

HOLOMORPHIC FAMILIES OF LINEAR OPERATORS IN BANACH SPACES

Victor Borisov and Noboru Okazawa

(Received September 27, 1997)

Abstract. Given two closed linear operators T and A in a Banach space, a sufficient condition is presented for the family $\{T(\kappa); \operatorname{Re} \kappa > a\} = \{T + \kappa A; \operatorname{Re} \kappa > a\}$, $a \in \mathbf{R}$, to be holomorphic of type (A). Detailed results are established when T and A are m -accretive in a reflexive Banach space. The results restricted to the Hilbert space case are almost identical with Kato's. As an application a simple first-order singular differential operator in the L^p -space ($1 < p < \infty$) is discussed. This is a generalization of Kato's result in the L^2 -case.

AMS 1991 Mathematics Subject Classification. Primary 47A56, Secondary 47B44.

Key words and phrases. Closed linear operators, holomorphic families of type (A), duality maps, m -accretive operators, singular differential operators of first-order.

Introduction

This paper is our first attempt to generalize Kato's theory [6] of holomorphic families of closed linear operators from the Hilbert space case to the (reflexive) Banach space case. We start with a brief review of Kato's theory.

Let T and A be linear m -accretive operators in a Hilbert space H . Then Kato assumes that A^{-1} exists (but not necessarily bounded) and there is a constant $a \in \mathbf{R}$ such that

$$(0.1) \quad \limsup_{\substack{\varepsilon \rightarrow 0 \\ (|\varepsilon|^{-1} \operatorname{Re} \varepsilon \geq \delta > 0)}} \operatorname{Re}((A + \varepsilon)^{-1}v, T^*v) \geq -a\|v\|^2 \quad \forall v \in D(T^*),$$

where T^* is the adjoint of T (and δ may depend on v). Under these conditions he proved among others that $\{T + \kappa A; \operatorname{Re} \kappa > a\}$ forms a holomorphic family of type (A) (see [6, Theorem 2.1]). Kato remarks that if A^{-1} is bounded, then (0.1) equals

$$(0.2) \quad \operatorname{Re}(A^{-1}v, T^*v) \geq -a\|v\|^2 \quad \forall v \in D(T^*)$$

and this condition is identical with Sohr's (see [11], [12]). For an interesting characterization of the condition (0.2) with $a = 0$ we refer to Miyajima [7].

Now let $\{A_\varepsilon; \varepsilon > 0\}$ be the Yosida approximation of A :

$$A_\varepsilon := A(1 + \varepsilon A)^{-1} = \varepsilon^{-1}[1 - (1 + \varepsilon A)^{-1}], \quad \varepsilon > 0.$$

Then the second author of the present paper introduced the following condition for $T + A$ (or its closure) to be m -accretive in H : there are constants $a \leq 1$ and $b, c \geq 0$ such that for all $u \in D(T)$,

$$(0.3) \quad \operatorname{Re}(Tu, A_\varepsilon u) \geq -a\|A_\varepsilon u\|^2 - b\|A_\varepsilon u\| \cdot \|u\| - c\|u\|^2$$

(see [8, Theorem 4.2 and Corollary 5.5]). It was shown in [8, Theorem 4.7] that if $a \geq 0$, then (0.2) implies (0.3) with $b = c = 0$. Here we should mention that the proof in [8] can be modified to include the case of $a < 0$. In fact, the inequality (4.10) in [8] can be replaced with (in the notation of this paper)

$$\operatorname{Re}(T_n u, A_\varepsilon u) \geq -a\|(1 + n^{-1}T^*)^{-1}A_\varepsilon u\|^2 \quad \forall u \in H,$$

where $\{T_n; n \in \mathbf{N}\}$ is the Yosida approximation of T (it remains to let $n \rightarrow \infty$). This is nothing but the inequality (3.1) in [6, Lemma 3.1] (with A replaced with A_ε). Therefore, we see that (0.3) is also a generalization of (0.2). It should be noted further that A need not be invertible in condition (0.3). Inequalities of the form (0.3) makes sense even in a (reflexive) Banach space if we replace the inner product $(Tu, A_\varepsilon u)$ with the semi-inner product $(Tu, F(A_\varepsilon u))$:

$$(0.4) \quad \operatorname{Re}(Tu, F(A_\varepsilon u)) \geq -a\|A_\varepsilon u\|^2 - b\|A_\varepsilon u\| \cdot \|u\| - c\|u\|^2,$$

where F is the duality map on the Banach space X to its adjoint X^* .

Thus the purpose of this paper is to reveal the usefulness of conditions of the form (0.4) in a (reflexive) Banach space. Namely, in Section 1 we consider the following inequality (introduced in [8]):

$$(0.5) \quad \operatorname{Re}(Tu, F(Au)) \geq -a\|Au\|^2 - b\|Au\| \cdot \|u\| - c\|u\|^2,$$

where T and A are simply assumed to be closed linear operators in a *general* Banach space. It ensures that $\{T + \kappa A; \operatorname{Re} \kappa > a\}$ forms a holomorphic family of type (A). In this connection we note that Borisov [2] considered the family $\{T + \kappa A\}$ for T and A in a Hilbert space, satisfying

$$\operatorname{Re}(Tu, Au) \geq -a\|Tu\|^2 - b\|Tu\| \cdot \|u\| - c\|u\|^2;$$

in this case the region of holomorphy is proved to be a circle of diameter a^{-1} (cf. [2, Lemma 1]). Section 2 is concerned with holomorphic families of linear m -accretive operators in a reflexive Banach space; we can use the fact that (0.4) implies (0.5). In the last Section 3 the first-order singular differential operator $d/dx + \kappa x^{-1}$ in $L^p(0, \infty)$, $1 < p < \infty$, will be analyzed in detail by using the theorems in the preceding sections. Roughly speaking, the operators in this application are not only m -accretive but also m -dispersive, that is, they are the generators of positive contraction semigroups. In other words, they are resolvent positive operators (cf. Arendt [1]).

Finally, we hope to deal with in a forthcoming paper typical examples of second-order singular differential operators in L^p by applying a generalization (Banach space version) of [6, Theorem 2.2].

1. Holomorphic families of closed linear operators

Let T and A be two closed linear operators from a Banach space X to another Y . The domain and range of an operator B from X to Y are denoted by $D(B)$ and $R(B)$, respectively. Then we consider the operator

$$(1.1) \quad T + \kappa A, \text{ with domain } D_0 := D(T) \cap D(A),$$

where κ is a complex parameter and D_0 is assumed to be non-trivial. We ask if $T + \kappa A$ forms a holomorphic family of type (A). An answer is given by Theorem 1.2 below.

First let us recall the definition (see Kato [4, VII-§2]). Let G_0 be a domain in \mathbf{C} . Then a family $\{T(\kappa); \kappa \in G_0\}$ is said to be *holomorphic of type (A)* if

i) $T(\kappa)$ is a closed linear operator (from X to Y) with domain $D(T(\kappa)) = D$ independent of κ ;

ii) $T(\kappa)u$ is holomorphic with respect to κ in G_0 for every $u \in D$.

