A REMARK ON TIME-DEPENDENT GINZBURG-LANDAU EQUATIONS

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Abstract. The purpose of this paper is to show the existence of unique global $C^1$-solutions to the time-dependent complex Ginzburg-Landau equation. We regard the equation as a genuinely nonlinear equation and simultaneously as a semilinear equation.

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1. Introduction

In this paper we consider the generalized complex Ginzburg-Landau equation (see e.g. Temam [3])

\[ \frac{\partial u}{\partial t} - (\lambda + i\alpha)\Delta u + (\kappa + i\beta)|u|^{p-1}u - \gamma u = 0, \quad (x,t) \in \Omega \times \mathbb{R}_+, \]

where $\Omega$ is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial \Omega$, $i = \sqrt{-1}$ and $u$ is a complex-valued unknown function. The equation will be supplemented with the homogeneous Dirichlet boundary condition

\[ (2-a) \quad u = 0 \quad \text{on} \quad \partial \Omega \times \mathbb{R}_+, \]

or the homogeneous Neumann boundary condition

\[ (2-b) \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial \Omega \times \mathbb{R}_+, \]

where $\nu$ is the unit outward normal on $\partial \Omega$, and the initial value of $u$:

\[ (3) \quad u(x,0) = u_0(x), \quad x \in \Omega. \]
Recently in [4], we proved that the initial-boundary value problem (1)–(3) has a unique strong global solution in \( X := L^2(\Omega; \mathbb{C}) \) under some conditions on the exponent \( p > 1 \) and the real parameters \( \lambda, \kappa, \alpha, \beta, \gamma \). We used the theory of nonlinear semi-groups in [4]. Therefore the solution \( u(t) \) to the problem (1)–(3) exists globally, but there is no guarantee that \( u(t) \) is differentiable for any \( t \in [0, \infty) \) (\( u(t) \) is differentiable for almost every \( t \in [0, \infty) \)).

The purpose of this paper is to show the differentiability for any \( t \in [0, \infty) \) of the solution \( u(t) \) to the problem (1)–(3) under additional restrictions on the exponent \( p \) and the dimension \( n \).

The basic idea is that we regard (1) as a genuinely nonlinear equation and simultaneously as a semilinear equation (Lipschitz perturbations of linear equations). As mentioned above, we can obtain a unique global strong solution by the theory of nonlinear semi-groups. On the other hand, we can prove the existence of a unique local \( C^1 \)-solution (continuously differentiable solution) by the theory of semilinear equations. Hence, by combining these two facts, namely the global existence by the theory of nonlinear semi-groups and the continuous differentiability by the theory of semilinear equations, we can prove the existence of a unique global \( C^1 \)-solution to the problem (1)–(3).

2. The Main Result and Proof

For the abstract setting we define three operators \( A_1, B, A \) in the complex Hilbert space \( X := L^2(\Omega; \mathbb{C}) \) with norm and inner product denoted by \( \| \cdot \| \) and \( (\cdot, \cdot) \), respectively:

\[
D(A_1) := H_0^2(\Omega; \mathbb{C}) \cap H^2(\Omega; \mathbb{C}) \quad \text{(in case of (2-a))},
\]

\[
D(A_1) := \{ u \in H^2(\Omega; \mathbb{C}); \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial \Omega \} \quad \text{(in case of (2-b))},
\]

\[
A_1 u := -\Delta u \quad \text{for } u \in D(A_1),
\]

\[
D(B) := \{ u \in X; |u|^{p-1} u \in X \} = L^{2p}(\Omega; \mathbb{C}),
\]

\[
B u := |u|^{p-1} u \quad \text{for } u \in D(B),
\]

\[
D(A) := D(A_1) \cap D(B),
\]

\[
A u := (\lambda + i\alpha)A_1 u + (\kappa + i\beta)B u - \gamma u \quad \text{for } u \in D(A),
\]

where \( H^2(\Omega; \mathbb{C}) \) and \( H_0^2(\Omega; \mathbb{C}) \) are the usual Sobolev Hilbert spaces.

