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THE EXPONENTIAL INTEGRAL AND THE CONVOLUTION

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Abstract. The exponential integral $\mathrm{ei}(\lambda x)$ and its associated functions $\mathrm{ei}_+(\lambda x)$ and $\mathrm{ei}_-(\lambda x)$ are defined as locally summable functions on the real line and their derivatives are found as distributions. Some convolution products of these distributions and other distributions are then found.

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The *exponential integral* $\mathrm{ei}(x)$ is defined for $x > 0$ by

$$\mathrm{ei}(x) = \int_x^\infty u^{-1} e^{-u} du, \tag{1}$$

see Sneddon [3], the integral diverging for $x \le 0$. It was pointed out in [1] that equation (1) can be rewritten in the form

$$\mathrm{ei}(x) = \int_x^\infty u^{-1}[e^{-u} - H(1-u)]\, du - H(1-x)\ln|x|,$$

where H denotes Heaviside's function. The integral in this equation is convergent for all x and so was used to define $\mathrm{ei}(x)$ on the real line.

More generally, if $\lambda \neq 0$, $\mathrm{ei}(\lambda x)$ was defined in the obvious way by

$$\mathrm{ei}(\lambda x) = \int_{\lambda x}^\infty u^{-1}[e^{-u} - H(1-u)]\, du - H(1-\lambda x)\ln|\lambda x|. \tag{2}$$

Further, $\mathrm{ei}_+(\lambda x)$ and $\mathrm{ei}_-(\lambda x)$ were defined by

$$\mathrm{ei}_+(\lambda x) = H(x)\,\mathrm{ei}(\lambda x), \quad \mathrm{ei}_-(\lambda x) = H(-x)\,\mathrm{ei}(\lambda x)$$

so that

$$\mathrm{ei}(\lambda x) = \mathrm{ei}_+(\lambda x) + \mathrm{ei}_-(\lambda x). \tag{3}$$

In particular, if $\lambda > 0$, we have

$$\begin{aligned}
\mathrm{ei}(\lambda x) &= \int_x^\infty u^{-1}[e^{-\lambda u} - H(1-\lambda u)]\,du - H(1-\lambda x)\ln|\lambda x|, && (4)\\
\mathrm{ei}_+(\lambda x) &= \int_x^\infty u^{-1}e^{-\lambda u}\,du, \quad x > 0, && (5)\\
\mathrm{ei}_-(\lambda x) &= -\gamma - \ln|\lambda| + \int_x^0 u^{-1}(e^{-\lambda u} - 1)\,du - \ln x_-, \quad x < 0, && (6)
\end{aligned}$$

where

$$\gamma = -\int_0^\infty u^{-1}[e^{-\lambda u} - H(1-\lambda u)]\,du$$

is Euler's constant.

If $\lambda < 0$, we have

$$\begin{aligned}
\mathrm{ei}(\lambda x) &= -\int_{-\infty}^x u^{-1}[e^{-\lambda u} - H(1-\lambda u)]\,du - H(1-\lambda x)\ln|\lambda x|, && (7)\\
\mathrm{ei}_+(\lambda x) &= -\gamma - \ln|\lambda| - \int_0^x u^{-1}(e^{-\lambda u} - 1)\,du - \ln x_+, \quad x > 0, && (8)\\
\mathrm{ei}_-(\lambda x) &= -\int_{-\infty}^x u^{-1}e^{-\lambda u}\,du, \quad x < 0. && (9)
\end{aligned}$$