In particular, if $T(\kappa)$ is a linear function of κ as in (1.1), then only the closedness of $T + \kappa A$ is required.

Now let Y^* be the adjoint space of Y . Then F denotes the duality map on Y to Y^* : for every $y \in Y$,

$$F(y) := \{g \in Y^*; (y, g) = \|y\|^2 = \|g\|^2\}.$$

The homogeneity of F is worth noticing: $F(ry) = rF(y)$, $r \geq 0$.

The next lemma is fundamental in this paper.

Lemma 1.1 ([8, Lemma 1.1]). *Let S, B be linear operators from X to Y . Set $D(S + B) := D(S) \cap D(B)$. Assume that for every $u \in D(S + B)$ there is $g \in F(Bu)$ such that*

$$(1.2) \quad \operatorname{Re}(Su, g) \geq -\gamma\|u\|^2 - \beta\|Bu\| \cdot \|u\| - \alpha\|Bu\|^2,$$

where $\alpha \in \mathbf{R}(\alpha < 1)$ and $\beta, \gamma \geq 0$ are constants.

Then B is $(S + B)$ -bounded :

$$\|Bu\| \leq (1 - \alpha)^{-1}\|(S + B)u\| + K_1\|u\|, \quad u \in D(S + B),$$

and hence S is also $(S + B)$ -bounded :

$$\|Su\| \leq \frac{2 - \alpha}{1 - \alpha}\|(S + B)u\| + K_1\|u\|, \quad u \in D(S + B),$$

where $K_1 := \beta(1 - \alpha)^{-1} + \sqrt{\gamma(1 - \alpha)^{-1}}$.

Our first result is the following

Theorem 1.2. *Let T, A be closed linear operators from X to Y . Assume that for every $u \in D_0$ there is $g \in F(Au)$ such that*

$$(1.3) \quad \operatorname{Re}(Tu, g) \geq -c\|u\|^2 - b\|Au\| \cdot \|u\| - a\|Au\|^2,$$

where $a \in \mathbf{R}$ and $b, c \geq 0$ are constants.

Then $T + \kappa A$ is closed for κ with $\operatorname{Re} \kappa > a$ and $\{T + \kappa A; \operatorname{Re} \kappa > a, \kappa \neq 0\}$ forms a holomorphic family of type (A); $\kappa = 0$ is an exceptional point even if $a < 0$.

Proof. Fix $r > 0$ arbitrarily. Then we see from (1.3) that for every $u \in D_0$ there is $g \in F(rAu)$ such that

$$\operatorname{Re}((T + aA)u, g) \geq -rc\|u\|^2 - b\|rAu\| \cdot \|u\|.$$

This is nothing but the inequality (1.2) with $S = T + aA$, $B = rA$ and $\alpha = 0$. Therefore it follows from Lemma 1.1 that

$$(1.4) \quad \|rAu\| \leq \|(T + (a + r)A)u\| + K_2\|u\|,$$

where $K_2 := b + \sqrt{rc}$, and

$$\|(T + aA)u\| \leq 2\|(T + (a + r)A)u\| + K_2\|u\|.$$

Consequently, we obtain

$$\|Tu\| \leq (2 + r^{-1}|a|)\|(T + (a + r)A)u\| + (1 + r^{-1}|a|)K_2\|u\|.$$

This inequality implies together with (1.4) that $T + (a + r)A$ is closed.

Next let $\kappa \in \mathbf{C}$ with $|\kappa - (a + r)| < r$. Then it follows from (1.4) that

$$\begin{aligned} \|(\kappa - (a + r))Au\| &= r^{-1}|\kappa - (a + r)| \cdot \|rAu\| \\ &\leq r^{-1}|\kappa - (a + r)| (\|(T + (a + r)A)u\| + K_2\|u\|). \end{aligned}$$

Since $r^{-1}|\kappa - (a + r)| < 1$, we see that

$$T + \kappa A = T + (a + r)A + (\kappa - (a + r))A$$

is closed; note that closedness is stable under relatively bounded small perturbation (see Kato [4, Theorem IV-1.1]). Noting further that

$$\begin{aligned} (1.5) \quad \{\kappa \in \mathbf{C}; \operatorname{Re} \kappa > a\} &= \cup_{r>0} \{\kappa \in \mathbf{C}; |\kappa - (a + r)| < r\} \\ &= \cup_{r>a_+} \{\kappa \in \mathbf{C}; |\kappa - (a + r)| < r\}, \end{aligned}$$

where $a_+ := \max \{a, 0\}$, we obtain the assertion of the theorem. \square

Remark 1.3. In particular, if $a < 0$ in (1.3), then we can take $r = -a$ in (1.4) :

$$(1.6) \quad \|Au\| \leq (-a)^{-1}\|Tu\| + K_3\|u\|, \quad u \in D_0,$$

where $K_3 = (-a)^{-1}K_2 = b(-a)^{-1} + \sqrt{c(-a)^{-1}}$. To conclude that A is T -bounded, it is necessary to know that D_0 is a core for T . This will be achieved in Theorem 2.2.

Proposition 1.4. *Let T, A be closed linear operators from X to Y . Assume that for every $u \in D_0$ there is $g \in F(Au)$ such that*

$$(1.7) \quad \operatorname{Re}(Tu, g) \geq -a\|Au\|^2,$$

where $a \in \mathbf{R}$ is a constant. Assume further that $T + tA$ is boundedly invertible for every $t > a_+$.

Then $T + \kappa A$ is also boundedly invertible for $\kappa \in \mathbf{C}$ with $\operatorname{Re} \kappa > a$.

Proof. Fix $r > a_- := \max\{-a, 0\}$ arbitrarily. Then as in Proof of Theorem 1.2 we have

$$\|(\kappa - (a + r))Au\| \leq r^{-1}|\kappa - (a + r)|\|(T + (a + r)A)u\|,$$

where $\kappa \in \mathbf{C}$ with $|\kappa - (a + r)| < r$ (note that $K_2 = 0$ by (1.7)). Since $a + r > a + a_- = a_+$, we see by assumption that $T + (a + r)A$ is boundedly invertible. Since $r^{-1}|\kappa - (a + r)| < 1$, it follows that

$$T + \kappa A = T + (a + r)A + (\kappa - (a + r))A$$

is also boundedly invertible; note that bounded invertibility is stable under relatively bounded small perturbation (see Kato [4, Theorem IV-1.16]). In view of (1.5) we obtain the assertion. \square

2. Holomorphic families of m -accretive operators

Let F be the duality map on a Banach space X to its adjoint X^* . Then a linear operator B in X is *accretive* if for every $u \in D(B)$ there is $f \in F(u)$ such that $\operatorname{Re}(Bu, f) \geq 0$. By definition an accretive operator B in X is *m -accretive* if $R(B + \xi) = X$ for $\xi > 0$.