The problem (1)–(3) is now equivalent to the following initial value problem for the abstract evolution equation

\[
\frac{d}{dt} u(t) + A u(t) = 0, \quad t \geq 0,
\]

\[
u(0) = u_0.
\]

For convenience we quote the existence theorem from [4]. It is summarized as follows:
**Theorem A ([4]).** Let \( \lambda > 0 \), \( \kappa > 0 \), \( p > 1 \), \( \frac{|\beta|}{\kappa} \leq \frac{2\sqrt{p}}{p-1} \), \( \lambda \kappa + \alpha \beta > 0 \).

Then for any \( T > 0 \) and \( u_0 \in D(A) \) there exists a unique strong solution \( u(t) \) (\( t \in [0, T] \)) to the problem (4) such that

(a) \( u(t) \in D(A) \) for \( t \in [0, T] \).

(b) \( u(t) \) is Lipschitz continuous for \( t \in [0, T] \).

(c) \( u(t) \) is strongly differentiable for almost every \( t \in [0, T] \) and satisfies (4).

(d) \( Au(t) \) is weakly continuous for \( t \in [0, T] \) (see [5, Theorem 31.A]).

At the same time we can regard (1)–(3) as a semilinear evolution equation

\[
(5) \quad \frac{d}{dt} u(t) + (\lambda + i\alpha)A_1 u(t) = - (\kappa + i\beta)Bu(t) + \gamma u(t), \quad t \geq 0,
\]

\[ u(0) = u_0. \]

Let \( n = 1, 2, 3 \). Then \( H^2(\Omega; \mathbb{C}) \) is embedded in \( L^\infty(\Omega; \mathbb{C}) \), and therefore \( D(A_1) \subset D(B) \) (consequently, \( D(A) = D(A_1) \cap D(B) = D(A_1) \)). Since the function \( f(s) = |s|^{p-1}s \) \( (p \geq 3) \) is three times continuously differentiable, we can see that the operator \( B \) is locally Lipschitz continuous on \( D(A_1) \) with graph norm \( || \cdot ||_{D(A_1)} \). Hence applying general theory of semilinear equations (see e.g. Otani [1, Theorem B] or Pazy [2, Remark after Theorem 6.1.7]), we have

**Theorem B.** Let \( n = 1, 2, 3 \). Assume that \( \lambda > 0 \) and \( p \geq 3 \). Then for any \( u_0 \in D(A_1) \) there exists \( T_m \) \( (0 < T_m \leq \infty) \) such that the problem (5) has a unique \( C^1 \)-solution \( u(\cdot) \in C^1([0, T_m] : X) \cap C([0, T_m] : D(A_1)) \cap C([0, T_m] : D(A)) \). Furthermore, if \( T_m < \infty \) then \( \lim_{t \uparrow T_m} ||u(t)|| + ||A_1 u(t)|| = \infty \).

As a combination of Theorem A and Theorem B, our theorem is stated as follows:

**Theorem.** Let \( n = 1, 2, 3 \). Assume that \( \lambda > 0 \), \( \kappa > 0 \), \( p \geq 3 \), \( \frac{|\beta|}{\kappa} \leq \frac{2\sqrt{p}}{p-1} \), and \( \lambda \kappa + \alpha \beta > 0 \). Then for any \( u_0 \in D(A) \) the problem (4) (or (5)) has a unique global \( C^1 \)-solution \( u(t) \) such that

\[ u(\cdot) \in C^1([0, \infty); X) \cap C([0, \infty); D(A)) \cap C([0, \infty); D(A_1)). \]

**Proof.** Under the assumption of our Theorem we can simultaneously apply Theorem A and Theorem B to the problem (4) (or (5)). Hence it is easy to see that the solution obtained by Theorem A coincides with the one obtained by Theorem B in the common time interval \( [0, T_m] \). Using the properties (a)-(d)
(especially (d)) of the solution \( u(t) \) \((0 \leq t < \infty)\) in Theorem A, we shall prove that \( T_m = \infty \). To this end, it suffices by Theorem B to show that if \( T_m < \infty \), then

\[
\sup_{0 \leq t < T_m} \left( \|u(t)\| + \|A_1 u(t)\| \right) < \infty.
\]

