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CERTAIN METRICS ON A PRINCIPAL FIBER BUNDLE AND VARIATIONAL PROBLEMS

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Abstract. We define a functional constructed from the scalar curvature of a certain metric on a principal fiber bundle and obtain some equations which correspond to the Einstein field equation, the Yang-Mills equation and the Brans-Dicke type wave equation by variations of this functional.

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§0. Introduction

The first model of the natural unification of gauge fields and gravitation goes back to the five-dimensional model of Kaluza and Klein. Their model extends in a reasonably straightforward way to the case of gauge potentials on principal fiber bundles with arbitrary structure groups. Let P be a principal bundle over a semi-Riemannian manifold (M, g) with a structure group G. If ω is a connection 1-form on P and k_0 is an ad-invariant metric on the Lie algebra of G, then a metric h on P is constructed from g, ω and k_0. It is called a bundle metric. In this case, the Einstein field equation and the Yang-Mills equation arise from a single variational principle, see [1], for example.

In this paper, we assume that an ad-invariant metric depends on a point of M. Let k be a map from M to the set of all ad-invariant metrics on the Lie algebra of G. In particular we consider the case where k is constructed from a fixed ad-invariant metric and a positive function on M. Physically this scalar field gives scales of the internal spaces. When G is compact and its Lie algebra is simple, a positive definite ad-invariant metric is unique up to multiplication by a constant [3]. We give a metric h on P similar to a bundle metric using such a map k. Because the projection $\pi : (P, h) \to (M, g)$ is a semi-Riemannian submersion, geometrical quantities are described by the fundamental tensors defined in [4].

We define a functional constructed from a scalar curvature of (P, h). By demanding that the integral of this functional be stationary under variations of the metric on M, we obtain the equation correspondence to the Einstein field equation. Similarly, variations of the connection 1-form lead to the equation correspondence to the Yang-Mills equation. Moreover we get the Brans-Dicke type wave equation [2] for a scalar field on M by variations of the function on M.

In Section 1, we will prepare the notation and terminology used in this paper. Section 2 is devoted to compute the fundamental tensors. In Section 3, using the lemmas in Section 2, the curvatures of (P, h) will be calculated. We will define some functional on M from the scalar curvature of h and consider variational problems with respect to the metric, connection and scalar field in Section 4. Finally in Section 5, the results in Section 4 will be applied to cosmology.

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§1. Preliminaries

Throughout this paper, all objects are assumed to be smooth. Let G be a Lie group and $\mathcal{G}$ its Lie algebra. Let P be a principal G-bundle over a manifold M and $\pi : P \to M$ the projection. We define the vertical space of TP by $\mathcal{V}(P) :=\mathrm{Ker}\pi_*$, where TP is the tangent bundle of P and π_* is the differential map of π. The set of connection 1- forms on P is denoted by $\mathcal{C}(P)$. For $\omega \in \mathcal{C}\ (P)$, we define the horizontal space of TP by $\mathcal{H}(P) :=\mathrm{Ker}\omega$. Then we have $TP = \mathcal{V}(P) \oplus \mathcal{H}(P)$ (direct sum). Let $\mathcal{V}$ (resp. $\mathcal{H}$) be the projection of TP onto $\mathcal{V}(P)$ (resp. $\mathcal{H}(P)$). For a vector field E on M, the horizontal lift of E is denoted by $\tilde{E}$ or $E^{\sim}$. For $A \in \mathcal{G}$, the fundamental vector field induced from A is denoted by A^*. For a vector space W, the set of W-valued k-forms on P is denoted by $\Lambda^k(P, W)$. The set of all smooth functions on a manifold M is denoted by $C^\infty(M)$.