Now let T and A be linear m -accretive operators in a *reflexive* Banach space X . As in Section 1 we consider the operator

$$(2.1) \quad T + \kappa A, \text{ with domain } D_0 := D(T) \cap D(A).$$

The m -accretivity of A allows us to use the Yosida approximation $\{A_\varepsilon; \varepsilon > 0\}$ of A (see Introduction). Accordingly we can state our basic assumption as follows.

(A1) For any $u \in D(T)$ and $\varepsilon > 0$ there is $f_\varepsilon \in F(A_\varepsilon u)$ such that

$$(2.2) \quad \operatorname{Re}(Tu, f_\varepsilon) \geq -c\|u\|^2 - b\|A_\varepsilon u\| \cdot \|u\| - a\|A_\varepsilon u\|^2,$$

where $a \in \mathbf{R}$ and $b, c \geq 0$ are constants.

The m -accretivity of $T+A$ depends on the size of the constant a in condition (A1).

Lemma 2.1 ([8, Theorem 4.2]). *Let T and A be m -accretive in reflexive X . Assume that condition (A1) (with $0 \leq a \leq 1$) is satisfied. If $a < 1$ then $T+A$ is m -accretive in X and D_0 is a core for A . In particular, if $a = 0$ then D_0 is a core for T . If $a = 1$ then $(T+A)^\sim$, the closure of $T+A$, is m -accretive in X .*

The next theorem is an immediate consequence of the consideration in [8] and Theorem 1.2.

Theorem 2.2. *Let T and A be m -accretive in reflexive X . Assume that condition (A1) is satisfied. Then*

(a) $T + tA$ is m -accretive in X for $t > a_+ := \max\{a, 0\}$; consequently, D_0 is dense in X . In particular, if $a > 0$ in (2.2), then $(T+aA)^\sim$ is also m -accretive in X .

(b) D_0 is a core for A ; consequently,

$$(2.3) \quad (A + \zeta)^{-1} = \operatorname{s-lim}_{t \rightarrow \infty} (t^{-1}T + A + \zeta)^{-1}, \quad \operatorname{Re} \zeta > 0.$$

(c) If $a \leq 0$ in (2.2), then D_0 is a core for T .

(d) If $a < 0$ in (2.2), then A is T -bounded with T -bound less than or equal to $(-a)^{-1}$ so that $D_0 = D(T)$.

(e) $T + \kappa A$ is closed for κ with $\operatorname{Re} \kappa > a$ and $\{T + \kappa A; \operatorname{Re} \kappa > a\}$ forms a holomorphic family of type (A).

Proof. Let $t > 0$. Then it follows from (2.2) that

$$\operatorname{Re}(t^{-1}Tu, f_\varepsilon) \geq -t^{-1}(c\|u\|^2 + b\|A_\varepsilon u\| \cdot \|u\|) - t^{-1}a_+\|A_\varepsilon u\|^2.$$

Since $t^{-1}T$ is m -accretive, we see from Lemma 2.1 that if $t^{-1}a_+ < 1$ then $T + tA = t(t^{-1}T + A)$ is m -accretive in X and D_0 is a core for A . For the convergence (2.3) see Kato [4, Theorem VIII-1.5]. Since X is reflexive, the m -accretivity of $T + tA$ implies that D_0 is dense in X (see Pazy [10, Theorem 1.4.6] or Yosida [13, VIII-§4]).

Now suppose that $a > 0$ in (2.2). Then we have

$$\operatorname{Re}(a^{-1}Tu, f_\varepsilon) \geq -a^{-1}(c\|u\|^2 + b\|A_\varepsilon u\| \cdot \|u\|) - \|A_\varepsilon u\|^2.$$

Since $a^{-1}T$ is also m -accretive, it follows from Lemma 2.1 that $(T + aA)^\sim = a(a^{-1}T + A)^\sim$ is m -accretive in X . Thus we obtain (a) and (b).

Next suppose that $a \leq 0$ in (2.2). Then we have

$$\operatorname{Re}(Tu, f_\varepsilon) \geq -c\|u\|^2 - b\|A_\varepsilon u\| \cdot \|u\|.$$

Therefore (c) follows also from Lemma 2.1. On the other hand, (d) is a non-selfadjoint generalization of [8, Remark 5.6]. But since (d) is an important information, we want to explain the relationship to Remark 1.3. First we note that the inequality (1.3) follows from (2.2). In fact, we can find a subsequence $\{f_{\varepsilon_n}\}$ of $\{f_\varepsilon\}$ and $g \in F(Au)$ such that

$$f_{\varepsilon_n} \rightarrow g \quad (n \rightarrow \infty) \text{ weakly}$$

(see [8, Proof of Theorem 4.2]). Thus we obtain (1.3) and hence (1.6):

$$\|Au\| \leq (-a)^{-1}\|Tu\| + K_3\|u\|, \quad u \in D_0.$$

Since D_0 is a core for T (as noted in (c)), we can give a complete proof of (d).

Finally, we prove (e). As noted above, (1.3) follows from (2.2). Therefore we see from Theorem 1.2 that $\{T + \kappa A; \operatorname{Re} \kappa > a (\kappa \neq 0)\}$ forms a holomorphic family of type (A). Now suppose that $a < 0$ in (2.2). Then we see from (d) that $D_0 = D(T)$. Therefore we do not need to exclude the origin $\kappa = 0$. Thus we can conclude that $\{T + \kappa A; \operatorname{Re} \kappa > a\}$ forms a holomorphic family of type (A). \square

Remark 2.3. If X^* is uniformly convex, then the assertions (a) and (d) of Theorem 2.2 are stated in Okazawa [9, Theorems 1.6 and 1.7] and applied to the “ m -accretivity” problem of Schrödinger operators in L^p ($1 < p < \infty$).

Now we are in a position to state the main theorem in this paper.

Theorem 2.4. *Let T and A be m -accretive in reflexive X . Assume that conditions (A1) above and (A2) below are satisfied.*

(A2) *For every $u \in D(A)$, $\operatorname{Im}(Au, g) = 0 \quad \forall g \in F(u)$ and*

$$(2.4) \quad (u, f) \geq 0 \quad \forall f \in F(Au).$$

Then

(i) *$\{T + \kappa A; \operatorname{Re} \kappa > a\}$ forms a holomorphic family of type (A), with*

$$(2.5) \quad \|Au\| \leq (\operatorname{Re} \kappa - a)^{-1} \|(T + \kappa A + \lambda)u\| + K(\operatorname{Re} \kappa)\|u\|,$$

where $u \in D_0, \lambda \in \mathbf{C}$ with $\operatorname{Re} \lambda \geq 0$ and

$$(2.6) \quad K(r) := b(r - a)^{-1} + \sqrt{c(r - a)^{-1}}, \quad r > a.$$

(ii) *The left half-plane \mathbf{C}_- is contained in the resolvent set of $T + \kappa A$ for $\operatorname{Re} \kappa > a$.*

- (iii) If $a \geq 0$ in (2.2), then $T + \kappa A$ is m -accretive in X for κ with $\operatorname{Re} \kappa > a$.
 If $a < 0$ in (2.2), then $T + \kappa A$ is m -accretive in X for κ with $\operatorname{Re} \kappa \geq 0$.
 (iv) If $D_{00} \subset D_0$ is a core for $T + \kappa_0 A$ for some $\kappa_0 > a_+$, then D_{00} is a core for A .

