^\infty Z_m(K)$. Similarly we can define $C_D$ and $C_R$ (because a parameter $\{t_1, t_2, \ldots, t_n\}$ of $R$ is a $p$-basis of $R$, see Lemma 1 of Ohi[6]). Moreover we can define $B_m(D)$, Lemma 2 of Suzuki[7] that

$$C_D \circ \mathrm{res}_{R,\underline{t}} = \mathrm{res}_{R,\underline{t}} \circ C_K.$$

The following composition of two maps:

$$Z_\infty(K) \xrightarrow[\mathrm{res}_{R,\underline{t}}]{} Z_\infty(D) \xrightarrow[\text{natural surjection}]{} Z_\infty(D)/B_\infty(D)$$

is independent on the choice of a parameter $\underline{t}$ and denoted by $\mathrm{res}_R$ (c.f. [7], also see [6]). Furthermore, when $R$ is the valuation ring of $W$, $\mathrm{res}_R$ is denoted by $\mathrm{res}_W$.

## 2. Auxiliary Theorem

Let $W$ be a prime divisor of $V$, $R$ its valuation ring and $\underline{t}$ a parameter of $R$. For a differential form $\omega \in \Omega(K)$, we define $\nu_R(\omega)$ as follows:

$$\nu_R(\omega) := -\min\{s \mid t_1^s \omega \in \Omega(R), s\colon \text{integer}\}.$$

It is clear that $\nu_R(\omega)$ is independent on the choice of a prime element $t_1$ of $R$. For an element $\omega$ of $Z_\infty(K)$, by Lemma 3 in [7], there exists an integer $N$ such that

$$\nu_R(C_K^m(\omega)) \geq -1 \quad (m \geq N).$$

A differential form $\omega \in \Omega^r(K)$ can be uniquely expressed in the form $\omega_1 + \omega_2$, where

$$\omega_1 = \sum_{1<i_1<\ldots<i_r} g_{i_1,\ldots,i_r} dt_{i_1} \wedge \cdots \wedge dt_{i_r},$$

$$\omega_2 = \sum_{1<i_2<\ldots<i_r} h_{i_2,\ldots,i_r} dt_1 \wedge dt_{i_2} \wedge \cdots \wedge dt_{i_r}, \quad (g_{i_1,\ldots,i_r}, h_{i_2,\ldots,i_r} \in K).$$

**Lemma 1.** *For a closed differential form $\omega \in \Omega^r(K)$, let $\omega_1 + \omega_2$ be the above expression of $\omega$. Then the inequality $\nu_R(\omega) \geq -1$ implies that $\omega_1$ belongs to $\Omega^r(R)$.*

*Proof.* The inequality $\nu_R(\omega) \geq -1$ means $t_1\omega \in \Omega^r(R)$, hence $d(t_1\omega) \in \Omega^{r+1}(R)$. Since $\omega$ is closed and $dt_1 \wedge \omega_2 = 0$, we have that

$$d(t_1\omega) = dt_1 \wedge \omega = dt_1 \wedge (\omega_1 + \omega_2) = dt_1 \wedge \omega_1.$$

Therefore we have $dt_1 \wedge \omega_1 \in \Omega^{r+1}(R)$, which implies $\omega_1 \in \Omega^r(R)$.

**Theorem A.** *Let $W$ be a prime divisor of $V$ and $R$ its valuation ring. For a differential form $\omega$ in $Z_\infty(K)$, the following three conditions are equivalent:*

*(1)* $\mathrm{res}_W(\omega) = 0$,

*(2) there exists an integer $N$ such that $C_K^N(\omega) \in \Omega(R)$,*

*(3) there exists $\omega_R \in B_\infty(K)$ such that $\omega - \omega_R \in \Omega(R)$.*

*Proof.* (1) ⇒ (2). We can assume that $\omega \in \Omega^r(K)$ by the linearity of $C_K$, $C_D$ and $\mathrm{res}_W$. By $\mathrm{res}_W(\omega) = 0$, we have $\mathrm{res}_{R,\underline{t}}(\omega) \in B_\infty(D)$ for any parameter $\underline{t}$ of $R$. Hence there exists an integer $N$ such that $C_D^N(\mathrm{res}_{R,\underline{t}}(\omega)) = 0$ and so we have $\mathrm{res}_{R,\underline{t}}(C_K^N(\omega)) = 0$. Moreover for a sufficiently large $N$, we can assume that $\nu_R(C_K^N(\omega)) \geq -1$. The differential form $C_K^N(\omega)$ can be expressed in the form $\omega_1 + \omega_2$, where

$$\omega_1 = \sum_{1<i_1<\ldots<i_r} g_{i_1,\ldots,i_r} dt_{i_1} \wedge \cdots \wedge dt_{i_r},$$

$$\omega_2 = \sum_{1<i_2<\ldots<i_r} h_{i_2,\ldots,i_r} dt_1 \wedge dt_{i_2} \wedge \cdots \wedge dt_{i_r}, \quad (g_{i_1,\ldots,i_r}, h_{i_2,\ldots,i_r} \in K)$$

$$h_{i_2,\ldots,i_r} = \sum_{k\geq -1} h_{i_2,\ldots,i_r,k} t_1^k, \quad (h_{i_2,\ldots,i_r,k} \in E).$$

From the relation above $\mathrm{res}_{R,\underline{t}}(C_K^N(\omega)) = 0$, we have

$$\sum_{1<i_2<\ldots<i_r} \overline{h}_{i_2,\ldots,i_r,-1} d\bar{t}_{i_2} \wedge \ldots \wedge d\bar{t}_{i_r} = 0,$$

which implies $\overline{h}_{i_2,\ldots,i_r,-1} = 0$ for any $1 < i_2 < \ldots < i_r$. Since the natural surjection: $\hat{R} \to D$ is injective on the coefficient field $E$ of $\hat{R}$, we have $h_{i_2,\ldots,i_r,-1} = 0$ and thus $\omega_2 \in \Omega(R)$. On the other hand, by Lemma 1 we have $\omega_1 \in \Omega(R)$. Therefore, we get $C_K^N(\omega) \in \Omega(R)$.