We know that \( u(t) \) satisfies the following equation

\[
\frac{d}{dt} u(t) - (\lambda + i\alpha) \Delta u(t) + (\kappa + i\beta) |u(t)|^{p-1} u(t) - \gamma u(t) = 0, \quad 0 \leq t < T_m.
\]

Dividing by \((\lambda + i\alpha)\) and taking inner product with \(|u|^{p-1}u\), we have

\[
\frac{1}{\lambda + i\alpha} \left( \frac{d}{dt} u(t), |u(t)|^{p-1} u(t) \right) - (\Delta u(t), |u(t)|^{p-1} u(t)) + \frac{\kappa + i\beta}{\lambda + i\alpha} \|u(t)\|_{L^2}^{2p} - \frac{\gamma}{\lambda + i\alpha} \int_{\Omega} |u(t)|^{p+1} \, dx = 0.
\]

Integration by parts yields

\[
-p \Re(\Delta u(t), |u(t)|^{p-1} u(t)) = \int_{\Omega} |u(t)|^{p-1} |\nabla u(t)|^2 \, dx + (p-1) \int_{\Omega} |u(t)|^{p-3} \sum_{j=1}^{n} \left\{ \Re(u(t) \cdot \frac{\partial}{\partial x_j} u(t)) \right\}^2 \, dx \geq 0,
\]

where \( \Re(\cdot) \) and \( \overline{u(t)} \) mean the real part of \((\cdot)\) and the complex conjugate of \( u(t) \), respectively. In view of (7), (8) we obtain for any \( \varepsilon > 0 \)

\[
\frac{\lambda \kappa + \alpha \beta}{\lambda^2 + \alpha^2} \|u(t)\|_{L^2}^{2p} \leq \left| \frac{1}{\lambda + i\alpha} \right| \int_{\Omega} \left| \frac{d}{dt} u(t) \right| \cdot |u(t)|^p \, dx + \frac{\lambda \gamma}{\lambda^2 + \alpha^2} \int_{\Omega} |u(t)|^{p+1} \, dx \leq \frac{1}{\sqrt{\lambda^2 + \alpha^2}} \int_{\Omega} \left( \frac{1}{4\varepsilon} \left| \frac{d}{dt} u(t) \right|^2 + \varepsilon |u(t)|^{2p} \right) \, dx + \frac{\lambda \gamma}{\lambda^2 + \alpha^2} \int_{\Omega} \left( \varepsilon |u(t)|^{2p} + \frac{1}{4\varepsilon} |u(t)|^2 \right) \, dx.
\]

Thus we have

\[
\frac{1}{\lambda^2 + \alpha^2} \left\{ (\lambda \kappa + \alpha \beta) - \varepsilon (\sqrt{\lambda^2 + \alpha^2} + \lambda |\gamma|) \right\} \|u(t)\|_{L^2}^{2p} \leq \frac{1}{4\varepsilon} \left( \frac{1}{\sqrt{\lambda^2 + \alpha^2}} \left| \frac{d}{dt} u(t) \right|^2 + \frac{\lambda |\gamma|}{\lambda^2 + \alpha^2} \|u(t)\|^2 \right) = \frac{1}{4\varepsilon} \left( \frac{1}{\sqrt{\lambda^2 + \alpha^2}} \|Au(t)\|^2 + \frac{\lambda |\gamma|}{\lambda^2 + \alpha^2} \|u(t)\|^2 \right).
\]
for $0 \leq t < T_m$. Choose $\varepsilon > 0$ small enough in such a way that the left hand side of (9) is positive. We know from Theorem A (d) that $\{\|Au(t)\|; t \in [0, T)\}$ is bounded for every $T > 0$. In particular, it follows that

$$\sup_{0 \leq t < T_m} \|Au(t)\| < \infty. \quad (10)$$

Moreover, it is not difficult to see that

$$\sup_{0 \leq t < T_m} \|u(t)\| \leq e^{\gamma T_m} \|u_0\| < \infty. \quad (11)$$

From (9), (10), (11) we have

$$\sup_{0 \leq t < T_m} \|u(t)\|_{L^p} < \infty.$$

Finally, in view of the definition of the operator $A$, we obtain (6). This completes the proof of Theorem. \square

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**References**


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