The derivatives of these functions were found as

$$\begin{aligned}
[\mathrm{ei}(\lambda x)]' &= -e^{-\lambda x}x^{-1} = -x^{-1} - \sum_{i=1}^\infty \frac{(-\lambda)^i}{i!}x^{i-1}, && (10)\\
[\mathrm{ei}_+(\lambda x)]' &= -e^{-\lambda x}x_+^{-1} - (\gamma + \ln|\lambda|)\delta(x)\\
&= -x_+^{-1} - \sum_{i=1}^\infty \frac{(-\lambda)^i}{i!}x_+^{i-1} - (\gamma + \ln|\lambda|)\delta(x), && (11)\\
[\mathrm{ei}_-(\lambda x)]' &= e^{-\lambda x}x_-^{-1} + (\gamma + \ln|\lambda|)\delta(x)\\
&= x_-^{-1} - \sum_{i=1}^\infty \frac{\lambda^i}{i!}x_-^{i-1} + (\gamma + \ln|\lambda|)\delta(x), && (12)
\end{aligned}$$

for all $\lambda \neq 0$.

We now note the following results obtained by replacing x by $-x$ in the functions $\mathrm{ei}(\lambda x)$, $\mathrm{ei}_+(\lambda x)$ and $\mathrm{ei}_-(\lambda x)$.

$$\begin{aligned}
\mathrm{ei}(\lambda(-x)) &= \mathrm{ei}((-\lambda)x), && (13)\\
\mathrm{ei}_+(\lambda(-x)) &= H(-x)\,\mathrm{ei}(\lambda(-x)) = \mathrm{ei}_-((-\lambda)x), && (14)\\
\mathrm{ei}_-(\lambda(-x)) &= H(x)\,\mathrm{ei}(\lambda(-x)) = \mathrm{ei}_+((-\lambda)x). && (15)
\end{aligned}$$

These results will be used to deduce results for $\lambda < 0$ from results proved for $\lambda > 0$.

The classical definition of the convolution product of two functions f and g is as follows:

Definition 1. *Let f and g be functions. Then the convolution product $f * g$ is defined by*

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)\,dt$$

for all points x for which the integral exist.

It follows easily from the definition that if $f * g$ exists then $g * f$ exists and

$$f * g = g * f \tag{16}$$

and if $(f * g)'$ and $f * g'$ (or $f' * g$) exists, then

$$(f * g)' = f * g' \quad (\text{or } f' * g). \tag{17}$$

Definition 1 can be extended to define the convolution product $f * g$ of two distributions f and g in $\mathcal{D}'$ with the following definition, see Gel'fand and Shilov [2].

Definition 2. *Let f and g be distributions in $\mathcal{D}'$. Then the convolution product $f * g$ is defined by the equation*

$$\langle (f * g)(x), \phi \rangle = \langle f(y), \langle g(x), \phi(x+y) \rangle \rangle$$

for arbitrary ϕ in $\mathcal{D}$, provided f and g satisfy either of the conditions
(a) either f or g has bounded support,
(b) the supports of f and g are bounded on the same side.

It follows that if the convolution product $f * g$ exists by this definition then equations (16) and (17) are satisfied. In the following, the locally summable functions $e_+^{\lambda x}$ and $e_-^{\lambda x}$ are defined for $\lambda \neq 0$ by

$$e_+^{\lambda x} = H(x)e^{\lambda x} \quad e_-^{\lambda x} = H(-x)e^{\lambda x}.$$

Note that

$$e^{\lambda(-x)} = e^{(-\lambda)x}, \quad e_+^{\lambda(-x)} = e_-^{(-\lambda)x}, \quad e_-^{\lambda(-x)} = e_+^{(-\lambda)x}. \tag{18}$$

These results will also be used to deduce results for $\lambda < 0$ from results proved for $\lambda > 0$.

We now prove the following theorem

Theorem 1. *If $\lambda \neq 0$ and $\mu \neq 0$, then the convolution product* $\mathrm{ei}_+(\lambda x) * e_+^{\mu x}$ *exists and*

$$\mathrm{ei}_+(\lambda x) * e_+^{\mu x} = \mu^{-1}\{e^{\mu x}\,\mathrm{ei}_+[(\lambda+\mu)x] + \ln|1+\mu/\lambda|e_+^{\mu x} - \mathrm{ei}_+(\lambda x)\} \tag{19}$$

if $\lambda + \mu \neq 0$ and

$$\mathrm{ei}_+(\lambda x) * e_+^{-\lambda x} = \lambda^{-1}[\mathrm{ei}_+(\lambda x) + (\gamma + \ln|\lambda|)e_+^{-\lambda x} + e^{-\lambda x}\ln x_+]. \tag{20}$$

if $\lambda + \mu = 0$.