For $\varphi \in \Lambda^i(P, \mathcal{G})$ and $\psi \in \Lambda^j(P, \mathcal{G})$, we define $[\varphi, \psi] \in \Lambda^{i+j}(P, \mathcal{G})$ by

$$[\varphi, \psi](X_1, ..., X_{i+j}) = \frac{1}{i!j!} \sum_{\sigma} (-1)^\sigma [\varphi(X_{\sigma(1)}, ..., X_{\sigma(i)}), \psi(X_{\sigma(i+1)}, ..., X_{\sigma(i+j)})],$$

where the sum is over the set of all permutations σ of $1, ..., i+j$ and $(-1)^\sigma = \pm 1$ is the sign of σ. For $\omega \in \mathcal{C}(P)$ and $\tau \in \Lambda^i(P, \mathcal{G})$, we define $\tau^H \in \Lambda^i(P, \mathcal{G})$ by $\tau^H(X_1, ..., X_i) = \tau(\mathcal{H}X_1, ..., \mathcal{H}X_i)$ and the exterior covariant derivative of τ by $D^\omega \tau := (d\tau)^H$. The curvature form $\Omega \in \Lambda^2(P, \mathcal{G})$ is defined by $\Omega := D^\omega \omega$. The equation $\Omega = d\omega + (1/2)[\omega, \omega]$ is called the structure equation. From the structure equation, we see that

$$\Omega(X, Y) = -\omega([X, Y]) \quad \text{for horizontal vector fields } X \text{ and } Y \tag{1.1}$$

and

$$d\Omega = [\Omega, \omega]. \tag{1.2}$$

The set of all metrics on a manifold M is denoted by $\mathcal{M}(M)$. For $a \in G$, let $Ad_a : G \to G$ be the adjoint isomorphism given by $Ad_a(b) = aba^{-1}$ and $ad(a) : \mathcal{G} \to \mathcal{G}$ the induced isomorphism of $\mathcal{G}$, that is, $ad(a) = (Ad_a)_{*e}$. The set of all ad-invariant metrics on $\mathcal{G}$ is denoted by $\mathcal{M}_{ad}(\mathcal{G})$. For $k_0 \in \mathcal{M}_{ad}(\mathcal{G})$, we see that

$$k_0([A, B], C) + k_0(B, [A, C]) = 0 \ for \ A,\ B,\ C \in \mathcal{G}. \tag{1.3}$$

§2. Fundamental tensors

Let P be a principal G-bundle over a manifold M and $\pi : P \to M$ the projection. We define a metric h on P as follows.

Definition. *For $k : M \to \mathcal{M}_{ad}(\mathcal{G})$, $g \in \mathcal{M}(M)$ and $\omega \in \mathcal{C}(P)$, we define $h \in \mathcal{M}(P)$ by*

$$h(E, F) = g(\pi_* E, \pi_* F) + (k \circ \pi)(\omega(E), \omega(F))$$

for any tangent vector E and F of P. When k is a constant map, it is called a bundle metric.

If P is the semi-Riemannian manifold with the metric h as above, then $\pi : (P, h) \to (M, g)$ is a semi-Riemannian submersion. The tensors T and A are defined for arbitrary vector fields E and F by

$$T_E F := \mathcal{H}\nabla_{\mathcal{V}E}(\mathcal{V}F) + \mathcal{V}\nabla_{\mathcal{V}E}(\mathcal{H}F)$$

and

$$A_E F := \mathcal{V}\nabla_{\mathcal{H}E}(\mathcal{H}F) + \mathcal{H}\nabla_{\mathcal{H}E}(\mathcal{V}F),$$

where ∇ is the covariant derivative of (P, h). They are called the fundamental tensors in [4] and [5]. For a fixed ad-invariant metric k_0 on $\mathcal{G}$ and a smooth function $K > 0$ on M, we set $k = \varepsilon K^2 k_0$ $(\varepsilon = \pm 1)$ in the definition of h and consider only this case in the present paper. We write $\bar{K} := K \circ \pi$ for simplicity. If the Lie algebra of a compact Lie group is simple, then the positive definite ad-invariant metric is unique up to multiplication by a constant [3].

To compute the fundamental tensors and the curvature tensor, we define some defferential operators in the usual way. For a function f on a manifold, $\mathrm{grad} f$ is the gradient vector field of f, H^f is the Hessian of f and Δf is the Laplacian of f defined by $\Delta f = -\mathrm{div}(\mathrm{grad} f)$, where div is a divergence. We have the following lemma.