Theorem 2.4 combined with Theorem 2.2 is regarded as a generalization of Kato [6, Theorem 2.1] from the Hilbert space case to the reflexive Banach space case.

To prove Theorem 2.4 we need two lemmas.

Lemma 2.5. *Let A be a linear m -accretive operator in a Banach space X and $\{A_\varepsilon\}$ its Yosida approximation. Assume that condition (2.4) is satisfied. Then for any $v \in X$ and $\varepsilon > 0$*

$$(2.7) \quad (v, f_\varepsilon) \geq 0 \quad \forall f_\varepsilon \in F(A_\varepsilon v).$$

Proof. Let $u \in D(A)$ and $\varepsilon > 0$. Then it follows from (2.4) that

$$(2.8) \quad ((1 + \varepsilon A)u, f) \geq 0, \quad f \in F(Au).$$

Now let $v \in X$. Then $(1 + \varepsilon A)^{-1}v \in D(A)$. So, we can obtain (2.8) with $u = (1 + \varepsilon A)^{-1}v$ for all $f_\varepsilon \in F(A(1 + \varepsilon A)^{-1}v) = F(A_\varepsilon v)$. \square

The next lemma is a modification of Lemma 1.1.

Lemma 2.6. *Under conditions (A1) and (2.4) one has*

$$(2.9) \quad \|A_\varepsilon u\| \leq (\operatorname{Re} \kappa - a)^{-1} \|(T + \kappa A_\varepsilon + \lambda)u\| + K(\operatorname{Re} \kappa) \|u\|,$$

where $u \in D(T)$, $\operatorname{Re} \lambda \geq 0$ and $K(\cdot)$ is defined by (2.6).

Proof. Let $u \in D(T)$ and $\operatorname{Re} \lambda \geq 0$. Then it follows from (2.7) and (2.2) that

$$\begin{aligned} (\operatorname{Re} \kappa) \|A_\varepsilon u\|^2 &= \operatorname{Re}(\kappa A_\varepsilon u, f_\varepsilon) \\ &\leq \operatorname{Re}((T + \kappa A_\varepsilon + \lambda)u, f_\varepsilon) + c\|u\|^2 + b\|A_\varepsilon u\| \cdot \|u\| + a\|A_\varepsilon u\|^2. \end{aligned}$$

So we have

$$(\operatorname{Re} \kappa - a) \|A_\varepsilon u\|^2 \leq [\|(T + \kappa A_\varepsilon + \lambda)u\| + b\|u\|] \|A_\varepsilon u\| + c\|u\|^2$$

which implies (2.9). \square

Proof of Theorem 2.4. (i) We have already proved that $\{T + \kappa A; \operatorname{Re} \kappa > a\}$ forms a holomorphic family of type (A) (see Theorem 2.2(e)). On the other hand, (2.5) follows directly from (2.9).

(ii) Let $t > a_+$. Then the resolvent of $T + \kappa A$ will be given by the Neumann series for $\operatorname{Re} \lambda > 0$

$$(2.10) \quad (T + \kappa A + \lambda)^{-1} = (T + tA + \lambda)^{-1} \sum_{n=0}^{\infty} (t - \kappa)^n [A(T + tA + \lambda)^{-1}]^n.$$

We will show that

$$(2.11) \quad \|A(T + tA + \lambda)^{-1}\| \leq (t - a)^{-1} [1 + K(t)(\operatorname{Re} \lambda)^{-1}], \quad \operatorname{Re} \lambda > 0,$$

where $K(t)$ is given by (2.6). Since $T + tA$ is accretive, it follows that

$$K(t)\|u\| \leq K(t)(\operatorname{Re} \lambda)^{-1} \|(T + tA + \lambda)u\|, \quad \operatorname{Re} \lambda > 0.$$

So, we see from (2.5) with $\kappa = t > a_+$ that

$$\|Au\| \leq (t - a)^{-1} [1 + K(t)(\operatorname{Re} \lambda)^{-1}] \|(T + tA + \lambda)u\|, \quad u \in D_0, \quad \operatorname{Re} \lambda > 0,$$

which is nothing but (2.11) because $T + tA$ is m -accretive in X . Hence the resolvent (2.10) exists for $\operatorname{Re} \lambda > 0$ and κ in the region:

$$|t - \kappa| < \frac{(t - a) \operatorname{Re} \lambda}{K(t) + \operatorname{Re} \lambda}.$$

Noting that $K(t) \rightarrow 0 (t \rightarrow \infty)$ (see (2.6)), we have

$$\{\kappa \in \mathbf{C}; \operatorname{Re} \kappa > a\} = \bigcup_{t > a_+} \left\{ \kappa \in \mathbf{C}; |\kappa - t| < \frac{(t - a) \operatorname{Re} \lambda}{K(t) + \operatorname{Re} \lambda} \right\}.$$

(iii) First we note that (ii) implies

$$R(T + \kappa A + \lambda) = X, \quad \operatorname{Re} \kappa > a, \quad \operatorname{Re} \lambda > 0.$$

On the other hand, we see from the first half of condition **(A2)** that $T + \kappa A$ is accretive in X for κ with $\operatorname{Re} \kappa \geq 0$. Put $P(a) := \{\kappa; \operatorname{Re} \kappa > a\} \cap \{\kappa; \operatorname{Re} \kappa \geq 0\}$. Then we have

$$P(a) = \begin{cases} \{\kappa; \operatorname{Re} \kappa > a\} & \text{if } a \geq 0, \\ \{\kappa; \operatorname{Re} \kappa \geq 0\} & \text{if } a < 0. \end{cases}$$

Therefore we obtain the assertion of (iii).

(iv) Let D_{00} be a core for $T + t_0 A$ for some $t_0 > a_+$. Then it suffices to show that $(A + 1)D_{00}$ is dense in X (see Kato [4, Problem III-5.19]). Since $t^{-1}T + A$ is m -accretive for $t > a_+$ (see Theorem 2.2(a)), for every $v \in X$ there is a unique solution $u(t) \in D_0$ to the equation

$$(2.12) \quad (t^{-1}T + A + 1)u(t) = v.$$

But since D_{00} is a core for $T + t_0A$, there is a sequence $\{u_n(t)\}$ in D_{00} such that in $X \times X$

$$[u_n(t), (T + t_0A)u_n(t)] \rightarrow [u(t), (T + t_0A)u(t)] \quad (n \rightarrow \infty).$$

Since A is $(T + t_0A)$ -bounded (see (2.5)), it follows that $Au_n(t) \rightarrow Au(t)$ ($n \rightarrow \infty$).