Proof. The convolution product $\mathrm{ei}_+(\lambda x) * e_+^{\mu x} = 0$ if $x < 0$ and so we suppose that $x > 0$. There are four cases to consider to prove equation (19).

Case (i). $\lambda > 0, \quad \lambda + \mu > 0.$

We first of all prove that

$$\begin{aligned}\mathrm{ei}_+(\lambda x) * e_+^{\mu x} &= \mu^{-1}e_+^{\mu x}\int_0^x u^{-1}[e^{-\lambda u} - e^{-(\lambda+\mu)u}]\,du + \\ &\quad +\mu^{-1}(e^{\mu x}-1)\,\mathrm{ei}_+(\lambda x).\end{aligned} \tag{21}$$

We have

$$\begin{aligned}\mathrm{ei}_+(\lambda x) * e_+^{\mu x} &= \int_0^x e^{\mu(x-t)}\int_t^\infty u^{-1}e^{-\lambda u}\,du\,dt \\ &= \int_0^x u^{-1}e^{-\lambda u}\int_0^u e^{\mu(x-t)}\,dt\,du + \int_x^\infty u^{-1}e^{-\lambda u}\int_0^x e^{\mu(x-t)}\,dt\,du \\ &= \mu^{-1}e_+^{\mu x}\int_0^x u^{-1}[e^{-\lambda u} - e^{-(\lambda+\mu)u}]\,du + \\ &\qquad +\mu^{-1}(e^{\mu x}-1)\,\mathrm{ei}_+(\lambda x),\end{aligned}$$

giving equation (21).

Further,

$$\begin{aligned}\int_0^x u^{-1}[e^{-\lambda u} - H(1-\lambda u)]\,du &= \int_0^\infty u^{-1}[e^{-\lambda u} - H(1-\lambda u)]\,du + \\ &\quad -\int_x^\infty u^{-1}e^{-\lambda u}\,du + \int_x^\infty u^{-1}H(1-\lambda u)\,du \\ &= -\gamma - \mathrm{ei}_+(\lambda x) + \int_x^\infty u^{-1}H(1-\lambda u)\,du.\end{aligned} \tag{22}$$

Similarly

$$\begin{aligned}\int_0^x u^{-1}[e^{-(\lambda+\mu)u} - H(1-(\lambda+\mu)u)]\,du &= -\gamma - \mathrm{ei}_+[(\lambda+\mu)x] + \\ &\quad + \int_x^\infty u^{-1}H[1-(\lambda+\mu)u]\,du.\end{aligned} \tag{23}$$

It follows from equations (22) and (23) that

$$\begin{aligned}\int_0^x u^{-1}(e^{-\lambda u} - e^{-(\lambda+\mu)u})\,du &= \mathrm{ei}_+[(\lambda+\mu)x] - \mathrm{ei}_+(\lambda x) + \\ &\quad + \int_0^\infty u^{-1}[H(1-\lambda u) - H(1-(\lambda+\mu)u)]\,du \\ &= \mathrm{ei}_+[(\lambda+\mu)x] - \mathrm{ei}_+(\lambda x) + \ln(1+\mu/\lambda). \quad (24)\end{aligned}$$

Equation (19) now follows from equations (21) and (24) for Case (i).

Case (ii). $\lambda > 0, \quad \lambda + \mu < 0.$

Equations (21) and (22) again hold in this case but since $\lambda + \mu < 0$, we have from equation (8)

$$\int_0^x u^{-1}(e^{-(\lambda+\mu)u} - 1)\,du = -\gamma - \ln|\lambda+\mu| - \mathrm{ei}_+[(\lambda+\mu)x] - \ln x_+. \quad (25)$$

It follows from equations (22) and (25) that equation (24) again holds. Equation (19) follows for Case (ii).