Lemma 2.1. *If U, V are vertical and X, Y are horizontal, then*

$$T_U V = -\varepsilon \bar{K} k_0(\omega(U), \omega(V)) \mathrm{grad} \bar{K}, \tag{2.1}$$

$$T_U X = \frac{X\bar{K}}{\bar{K}} U = \frac{h(\mathrm{grad}\bar{K}, X)}{\bar{K}} U, \tag{2.2}$$

$$A_X Y = \frac{1}{2} \mathcal{V}[X, Y] \tag{2.3}$$

and

$$A_X U = -\frac{\varepsilon}{2} \bar{K}^2 \overline{\Omega^{\omega(U)}(X)}, \tag{2.4}$$

where $\overline{\Omega^{\omega(U)}(X)}$ is defined by $h(\overline{\Omega^{\omega(U)}(X)}, E) = k_0(\omega(U), \Omega(E, X))$ for any vector field E on P.

Moreover, to compute the curvature of (P, h), we show the following lemma by using Lemma 2.1.

Lemma 2.2. *If U, V, W are vertical and X, Y, Z are horizontal, then*

$$\mathcal{H}((\nabla_V T)_U W) = \frac{1}{2} \bar{K}^3 k_0(\omega(U), \omega(W)) \overline{\Omega^{\omega(V)}(\mathrm{grad}\bar{K})}, \tag{2.5}$$

$$\begin{aligned} \mathcal{H}((\nabla_X T)_V W) = & -\varepsilon \bar{K} k_0(\omega(V), \omega(W)) \mathcal{H}(\nabla_X \mathrm{grad}\bar{K}) \\ & + \varepsilon k_0(\omega(V), \omega(W)) h(X, \mathrm{grad}\bar{K}) \mathrm{grad}\bar{K}, \end{aligned} \tag{2.6}$$

$$\omega((\nabla_V A)_X Y) = \frac{1}{4}[\Omega, \omega](X, Y, V) - \frac{1}{2}(\nabla_V \Omega)(X, Y) \tag{2.7}$$

and

$$\omega((\nabla_Z A)_X Y) = -\frac{1}{2}\frac{h(Z, \mathrm{grad}\bar{K})}{\bar{K}}\Omega(X, Y) - \frac{1}{2}(\nabla_Z \Omega)(X, Y), \tag{2.8}$$

where $(\nabla_E \Omega)$ *is defined by* $(\nabla_E \Omega)(F_1, F_2) = E\Omega(F_1, F_2) - \Omega(\nabla_E F_1, F_2) - \Omega(F_1, \nabla_E F_2)$.

§3. The Curvature tensors

Let $\hat{\nabla}^x$ be the covariant derivative of $\pi^{-1}(x)$ with respect to the induced metric from h and ∇^* the covariant derivative of (M, g). Let R (resp. $\hat{R}^x$, R^*) be the curvature tensor of ∇ (resp. $\hat{\nabla}^x$, ∇^*). However we omit the superscript x for simplicity. Let $R_*(X, Y)Z$ be the horizontal vector field such that $\pi_*(R_*(X, Y)Z) = R^*(\pi_* X, \pi_* Y)\pi_* Z$ at each point of P. We can compute the curvature of (P, h) by using Lemmas 2.1 and 2.2.

Proposition 3.1. *For vertical vector fields* U, V, W, F *and horizontal vector fields* X, Y, Z, H, *we obtain*

$$\begin{aligned} h(R(U, V)W, F) &= \frac{\varepsilon}{4}\bar{K}^2 k_0([\omega(U), \omega(V)], [\omega(W), \omega(F)]) \\ &\quad -\bar{K}^2 k_0(\omega(U), \omega(W)) k_0(\omega(V), \omega(F)) g(\mathrm{grad}K, \mathrm{grad}K) \circ \pi \\ &\quad +\bar{K}^2 k_0(\omega(V), \omega(W)) k_0(\omega(U), \omega(F)) g(\mathrm{grad}K, \mathrm{grad}K) \circ \pi, \end{aligned} \tag{3.1}$$