Now suppose that $g \in X^*$ annihilates $(A + 1)D_{00}$. Then we have

$$((A + 1)u(t), g) = \lim_{n \rightarrow \infty} ((A + 1)u_n(t), g) = 0.$$

This implies together with (2.12) that

$$(2.13) \quad (v, g) = t^{-1}(Tu(t), g).$$

So, it remains to show that

$$(2.14) \quad t^{-1}Tu(t) \rightarrow 0 \quad (t \rightarrow \infty) \text{ weakly.}$$

First we note that (2.12) is written as $(T + tA + t)u(t) = tv$. Since $\|u(t)\| \leq \|v\|$, it follows from (2.5) (with $\kappa = \lambda = t$) that $\|Au(t)\| \leq [K(t) + (t - a)^{-1}t]\|v\|$. Therefore we see again from (2.12) that $\{t^{-1}Tu(t); t \geq 1 + a_+\}$ is bounded:

$$\|t^{-1}Tu(t)\| \leq \left(3 + K(t) + \frac{a}{t - a}\right)\|v\|.$$

Noting further that $D(T^*)$ is dense in X^* (see Pazy [10, Lemma 1.10.5]) and for every $h \in D(T^*)$

$$|t^{-1}(Tu(t), h)| \leq t^{-1}\|v\| \cdot \|T^*h\|,$$

we obtain (2.14). It then follows from (2.13) that $(v, g) = 0$ for all $v \in X$ and hence $g = 0$. \square

Remark 2.7. (a) In particular, if $b = c = 0$ in (2.2), then Theorem 2.4(ii) is a consequence of Proposition 1.4. In fact, let $\lambda \in \mathbf{C}$ with $\operatorname{Re} \lambda > 0$. Then, since (2.2) implies (1.3), we see from (2.4) that

$$\operatorname{Re}((T + \lambda)u, f) \geq -a\|Au\|^2, \quad f \in F(Au), \quad u \in D_0.$$

Furthermore, $T + \lambda + tA$ is boundedly invertible for $t > a_+$.

(b) If A is m -accretive in a Hilbert space, then condition **(A2)** means that A is nonnegative selfadjoint. We shall see the usefulness of Theorem 2.4 (condition **(A2)**) in the next section, however, we note that condition **(A2)** can be replaced with

$$(\mathbf{A2}') \quad \text{Given } u \in D(A), \operatorname{Re}(u, f) \geq 0 \text{ for all } f \in F(Au).$$

In a Hilbert space **(A2')** is automatically satisfied.

3. A first-order differential operator in L^p

As the simplest example of singular differential operators, we consider

$$(3.1) \quad \frac{d}{dx} + \frac{\kappa}{x}, \quad 0 < x < \infty,$$

in the reflexive Banach space $X_p := L^p(0, \infty)$, $1 < p < \infty$.

Let $W_0^{1,p} = W_0^{1,p}(0, \infty)$ be the usual Sobolev space. Then the operator $T_p := d/dx$ with domain $W_0^{1,p}$ is m -accretive in X_p (see Kato [4, Example IX-1.7]), with resolvent

$$(3.2) \quad (T_p - \zeta)^{-1}v(x) = \int_0^x e^{\zeta(x-y)}v(y) dy, \quad \operatorname{Re} \zeta < 0$$

(see [4, Problem III-6.9]). If $-\zeta = \xi > 0$, then $(T_p + \xi)^{-1}$ is positive (more precisely, positivity preserving). Therefore, $-T_p$ is m -dispersive in (real) X_p . The perturbing operator $A_p := x^{-1}$ is also m -accretive as a maximal multiplication operator in X_p , with

$$(3.3) \quad \operatorname{Im}(A_p u, F(u)) = 0 \quad \text{and} \quad (u, F(A_p u)) \geq 0 \quad \forall u \in D(A_p),$$

where $F(v)(x) := \|v\|^{2-p}|v(x)|^{p-2}v(x)$, $v \in X_p$. Thus condition **(A2)** is clearly satisfied. The Yosida approximation of A_p is given by

$$A_\varepsilon = A_{p,\varepsilon} = (x + \varepsilon)^{-1}, \quad \varepsilon > 0.$$

Since $(A_p + \xi)^{-1} = x(1 + \xi x)^{-1}$, it follows that $-A_p$ is also m -dispersive in (real) X_p .

Let p' be the conjugate exponent of p : $p'^{-1} + p^{-1} = 1$. Then a simple computation gives

$$(3.4) \quad \operatorname{Re}(T_p u, F(A_{p,\varepsilon} u)) = p'^{-1} \|A_{p,\varepsilon} u\|^2, \quad u \in W_0^{1,p}.$$

In fact, we have for $u \in C_0^1(0, \infty)$

$$(3.5) \quad \begin{aligned} & (T_p u, |A_\varepsilon u|^{p-2} A_\varepsilon u) \\ &= \lim_{\delta \downarrow 0} \int_0^\infty u'(x) (x + \varepsilon)^{-(p-1)} (|u(x)|^2 + \delta e^{-x})^{(p-2)/2} \overline{u(x)} dx; \end{aligned}$$

note that we can take $\delta = 0$ when $p \geq 2$. Hence it follows that

$$\begin{aligned} & \operatorname{Re}(T_p u, |A_\varepsilon u|^{p-2} A_\varepsilon u) \\ &= \frac{1}{p} \lim_{\delta \downarrow 0} \int_0^\infty (x + \varepsilon)^{-(p-1)} \frac{d}{dx} (|u(x)|^2 + \delta e^{-x})^{p/2} dx \\ &+ \frac{1}{2} \lim_{\delta \downarrow 0} \int_0^\infty \delta e^{-x} (x + \varepsilon)^{-(p-1)} (|u(x)|^2 + \delta e^{-x})^{(p-2)/2} dx \\ &= \frac{p-1}{p} \int_0^\infty (x + \varepsilon)^{-p} |u(x)|^p dx. \end{aligned}$$

Since $C_0^1(0, \infty)$ is dense in $W_0^{1,p}(0, \infty)$, we obtain (3.4) (see [9, Remark 2.11]).

Thus (2.2) is true with $a = -p'^{-1}$ and $b = c = 0$. Since $a < 0$, we see from Theorem 2.2(d) that $D(T_p) \subset D(A_p)$ and

$$(3.6) \quad \|A_p u\| \leq p' \|T_p u\|, \quad u \in D(T_p) = W_0^{1,p}.$$

This is a form of the Hardy inequality (see e.g. Ziemer [14, Lemma 1.8.11]). According to Theorem 2.4(i), $\{T_p + \kappa A_p; \operatorname{Re} \kappa > -p'^{-1}\}$ (with domain $D_0 = W_0^{1,p}$) forms a holomorphic family of type (A).

In particular, we see from Theorem 2.4(iii) that $T_p + \kappa A_p$ is m -accretive in X_p for $\operatorname{Re} \kappa \geq 0$.

On the other hand, the operator $S_p := -d/dx$ with domain $W^{1,p} = W^{1,p}(0, \infty)$ is also m -accretive in X_p (see Kato [4, Example IX-1.8]), that is, $-S_p$ is m -dissipative in X_p . The resolvent of $-S_p$ is given by

$$(3.7) \quad (-S_p - \zeta)^{-1} v(x) = - \int_x^\infty e^{\zeta(x-y)} v(y) dy, \quad \operatorname{Re} \zeta > 0$$

(see [4, III-Problem 6.9]). Therefore $-S_p$ is m -dispersive in (real) X_p .