Case (iii). $\lambda < 0, \quad \lambda + \mu < 0.$

This time we have

$$\begin{aligned}\mathrm{ei}_+(\lambda x) * e_+^{\mu x} &= -(\gamma + \ln|\lambda|)\int_0^x e^{\mu t}dt - \int_0^x e^{\mu(x-t)}\int_0^t u^{-1}(e^{-\lambda u} - 1)\,du\,dt + \\ &\quad - \int_0^x e^{\mu(x-u)}\ln u\,du \\ &= -\mu^{-1}(\gamma + \ln|\lambda|)(e^{\mu x} - 1) - \int_0^x u^{-1}(e^{-\lambda u} - 1)\int_u^x e^{\mu(x-t)}\,dt\,du + \\ &\quad + \mu^{-1}e^{\mu x}\int_0^x \ln u\,d(e^{-\mu u} - 1) \\ &= -\mu^{-1}(\gamma + \ln|\lambda|)(e^{\mu x} - 1) + \\ &\quad + \mu^{-1}\int_0^x u^{-1}(e^{-\lambda u} - 1)(1 - e^{\mu(x-u)})\,du + \\ &\quad + \mu^{-1}(1 - e^{\mu x})\ln x - \mu^{-1}e^{\mu x}\int_0^x u^{-1}(e^{-\mu u} - 1)\,du \\ &= -\mu^{-1}\,\mathrm{ei}_+(\lambda x) - \mu^{-1}e^{\mu x}\int_0^x u^{-1}(e^{-(\lambda+\mu)u} - e^{-\mu u})\,du + \\ &\quad - \mu^{-1}(\gamma + \ln|\lambda|)e^{\mu x} - \mu^{-1}e^{\mu x}\ln x - \mu^{-1}e^{\mu x}\int_0^x u^{-1}(e^{-\mu u} - 1)\,du \\ &= -\mu^{-1}\,\mathrm{ei}_+(\lambda x) - \mu^{-1}e^{\mu x}\int_0^x u^{-1}(e^{-(\lambda+\mu)u} - 1)\,du + \\ &\quad - \mu^{-1}(\gamma + \ln|\lambda|)e^{\mu x} - \mu^{-1}e^{\mu x}\ln x \quad (26)\end{aligned}$$

and equation (19) follows for Case (iii).

Case (iv). $\lambda < 0,\ \lambda + \mu > 0$.

Equation (26) still holds for this case but this time we have

$$\begin{aligned}\int_0^x u^{-1}(e^{-(\lambda+\mu)u} - 1)\,du &= \int_0^\infty u^{-1}[e^{-(\lambda+\mu)u} - H(1-(\lambda+\mu)u)]\,du + \\ &\quad - \int_x^\infty u^{-1}e^{-(\lambda+\mu)u}\,du + \int_x^{(\lambda+\mu)^{-1}} u^{-1}\,du \\ &= -\gamma - \mathrm{ei}_+[(\lambda+\mu)x] - \ln[(\lambda+\mu)x]\end{aligned}$$

and equation (19) now follows from this equation and equation (26) for Case (iv).

We now have a further two cases to consider when $\lambda + \mu = 0$.

Case (v). $\lambda > 0,\ \lambda + \mu = 0$.

Equation (21) holds for this case. Further, replacing $\lambda+\mu$ by μ in equation (25) we have

$$\int_0^x u^{-1}(e^{-\lambda u} - 1)\,du = -\gamma - \mathrm{ei}_+(\lambda x) - \ln(\lambda x) \tag{27}$$

and equation (20) now follows from equation (21) for Case (v).

Case (vi). $\lambda < 0,\ \lambda + \mu = 0$.