$$\begin{aligned} h(R(U, V)W, X) &= -\frac{1}{2}\bar{K}^3 k_0(\omega(U), \omega(W)) k_0(\omega(V), \Omega(\mathrm{grad}\bar{K}, X)) \\ &\quad +\frac{1}{2}\bar{K}^3 k_0(\omega(V), \omega(W)) k_0(\omega(U), \Omega(\mathrm{grad}\bar{K}, X)), \end{aligned} \tag{3.2}$$

$$\begin{aligned} h(R(X, V)Y, W) &= -\varepsilon \bar{K} k_0(\omega(V), \omega(W)) H^{\bar{K}}(X, Y) \\ &\quad +\frac{\varepsilon}{4}\bar{K}^2 k_0([\Omega, \omega](X, Y, V), \omega(W)) - \frac{\varepsilon}{2}\bar{K} k_0((\nabla_V \Omega)(X, Y), \omega(W)) \end{aligned} \tag{3.3}$$

$$+\frac{1}{4}\bar{K}^2 h(\overline{\Omega^{\omega(V)}(X)}, \overline{\Omega^{\omega(W)}(Y)}),$$

(3.4)
$$h(R(X,Y)Z,V) = -\frac{\varepsilon}{2}\bar{K}^2 k_0((\nabla_Z \Omega)(X,Y), \omega(V))$$
$$+\frac{\varepsilon}{2}\bar{K}k_0(\Omega(Y,Z),\omega(V))h(X,\mathrm{grad}\bar{K}) + \frac{\varepsilon}{2}\bar{K}k_0(\Omega(Z,X),\omega(V))h(Y,\mathrm{grad}\bar{K})$$
$$-\varepsilon\bar{K}k_0(\Omega(X,Y),\omega(V))h(Z,\mathrm{grad}\bar{K})$$

and

(3.5)
$$h(R(X,Y)Z,H) = h(R_*(X,Y)Z,H) - \frac{\varepsilon}{2}\bar{K}^2 k_0(\Omega(X,Y),\Omega(Z,H))$$
$$+\frac{\varepsilon}{2}\bar{K}^2 k_0(\Omega(Y,Z),\Omega(X,H)) + \frac{\varepsilon}{4}\bar{K}^2 k_0(\Omega(Z,X),\Omega(Y,H)).$$

Note that for (3.1), we used (1.3) and

$$\hat{R}(A^*,B^*)C^* = \frac{1}{4}[[A^*,B^*],C^*] = \frac{1}{4}[[A,B],C]^*$$

for fundamental vector fields A^*, B^* and C^*.

By Proposition 3.1, we can compute the sectional curvature. Let Π_{ab} be the nondegenerate space spanned by tangent vectors a, b.

Corollary 3.2. *Let $\mathcal{K}$, $\mathcal{K}_*$ and $\hat{\mathcal{K}}$ be the sectional curvature of (P,h), (M,g) and the fibers with the induced metrics from h. If x and y are horizontal vectors at $p \in P$, and v and w are vertical, then*

(3.6)
$$\mathcal{K}(\Pi_{vw}) = \hat{\mathcal{K}}(\Pi_{vw}) - \frac{1}{4}\frac{g_{\pi(p)}(\mathrm{grad}K,\mathrm{grad}K)}{K^2(\pi(p))},$$

(3.7)
$$\mathcal{K}(\Pi_{xv}) = -\frac{g_{\pi(p)}((\nabla^*_{x_*}\mathrm{grad}K), x_*)}{K(\pi(p))g_{\pi(p)}(x_*,x_*)}$$
$$+\frac{\varepsilon K^2(\pi(p))}{4}\frac{g_{\pi(p)}(\pi_*\overline{\Omega^{\omega(v)}(x)}, \pi_*\overline{\Omega^{\omega(v)}(x)})}{g_{\pi(p)}(x_*,x_*)k_0(\omega(v),\omega(v))}$$

and

(3.8)
$$\mathcal{K}(\Pi_{xy}) = \mathcal{K}_*(\Pi_{x_*y_*}) - \frac{3}{4}\frac{\varepsilon K^2(\pi(p))k_0(\Omega(x,y),\Omega(x,y))}{g_p(x_*,x_*)g_p(y_*,y_*) - g_p(x_*,y_*)^2},$$

where

$$\hat{\mathcal{K}}(\Pi_{vw}) = \frac{\varepsilon k_0([\omega(v),\omega(w)],[\omega(v),\omega(w)])}{K^2(\pi(p))(k_0(\omega(v),\omega(v))k_0(\omega(w),\omega(w)) - k_0(\omega(v),\omega(w))^2)}$$

and $x_* = \pi_* x$ and $y_* = \pi_* y$.