Another computation gives

$$(3.8) \quad \operatorname{Re}(S_p u, F(A_{p,\varepsilon} u)) \geq -p'^{-1} \|A_{p,\varepsilon} u\|^2, \quad u \in W^{1,p}.$$

In fact, let $u := u^*|_{[0,\infty)}$ for $u^* \in C_0^1(\mathbf{R})$. Then we have (3.5) with $T_p u$ and $u'(x)$ replaced by $S_p u$ and $-u'(x)$, respectively. Hence it follows that

$$\operatorname{Re}(S_p u, |A_\varepsilon u|^{p-2} A_\varepsilon u) = \frac{1}{p} \varepsilon^{-(p-1)} |u(0)|^p - \frac{p-1}{p} \|A_\varepsilon u\|^p.$$

Since the restriction of $C_0^1(\mathbf{R})$ to $[0, \infty)$ is dense in $W^{1,p}(0, \infty)$, we obtain (3.8), that is, (2.2) is true with $a = p'^{-1}$ and $b = c = 0$. In this case A_p is not S_p -bounded. But since $W^{1,p} \cap D(x^{-1}) = W_0^{1,p}$ (see Lemma 3.1 below), it follows from Theorem 2.4(i) that $\{S_p + \kappa A_p; \operatorname{Re} \kappa > p'^{-1}\}$ (with domain $D_0 = W_0^{1,p}$) forms a holomorphic family of type (A). Accordingly,

$$(3.9) \quad \{-(S_p - \kappa A_p); \operatorname{Re} \kappa < -p'^{-1}\} \quad (\text{with domain } W_0^{1,p})$$

is holomorphic of type (A). In other words, the family (3.1) is also holomorphic of type (A) for $\operatorname{Re} \kappa < -p'^{-1}$ with domain $W_0^{1,p}$.

In this connection it is worth noticing that $D(S_p) \cap D(A_p)$ is not a core for S_p (cf. Theorem 2.2(c)).

Lemma 3.1. $W_0^{1,p}(0, \infty) = W^{1,p}(0, \infty) \cap D(x^{-1})$. Furthermore one has

- (i) $C_0^\infty(0, \infty)$ is a core for $T_p + \kappa A_p$ for κ with $\operatorname{Re} \kappa > -p'^{-1}$.
- (ii) $C_0^\infty(0, \infty)$ is a core for $-S_p + \kappa A_p$ for κ with $\operatorname{Re} \kappa < -p'^{-1}$.

Proof. Let $\phi \in C^\infty(0, \infty)$ with $0 \leq \phi \leq 1$ and

$$\phi(x) = 0 \ (x \leq 1), \quad \phi(x) = 1 \ (x \geq 2).$$

For $u \in W^{1,p}(0, \infty) \cap D(x^{-1})$ set

$$u_n(x) := \phi_n(x)u(x) := \phi(nx)u(x) \quad (x > 0), \quad n \in \mathbf{N}.$$

Then $u_n \in W_0^{1,p}(0, \infty)$ and $u_n \rightarrow u$ ($n \rightarrow \infty$) in $W^{1,p}(0, \infty)$; note that

$$\int_{1/n}^{2/n} |\phi'_n(x)u(x)|^p dx \leq M^p \int_{1/n}^{2/n} x^{-p} |u(x)|^p dx \rightarrow 0 \ (n \rightarrow \infty),$$

where $M := \max\{s|\phi'(s)|; 1 \leq s \leq 2\}$. Hence $W^{1,p} \cap D(x^{-1}) \subset W_0^{1,p}$. The opposite inclusion follows from the Hardy inequality (3.6).

(i) By definition we have $T_p = T_{p,\min}$ (the closure of d/dx with domain $C_0^\infty(0, \infty)$). It follows from (3.6) that $C_0^\infty(0, \infty)$ is also a core for $T_p + \kappa A_p$ for $\operatorname{Re} \kappa > -p'^{-1}$.

(ii) Noting that

$$\begin{aligned} \|(-S_p + \kappa A_p)u\| &\leq \|T_p u\| + |\kappa| \|A_p u\| \\ &\leq (1 + |\kappa|p') \|T_p u\|, \quad u \in W_0^{1,p}, \end{aligned}$$

we see that $C_0^\infty(0, \infty)$ is a core for $-S_p + \kappa A_p$ for $\operatorname{Re} \kappa < -p'^{-1}$. \square

Thus (3.1) gives two separate families of type (A) for $\operatorname{Re} \kappa > -p'^{-1}$ and for $\operatorname{Re} \kappa < -p'^{-1}$, both with domain $W_0^{1,p}$. Actually the second family can be continued analytically across the line $\operatorname{Re} \kappa = -p'^{-1}$ up to $\operatorname{Re} \kappa < p^{-1}$, though it is no longer of type (A). To see this we have only to consider the adjoint of the first family, with κ replaced with $\bar{\kappa}$.

In this way we can prove an L^p -generalization of Kato [6, Theorem 4.1].

Theorem 3.2. *There are two holomorphic families $\{T_p^\pm(\kappa)\}$ of realization of (3.1) in $X_p = L^p(0, \infty)$, and the rest part of the statement is divided into two parts.*

I. $T_p^+(\kappa) := T_p + \kappa A_p = d/dx + \kappa x^{-1}$, with domain $W_0^{1,p}$, is closed for $\operatorname{Re} \kappa > -p'^{-1}$. $T_p^+(\kappa)$ has the following properties:

- (i)₊ $T_p^+(\kappa) = T_p^{\min}(\kappa)$ (the closed minimal realization of (3.1)).
- (ii)₊ $T_p^+(\kappa)$ has resolvent set \mathbf{C}_- and residual spectrum \mathbf{C}_+ .

(iii)₊ For $\operatorname{Re} \kappa \geq 0$, $T_p^+(\kappa)$ is m -accretive in X_p , with resolvent

$$(3.10) \quad (T_p^+(\kappa) - \zeta)^{-1}v(x) = x^{-\kappa} \int_0^x e^{\zeta(x-y)} y^\kappa v(y) dy, \quad \operatorname{Re} \zeta < 0;$$

consequently, $-T_p^+(\kappa)$ is m -dispersive in (real) X_p for $\kappa \geq 0$.

(iv)₊ For $\operatorname{Re} \kappa > p^{-1}$, $T_p^+(\kappa) = T_p^{\max}(\kappa)$ (the maximal realization of (3.1)).

(v)₊ $\{T_p^+(\kappa); \operatorname{Re} \kappa > -p'^{-1}\} = \{T_p + \kappa A_p; \operatorname{Re} \kappa > -p'^{-1}\}$ forms a holomorphic family of type (A).

II. $T_p^-(\kappa) := -(T_{p'} - \bar{\kappa} A_{p'})^*$ is defined for $\operatorname{Re} \kappa < p^{-1}$. $T_p^-(\kappa)$ has the following properties:

(i)₋ $T_p^-(\kappa) = T_p^{\max}(\kappa)$.