Equation (26) holds when $\mu = -\lambda$ but it reduces to

$$\mathrm{ei}_+(\lambda x) * e_+^{-\lambda x} = \lambda^{-1}\,\mathrm{ei}_+(\lambda x) + \lambda^{-1}(\gamma + \ln|\lambda|)e^{-\lambda x} + \lambda^{-1}e^{-\lambda x}\ln x$$

and equation (20) follows for Case (vi). □

Corollary 1.1. *If $\lambda \neq 0$ and $\mu \neq 0$, then the convolution product $(e^{-\lambda x}x_+^{-s}) * e_+^{\mu x}$ exists for $s = 1, 2, \ldots$. In particular, if $\lambda + \mu \neq 0$, then*

$$(e^{-\lambda x}x_+^{-1}) * e_+^{\mu x} = -e^{\mu x}\,\mathrm{ei}_+[(\lambda+\mu)x] - (\gamma + \ln|\lambda+\mu|)e_+^{\mu x} \tag{28}$$

and if $\lambda + \mu = 0$, then

$$(e^{-\lambda x}x_+^{-1}) * e_+^{-\lambda x} = e^{-\lambda x}\ln x_+. \tag{29}$$

Proof. The convolution product $(e^{-\lambda x}x_+^{-s}) * e_+^{\mu x}$ exists by Definition 2 for $s = 1, 2, \ldots$ since $e^{-\lambda x}x_+^{-s}$ and $e_+^{\mu x}$ are both bounded on the left. In particular, we have from equations (11), (17) and (19)

$$\begin{aligned}[-e^{-\lambda x}x_+^{-1} - (\gamma + \ln|\lambda|)\delta(x)] * e_+^{\mu x} &= \mathrm{ei}_+(\lambda x) * [\mu e_+^{\mu x} + \delta(x)] \\ &= e^{\mu x}\,\mathrm{ei}_+[(\lambda+\mu)x] + \ln|1 + \mu/\lambda|e_+^{\mu x}\end{aligned}$$

and equation (28) follows.

Similarly, using equations (11), (17) and (20), we have

$$
\begin{aligned}
[-e^{-\lambda x}x_+^{-1} - (\gamma + \ln|\lambda|)\delta(x)] * e_+^{-\lambda x} &= \operatorname{ei}_+(\lambda x) * [-\lambda e_+^{-\lambda x} + \delta(x)] \\
&= -\operatorname{ei}_+(\lambda x) - (\gamma + \ln|\lambda|)e_+^{-\lambda x} - e^{-\lambda x}\ln x_+ + \operatorname{ei}_+(\lambda x)
\end{aligned}
$$

and equation (29) follows. □

Theorem 2. *If $\lambda \neq 0$ and $\mu \neq 0$, then the convolution product $\operatorname{ei}_-(\lambda x) * e_-^{\mu x}$ exists and*

$$\operatorname{ei}_-(\lambda x) * e_-^{\mu x} = -\mu^{-1}\{e^{\mu x}\operatorname{ei}_-[(\lambda+\mu)x] + \ln|1+\mu/\lambda|e_-^{\mu x} - \operatorname{ei}_-(\lambda x)\} \quad (30)$$

if $\lambda + \mu \neq 0$, and

$$\operatorname{ei}_-(\lambda x) * e_-^{-\lambda x} = -\lambda^{-1}[\operatorname{ei}_-(\lambda x) + (\gamma + \ln|\lambda|)e_-^{-\lambda x} + e^{-\lambda x}\ln x_-] \quad (31)$$

if $\lambda + \mu = 0$.

Proof. Replacing λ by $-\lambda$ and μ by $-\mu$ in equation (19) we get

$$
\begin{aligned}
\operatorname{ei}_+((-\lambda)x) * e_+^{(-\mu)x} &= -\mu^{-1}\{e^{(-\mu)x}\operatorname{ei}_+[((-\lambda-\mu)x] + \ln|1+\mu/\lambda)|e_+^{(-\mu)x} + \\
&\quad - \operatorname{ei}_+((-\lambda)x)\}
\end{aligned}
$$

and equation (30) follows on replacing x by $-x$ in this equation.