Next we calculate the Ricci and scalar curvatures of (P,h) by using Proposition 3.1. Especially we will form the functional from scalar curvature h in the next section. Let n (resp. l) be the dimension of M (resp. G). Assume that $E_{1*},...,E_{n*}$ is orthonormal base fields relative to g on a neighborhood $\mathcal{U}\subset M$ and $E_1,...,E_n$ their horizontal lifts. Let $e_1,...,e_l$ be an orthonormal base on $\mathcal{G}$ relative to the fixed metric k_0 and we set $E_{n+1}:=\bar{K}^{-1}e_1^*,...,E_{n+l}:=\bar{K}^{-1}e_l^*$. Then $E_1,...,E_n,E_{n+1},...,E_{n+l}$ is orthonormal base fields on $\pi^{-1}(\mathcal{U})\subset P$ with respect to h. The indices $i,\ j,...$(resp. $\alpha,\ \beta,...$) range from 1 to n (resp. from $n+1$ to $n+l$) and we set $\varepsilon_i:=h(E_i,E_i)=g(E_{i*},E_{i*})$ and $\varepsilon_\alpha:=h(E_\alpha,E_\alpha)=\varepsilon k_0(e_\alpha,e_\alpha)$. Let Ric (resp. Ric^*) be the Ricci tensor of h (resp. g) and $(Ric)_*$ the symmetric 2-form on P such that $(Ric)_*(X,Y)=Ric^*(\pi_*X,\pi_*Y)$ at each point of P.

Proposition 3.3. *If V, W are vertical and X, Y are horizontal, then*

$$Ric(V,W)=\varepsilon\bar{K}k_0(\omega(V),\omega(W))((\Delta K)\circ\pi) \tag{3.9}$$
$$+\frac{1}{4}\bar{K}^4\sum_i\varepsilon_i h(\overline{\Omega^{\omega(V)}(E_i)},\overline{\Omega^{\omega(W)}(E_i)})+\frac{\varepsilon}{4}\sum_\alpha\varepsilon_\alpha k_0([\omega(V),e_\alpha],[\omega(W),e_\alpha])$$
$$-\varepsilon(l-1)k_0(\omega(V),\omega(W))g(\mathrm{grad}K,\mathrm{grad}K)\circ\pi,$$

$$Ric(V,X)=\frac{\varepsilon}{2}\bar{K}^2\sum_i\varepsilon_i k_0((\nabla_{E_i}\Omega)(X,E_i),\omega(V)) \tag{3.10}$$
$$-\frac{l+2}{2}\varepsilon\bar{K}k_0(\omega(V),\Omega(\mathrm{grad}\bar{K},X))$$

and

$$Ric(X,Y)=(Ric)_*(X,Y)-\frac{\varepsilon}{2}\bar{K}^2\sum_i\varepsilon_i k_0(\Omega(X,E_i),\Omega(Y,E_i)) \tag{3.11}$$
$$+\varepsilon\sum_\alpha\varepsilon_\alpha\{\frac{1}{4}k_0([\Omega,\omega](X,Y,e_\alpha),e_\alpha)-\frac{1}{2}k_0((\nabla_{e_{\alpha*}}\Omega)(X,Y),e_\alpha))\}-\frac{l}{\bar{K}}H^{\bar{K}}(X,Y).$$

By Proposition 3.3, we have

Proposition 3.4. *Let S (resp. S^*) be the scalar curvature of (P,h) (resp. (M,g)). Then*

$$S=S^*\circ\pi-\frac{\varepsilon}{4}\bar{K}^2\sum_{i,j}\varepsilon_i\varepsilon_j k_0(\Omega(E_i,E_j),\Omega(E_i,E_j))+2l\frac{(\Delta K)\circ\pi}{\bar{K}} \tag{3.12}$$
$$-l(l-1)\frac{g(\mathrm{grad}K,\mathrm{grad}K)\circ\pi}{\bar{K}^2}+\frac{1}{4}\frac{\varepsilon}{\bar{K}^2}\sum_{\alpha,\beta}\varepsilon_\alpha\varepsilon_\beta k_0([e_\alpha,e_\beta],[e_\alpha,e_\beta]).$$