(ii)₋ $T_p^-(\kappa)$ has resolvent set \mathbf{C}_+ and point spectrum \mathbf{C}_- , with eigenfunctions $x^{-\kappa} e^{\lambda x}$ with $\operatorname{Re} \lambda < 0$.

(iii)₋ For $\operatorname{Re} \kappa \leq 0$, $T_p^-(\kappa)$ is m -dissipative in X_p , with resolvent

$$(3.11) \quad (T_p^-(\kappa) - \zeta)^{-1}v(x) = -x^{-\kappa} \int_x^\infty e^{\zeta(x-y)} y^\kappa v(y) dy, \quad \operatorname{Re} \zeta > 0;$$

consequently, $T_p^-(\kappa)$ is m -dispersive in (real) X_p for $\kappa \geq 0$.

(iv)₋ For $\operatorname{Re} \kappa < -p'^{-1}$, $T_p^-(\kappa) = T_p^{\min}(\kappa) = -(S_p - \kappa A_p)$.

(v)₋ $\{T_p^-(\kappa); \operatorname{Re} \kappa < -p'^{-1}\} = \{-S_p + \kappa A_p; \operatorname{Re} \kappa < -p'^{-1}\}$ forms a holomorphic family of type (A) with domain $W_0^{1,p}$.

Proof. We have already proved basic inequalities (3.4) and (3.8). As mentioned above, the closedness of $T_p^+(\kappa)$ as well as (v)₊ is a direct consequence of (3.4) (see Theorem 2.2(e)).

(i)₊ is nothing but Lemma 3.1(i). The first half of (ii)₊ is a consequence of Theorem 2.4(ii). We can prove the second half by a direct computation. The m -accretivity of $T_p^+(\kappa)$ in (iii)₊ is also a consequence of (3.3) and (3.4) (see Theorem 2.4(iii)). It is not difficult to prove (3.10); compare with (3.2).

To prove (iv)₊ we consider $T_p^{\max}(\kappa)$. By definition $v = T_p^{\max}(\kappa)u$, $u \in D(T_p^{\max}(\kappa))$, is equivalent to

$$(3.12) \quad (u, (S_{p'} + \bar{\kappa} A_{p'})f) = (v, f) \quad \forall f \in C_0^\infty(0, \infty);$$

note that $T_p^* = S_{p'}$ and $A_p^* = A_{p'}$ ($p^{-1} + p'^{-1} = 1$). Since $C_0^\infty(0, \infty)$ is a core for $S_{p'} + \bar{\kappa} A_{p'}$ for $\operatorname{Re} \kappa > (p')'^{-1}$ (see Lemma 3.1(ii)), we have

$$(u, (S_{p'} + \bar{\kappa} A_{p'})f) = (v, f) \quad \forall f \in W_0^{1,p'}, \quad \operatorname{Re} \kappa > p^{-1}.$$

Noting further that $S_{p'} + \bar{\kappa} A_{p'}$ with domain $W_0^{1,p'}$ is m -accretive in $X_{p'}$ for $\operatorname{Re} \kappa > p^{-1}$, we see from the definition of the adjoint that

$$T_p^{\max}(\kappa) = (S_{p'} + \bar{\kappa} A_{p'})^*, \quad \operatorname{Re} \kappa > p^{-1}.$$

Since $T_p + \kappa A_p \subset (S_{p'} + \bar{\kappa} A_{p'})^*$ and $(S_{p'} + \bar{\kappa} A_{p'})^*$ is accretive, it follows from the m -accretivity of $T_p + \kappa A_p$ that

$$T_p + \kappa A_p = (S_{p'} + \bar{\kappa} A_{p'})^*, \quad \operatorname{Re} \kappa > p^{-1}.$$

This completes the proof of Part **I**.

It remains to prove Part **II**. To define $T_p^-(\kappa)$ for $\operatorname{Re} \kappa < p^{-1}$ it suffices to consider $T_{p'} - \bar{\kappa} A_{p'}$. In fact, $T_{p'} - \bar{\kappa} A_{p'} = d/dx - \bar{\kappa} x^{-1}$, with domain $W_0^{1,p'}$, is densely defined and closed for $\operatorname{Re}(-\kappa) > -(p')'^{-1}$, that is, for $\operatorname{Re} \kappa < p^{-1}$ (other properties are stated in Part **I**). Noting that $T_{p'}^* = S_p$ and $A_{p'}^* = A_p$, we have

$$(3.13) \quad S_p - \kappa A_p \subset (T_{p'} - \bar{\kappa} A_{p'})^*, \quad \operatorname{Re} \kappa < p^{-1}.$$

In view of (3.9) we are led to the definition

$$(3.14) \quad T_p^-(\kappa) := -(T_{p'} - \bar{\kappa} A_{p'})^* \quad \text{for } \operatorname{Re} \kappa < p^{-1}.$$

To prove (i)₋ and (ii)₋ let $v = T_p^{\max}(\kappa)u$, $u \in D(T_p^{\max}(\kappa))$. Then (3.12) yields that

$$(u, -(T_{p'} - \bar{\kappa} A_{p'})f) = (v, f) \quad \forall f \in C_0^\infty(0, \infty).$$

Since $C_0^\infty(0, \infty)$ is a core for $T_{p'} - \bar{\kappa} A_{p'}$ for $\operatorname{Re}(-\kappa) > -(p')'^{-1}$ (see Lemma 3.1(i)), we have

$$(3.15) \quad (u, -(T_{p'} - \bar{\kappa} A_{p'})f) = (v, f) \quad \forall f \in W_0^{1,p'}, \quad \operatorname{Re} \kappa < p^{-1}.$$

This proves (i)₋. Let $\lambda \in \mathbf{C}$ with $\operatorname{Re} \lambda > 0$. Then we see from (3.15) that

$$(u, -(T_{p'} - \bar{\kappa} A_{p'} + \bar{\lambda})f) = (v - \lambda u, f) \quad \forall f \in W_0^{1,p'}.$$

Since $-\bar{\lambda} \in \rho(T_{p'} - \bar{\kappa} A_{p'})$ (see (ii)₊), it follows that $-\lambda \in \rho((T_{p'} - \bar{\kappa} A_{p'})^*)$ and

$$-(T_{p'} - \bar{\kappa} A_{p'})^* u - \lambda u = T_p^{\max}(\kappa)u - \lambda u, \quad \operatorname{Re} \kappa < p^{-1},$$

where $\rho(T)$ is the resolvent set of T . This proves the first half of (ii)₋: $\lambda \in \rho(T_p^-(\kappa))$. We can prove the second half of (ii)₋ by a direct computation.

Now we prove (iii)₋. We see from (iii)₊ that for $\operatorname{Re} \kappa \leq 0$, $T_{p'}^+(\bar{\kappa}) = T_{p'} - \bar{\kappa} A_{p'}$ is m -accretive in $X_{p'}$. Therefore $(T_{p'} - \bar{\kappa} A_{p'})^*$ is also m -accretive in X_p , that is, $T_p^-(\kappa) = -(T_{p'} - \bar{\kappa} A_{p'})^*$ is m -dissipative in X_p for $\operatorname{Re} \kappa \leq 0$. It is not difficult to prove (3.11); compare with (3.7).