Equation (31) follows similarly. □

Corollary 2.1. *If $\lambda \neq 0$ and $\mu \neq 0$, then the convolution product $(e^{-\lambda x}x_-^{-s}) * e_-^{\mu x}$ exists for $s = 1, 2, \ldots$. In particular, if $\lambda + \mu \neq 0$, then*

$$(e^{-\lambda x}x_-^{-1}) * e_-^{\mu x} = -e^{\mu x}\operatorname{ei}_-[(\lambda+\mu)x] - (\gamma + \ln|\lambda+\mu|)e_-^{\mu x} \quad (32)$$

and if $\lambda + \mu = 0$ then

$$(e^{-\lambda x}x_-^{-1}) * e_-^{-\lambda x} = e^{-\lambda x}\ln x_-. \quad (33)$$

Proof. The existence of convolution product $(e^{-\lambda x}x_-^{-s}) * e_-^{\mu x}$ follows from equations (11), (17) and (30). In particular, we have from equations (11), (17) and (30)

$$
\begin{aligned}
[e^{-\lambda x}x_-^{-1} + (\gamma + \ln|\lambda|)\delta(x)] * e_-^{\mu x} &= \operatorname{ei}_-(\lambda x) * [\mu e_-^{\mu x} - \delta(x)] \\
&= -e^{\mu x}\operatorname{ei}_-[(\lambda+\mu)x] - \ln(1+\mu/\lambda)e_-^{\mu x}
\end{aligned}
$$

and equation (32) follows. Similarly, using equations (11), (17) and (31), we have

$$
\begin{aligned}
[e^{-\lambda x}x_-^{-1} + (\gamma + \ln|\lambda|)\delta(x)] * e_-^{-\lambda x} &= \operatorname{ei}_-(\lambda x) * [-\lambda e_-^{\lambda x} - \delta(x)] \\
&= (\gamma + \ln|\lambda|)e_-^{-\lambda x} + e^{-\lambda x}\ln x_-
\end{aligned}
$$

and equation (33) follows. □

Theorem 3. *If $\lambda, \lambda + \mu > 0$ and $\mu \neq 0$, then the convolution product* $\mathrm{ei}_+(\lambda x) * e^{\mu x}$ *exists and*

$$\mathrm{ei}_+(\lambda x) * e^{\mu x} = \mu^{-1} \ln(1 + \mu/\lambda) e^{\mu x}. \tag{34}$$

Proof. We have

$$\begin{aligned} \mathrm{ei}_+(\lambda x) * e^{\mu x} &= \int_0^\infty e^{\mu(x-t)} \int_t^\infty u^{-1} e^{-\lambda u} \, du \, dt \\ &= \int_0^\infty u^{-1} e^{-\lambda u} \int_0^u e^{\mu(x-t)} \, dt \, du \\ &= \mu^{-1} e^{\mu x} \int_0^\infty u^{-1} [e^{-\lambda u} - e^{-(\lambda+\mu)u}] \, du. \end{aligned}$$

Now

$$\begin{aligned} \int_0^\infty u^{-1}[e^{-\lambda u} - e^{-(\lambda+\mu)u}] \, du &= \int_0^\infty u^{-1}[e^{-\lambda u} - H(1 - \lambda u)] \, du + \\ &\quad - \int_0^\infty u^{-1}[e^{-(\lambda+\mu)u} - H(1 - (\lambda + \mu)u)] \, du + \\ &\quad + \int_0^\infty u^{-1}[H(1 - \lambda u) - H(1 - (\lambda + \mu)u)] \, du \\ &= -\gamma + \gamma + \ln(1 + \mu/\lambda) \end{aligned}$$

and equation (34) follows. □

Note 1. *Theorem 3 is equivalent to the van der Pol formula [4]*

$$\int_0^\infty e^{-px} \, \mathrm{ei}(\lambda x) \, dx = p^{-1} \ln(1 + p/\lambda).$$