(2) ⇒ (3). Since the map $C_R^N \colon Z_N(R) \to \Omega(R)$ is surjective, there exist $\eta \in Z_N(R)$ such that $C_R^N(\eta) = C_K^N(\omega)$. We put $\omega_R = \omega - \eta$. Since $C_R^N(\eta) = C_K^N(\eta)$, we have that $\omega \in B_N(K) \subset B_\infty(K)$ and $\omega - \omega_R = \eta \in Z_N(R) \subset \Omega(R)$.

(3) ⇒ (1). By $\omega - \omega_R \in \Omega(R)$, we have $\mathrm{res}_{R,\underline{t}}(\omega - \omega_R) = 0$ and hence $\mathrm{res}_{R,\underline{t}}(\omega) = \mathrm{res}_{R,\underline{t}}(\omega_R)$. Since $\omega_R \in B_\infty(K)$, there exists an integer $N$ such that $C_K^N(\omega_R) = 0$. Therefore, we have $C_D^N(\mathrm{res}_{R,\underline{t}}(\omega)) = \mathrm{res}_{R,\underline{t}}(C_K^N(\omega_R)) = 0$ and thus we get $\mathrm{res}_{R,\underline{t}}(\omega) \in B_N(K)$, which implies $\mathrm{res}_W(\omega) = 0$.

## 3. Main Theorem

In this section, we denote by $R_W$ the valuation ring $R$ of a prime divisor $W$.

**Definition 1.** For a differential form $\omega \in Z_\infty(K)$, we define $\omega$ is residue free on $V$ if $\mathrm{res}_W(\omega) = 0$, for any prime divisor $W$ of $V$.

We set $G_1(V) = \bigcap_W \Omega(R_W)$. A differential form in $G_1(V)$ is said to be the first kind.

**Definition 2.** We define the subsets $D_N(V)$ and $D(V)$ of $\Omega(K)$ as follows: for a differential form $\omega \in \Omega(K)$, $\omega$ belongs to $D_N(V)$ if and only if for any prime divisor $W$ of $V$, there exists $\omega_{R_W} \in B_N(K)$ such that $\omega - \omega_{R_W} \in \Omega(R_W)$. We put $D(V) = \bigcup_{N=1}^{\infty} D_N(V)$.

**Lemma 2.** *Let $f$ be an element of $K$. Then $f$ belongs to $R_W$ for almost all of $W$.*

*Proof.* Let $\mathrm{spec}(A)$ be an affine open subset of $V$. Since $V - \mathrm{spec}(A)$ has only finite irreducible componets, almost all of the prime divisors meet to $\mathrm{spec}(A)$. We consider a prime divisor $W$ such that $W \cap \mathrm{spec}(A) \neq \emptyset$. We can put $f = b/a$ $(a, b \in A)$. The closed subset $V(a)$ of $\mathrm{spec}(A)$ has only finite irreducible components and thus $f \in R_W$ for almost all of $W$.

Let $\{x_1, \dots, x_n\}$ be a $p$-basis of $K/k$. Then any element $\omega$ of $\Omega^r(K)$ can be expressed in the form

$$\omega = \sum_{i_1 < \dots < i_r} h_{i_1,\dots,i_r} dx_{i_1} \wedge \cdots \wedge dx_{i_r} \quad (h_{i_1,\dots,i_r} \in K).$$

**Lemma 3.** *Let $\omega$ be an element of $\Omega(K)$. Then $\omega$ belongs to $\Omega(R_W)$ for almost all of $W$.*

*Proof.* We assume $\omega \in \Omega^r(K)$ and use the above expression of $\omega$. By Lemma 2, for almost all of $W$, $R_W$ contains all of the elements $h_{i_1,\dots,i_r}$ $(i_1 < \dots < i_r)$ and $x_i$ $(1 \leq i \leq n)$, and thus $\omega \in \Omega(R_W)$ for almost all of $W$.

**Theorem B.** *Let $\omega$ be a differential form in $Z_\infty(K)$. Then the following three conditions are equivalent:*
*(1) $\omega$ is residue free on $V$,*
*(2) there exists an integer $N$ such that $C_K^N(\omega) \in G_1(V)$,*
*(3) $\omega \in D(V)$.*

*Proof.* (1) $\Rightarrow$ (2). By Lemma 3, the set $S$ of prime divisor $W$ such that $\omega \notin \Omega(R_W)$ is finite. Put $S = \{W_1, \dots, W_S\}$. By Lemma A, there exists $N_i$ such that $C_K^{N_i}(\omega) \in \Omega(R_{W_i})$ for each $i$. We put $N = \max\{N_1, \dots, N_S\}$, then we have $C_K^N(\omega) \in G_1(V)$. The implications (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) are immediately proved by Theorem A.

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