On the other hand, it follows from (3.3) and (3.8) that $S_p - \kappa A_p$ is m -accretive in X_p for $\operatorname{Re}(-\kappa) > p'^{-1}$ (see Theorem 2.4(iii)), that is, for $\operatorname{Re} \kappa < -p'^{-1}$. In view of (3.13) we see from (iii)₋ that

$$(3.16) \quad S_p - \kappa A_p = (T_{p'} - \bar{\kappa} A_{p'})^*, \quad \operatorname{Re} \kappa < -p'^{-1}.$$

Since $-(S_p - \kappa A_p) = T_p^{\min}(\kappa)$ (see Lemma 3.1(ii)), (iv)₋ follows from (3.14) and (3.16). Therefore (v)₋ is clear from (3.8) (see Theorem 2.2(e)). \square

Remark 3.3. (a) We have

$$T_p^{\max}(\kappa) = T_p^{\min}(\kappa) \text{ for } \kappa \text{ with } \operatorname{Re} \kappa < -p'^{-1} \text{ or } p^{-1} < \operatorname{Re} \kappa.$$

Both $T_p^{\pm}(\kappa)$ are defined on the strip

$$(3.17) \quad S(p', p) := \left\{ \kappa; -\frac{1}{p'} < \operatorname{Re} \kappa < \frac{1}{p} \right\},$$

where

$$T_p^{\min}(\kappa) = T_p^+(\kappa) \subset T_p^-(\kappa) = T_p^{\max}(\kappa);$$

in particular $T_p^{\min}(0) = T_p \subsetneq -S_p = -(T_{p'})^* = T_p^{\max}(0)$. Note that we obtain the strip $0 < \operatorname{Re} \kappa < 1$ as the limit of $p \rightarrow 1$ and the strip $-1 < \operatorname{Re} \kappa < 0$ as the limit of $p \rightarrow \infty$.

(b) $-T_p^+(\kappa)$ generates a contraction semigroup for $\operatorname{Re} \kappa \geq 0$, and $T_p^-(\kappa)$ does for $\operatorname{Re} \kappa \leq 0$. The semigroups generated by $-T_p^+(\kappa)$ and $T_p^-(\kappa)$ are holomorphic in κ in the half-planes $\{\kappa; \operatorname{Re} \kappa > 0\}$ and $\{\kappa; \operatorname{Re} \kappa < 0\}$, respectively. To see this we can employ a recent result of Kantorovitz [3]. In fact, $\{-T_p^+(\kappa)\}$ and $\{T_p^-(\kappa)\}$ have resolvent analyticity (in the sense of Kantorovitz) with respect to κ . Therefore the desired assertion follows from the equivalence of semigroup analyticity and resolvent analyticity (see [3, Theorem 1]). It appears that neither $-T_p^+(\kappa)$ nor $T_p^-(\kappa)$ generates a C_0 -semigroup for other values of κ . The same question arises even if $L^p(0, \infty)$ is replaced with $L^p(0, 1)$. But the question in $L^p(0, 1)$, $1 \leq p < \infty$, has been solved by Arendt [1, Examples 3.3 and 3.5].

(c) The family $\{T_p^-(\kappa); \kappa \in S(p', p)\}$, where $S(p', p)$ is defined by (3.17), is not holomorphic of type (A) or of any familiar type dealt with in [4], as is seen from the behavior of its eigenfunctions $x^{-\kappa}e^{\lambda x}$. In fact, let $\kappa, \nu \in S(p', p)$. Then $x^{-\kappa}e^{\lambda x}$ does not belong to $D(T_p^-(\nu))$ for $\nu \neq \kappa$. This implies that $D(T_p^-(\kappa)) \neq D(T_p^-(\nu))$ for $\kappa, \nu \in S(p', p)$ with $\kappa \neq \nu$.

Remark 3.4. Dirac operators are typical examples of first-order differential operators in $(L^2(\mathbf{R}^N))^4$. But Theorems 2.2 and 2.4 (in which X is a Hilbert space) do not yield satisfactory results (see [5], [6]).

Acknowledgement. The authors would like to thank the referee for his careful reading of the manuscript.

References

1. W. Arendt, *Resolvent positive operators*, Proc. London Math. Soc. **54** (1987), 321–349.
2. V. Borisov, *Uniform resolvent convergence of linear operators under singular perturbation*, SUT J. Math. **33** (1997), 87–97.

3. S. Kantorovitz, *Analytic families of semigroups*, Semigroup Forum **54** (1997), 356–363.
4. T. Kato, *Perturbation Theory for Linear Operators*, Grundlehren math. Wissenschaften, Band 132, Springer-Verlag, Berlin and New York, 1966; second ed., 1976.
5. T. Kato, *Holomorphic families of Dirac operators*, Math. Z. **183** (1983), 399–406.
6. T. Kato, *Remarks on holomorphic families of Schrödinger and Dirac operators*, Differential Equations, Mathematics Studies **92**, North-Holland, Amsterdam, 1984, pp. 341–352.
7. S. Miyajima, *On conditions $\operatorname{Re}(Au, Su) \geq 0$ and $\operatorname{Re}(A^*u, S^{-1}u) \geq 0$ that lead to m -accreitivity*, RIMS Kokyuroku **860**, RIMS Kyoto Univ., Kyoto, 1994, pp. 12–18 (in Japanese).
8. N. Okazawa, *On the perturbation of linear operators in Banach and Hilbert spaces*, J. Math. Soc. Japan **34** (1982), 677–701.
9. N. Okazawa, *L^p -theory of Schrödinger operators with strongly singular potentials*, Japanese J. Math. **22** (1996), 199–239.
10. A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Applied Math. Sciences, vol. 44, Springer-Verlag, Berlin and New York, 1983.
11. H. Sohr, *Über die Selbstadjungiertheit von Schrödinger-Operatoren*, Math. Z. **160** (1978), 255–261.
12. H. Sohr, *Störungstheoretische Regularitätsuntersuchungen*, Math. Z. **179** (1982), 179–192.
13. K. Yosida, *Functional Analysis*, Grundlehren Math. Wissenschaften, Band. 123, Springer-Verlag, Berlin and New York, 1965; 6th ed., 1980.
14. W.P. Ziemer, *Weakly Differentiable Functions : Sobolev Spaces and Functions of Bounded Variation*, Graduate Texts in Math., vol. 120, Springer-Verlag, Berlin and New York, 1989.

Victor Borisov

Department of Mathematics, Ryazan Radio Engineering Academy
Ryazan 390005, Russia

and

Department of Mathematics, Science University of Tokyo
Wakamiya-cho, Shinjuku-ku, Tokyo 162, Japan

Noboru Okazawa

Department of Mathematics, Science University of Tokyo
Wakamiya-cho, Shinjuku-ku, Tokyo 162, Japan