Corollary 3.1. *If $\lambda, \lambda + \mu > 0$ and $\mu \neq 0$, then the convolution product* $(e^{-\lambda x} x_+^{-1}) * e^{\mu x}$ *exists and*

$$(e^{-\lambda x} x_+^{-1}) * e^{\mu x} = -(\gamma + \ln|\lambda + \mu|) e^{\mu x}. \tag{35}$$

Proof. Differentiating equation (34) we get

$$[-e^{-\lambda x} x_+^{-1} - (\gamma + \ln|\lambda|)\delta(x)] * e^{\mu x} = \ln(1 + \mu/\lambda) e^{\mu x}$$

and equation (35) follows. □

Note 2. *Corollary 3.1 is equivalent to*

$$\int_{-\infty}^\infty e^{-px} x_+^{-1} \, dx = -\gamma - \ln p,$$

due to Gel'fand and Shilov [2].

Corollary 3.2. *If $\lambda, \lambda+\mu > 0$ and $\mu \neq 0$, then the convolution products* $\mathrm{ei}_+(\lambda x) * e_-^{\mu x}$ *and* $(e^{-\lambda x} x_+^{-1}) * e_-^{\mu x}$ *exist and*

$$\mathrm{ei}_+(\lambda x) * e_-^{\mu x} = \mu^{-1}\{\mathrm{ei}_+(\lambda x) - e^{\mu x}\,\mathrm{ei}_+[(\lambda+\mu)x] + \ln(1+\mu/\lambda) e_-^{\mu x}\} \tag{36}$$

$$(e^{-\lambda x} x_+^{-1}) * e_-^{\mu x} = e^{\mu x}\,\mathrm{ei}_+[(\lambda+\mu)x] - (\gamma + \ln|\lambda+\mu|]) e_-^{\mu x}. \tag{37}$$

Proof. Equation (36) follows from equations (19) and (34). Equation (37) then follows from equations (28) and (35). □

Theorem 4. *If $\lambda, \lambda+\mu < 0$ and $\mu \neq 0$, then the convolution product* $\mathrm{ei}_-(\lambda x) * e^{\mu x}$ *exists and*

$$\mathrm{ei}_-(\lambda x) * e^{\mu x} = -\mu^{-1} \ln(1+\mu/\lambda) e^{\mu x}. \tag{38}$$

Proof. Replacing λ by $-\lambda$ and μ by $-\mu$ in equation (34) we get

$$\mathrm{ei}_+[(-\lambda) x)] * e^{-\mu x} = -\mu^{-1} \ln(1+\mu/\lambda) e^{-\mu x}$$

and equation (38) follows on replacing x by $-x$ in this equation. □

The results of the corollaries follow easily.

Corollary 4.1. *If $\lambda, \lambda+\mu < 0$ and $\mu \neq 0$, then the convolution product* $(e^{-\lambda x} x_-^{-1}) * e^{\mu x}$ *exists and*

$$(e^{-\lambda x} x_-^{-1}) * e^{\mu x} = -(\gamma + \ln|\lambda+\mu|) e^{\mu x}.$$

Corollary 4.2. *If $\lambda, \lambda+\mu < 0$ and $\mu \neq 0$, then the convolution products* $\mathrm{ei}_-(\lambda x) * e_+^{\mu x}$ *and* $(e^{-\lambda x} x_-^{-1}) * e_+^{\mu x}$ *exist and*

$$\mathrm{ei}_-(\lambda x) * e_+^{\mu x} = \mu^{-1}\{\mathrm{ei}_-(\lambda x) - e^{\mu x}\,\mathrm{ei}_-[(\lambda+\mu)x] + \ln(1+\mu/\lambda) e_+^{\mu x}\}$$
$$(e^{-\lambda x} x_-^{-1}) * e_+^{\mu x} = e^{\mu x}\,\mathrm{ei}_-[(\lambda+\mu)x] - (\gamma + \ln|\lambda+\mu|) e_+^{\mu x}.$$

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