§4. Variational problems

In this section, we consider the variational problems for the integral of a functional constructed from the scalar curvature of (P,h). Let $\bar{\Lambda}^i(P,\mathcal{G})$ be the space of $\mathcal{G}$-valued i-forms φ on P such that $R_{a*}\varphi = ad(a^{-1})\varphi$ and $\varphi(X_1,...,X_i)=0$ when one of $X_1,...,X_i$ is vertical. For $\tau \in \bar{\Lambda}^i(P,\mathcal{G})$, we have $D^\omega\tau = d\tau + [\omega,\tau]$. The metric g_x on the tangent space at $x \in M$ induces the metric $\bar{g}_p$ on horizontal subspace $\mathcal{H}(P)_p \subset T_pP$ $(p \in \pi^{-1}(x))$ via the isomorphism $\pi_*|_{\mathcal{H}(P)_p} : \mathcal{H}(P)_p \to\ T_xM$ (i.e., $\bar{g}_p(X,Y) := g_x(\pi_*X,\pi_*Y)$ for $X,\ Y \in \mathcal{H}(P)_p$). Let $\tilde{\mu}_p$ be the volume element on $\mathcal{H}(P)_p$ relative to this induced metric and we can define the star operator $\tilde{*}_p : \Lambda^i(\mathcal{H}(P)_p) \to \Lambda^{n-i}(\mathcal{H}(P)_p)$ (n =dim$M$). Moreover we define $\bar{*} : \bar{\Lambda}^i(P,\mathcal{G}) \to \bar{\Lambda}^{n-i}(P,\mathcal{G})$ by setting (for $\varphi \in \bar{\Lambda}^i(P,\mathcal{G})$) $(\bar{*}\varphi)_p$ equal to the unique extension of $\tilde{*}_p(\varphi|_{\mathcal{H}(P)_p})$ to a $\mathcal{G}$-valued $(n-i)$-form vanishing on vertical vectors. Let $\partial_1,...,\partial_n$ be coordinate vector fields on $\mathcal{U} \subset M$. The covariant codifferential $\delta^\omega : \bar{\Lambda}^i(P,\mathcal{G}) \to \bar{\Lambda}^{i-1}(P,\mathcal{G})$ is defined, for $\varphi \in \bar{\Lambda}^i(P,\mathcal{G})$, by $\delta^\omega(\varphi) := -(-1)^g(-1)^{n(i+1)}\bar{*}D^\omega(\bar{*}\varphi)$, where $(-1)^g$ is the sign of determinant of the matrix $(g(\partial_l,\partial_m))$. The self- action of the connection ω relative to the fixed ad-invariant metric k_0 is defined by

$$\mathcal{S}_0(g,\omega) := -\frac{1}{2}(\bar{g}k_0)(\Omega,\Omega) = -\frac{1}{4}g^{hj}g^{im}k_0(\Omega(\tilde{\partial}_h,\tilde{\partial}_i),\Omega(\tilde{\partial}_j,\tilde{\partial}_m)),$$

where $\bar{g}k_0$ is the metric on $\bar{\Lambda}^i(P,\mathcal{G})$ induced from $\bar{g}$ and k_0. Note that $\mathcal{S}_0(g,\omega)$ is a smooth function on M. The ad-invariant metric k_0 induces the bi-invariant metric $\overline{k_0}$ on G as follows. For $a \in G$ and $A,\ B \in T_aG$, we set $\overline{k_0}(A,B) := k_0(L_{a*}^{-1}A, L_{a*}^{-1}B)$, where L_a is the left action on G. Then $(G,\overline{k_0})$ has the constant scalar curvature

$$c_0 = \frac{1}{4}\sum_{\alpha,\beta}\varepsilon_\alpha\varepsilon_\beta k_0([e_\alpha,e_\beta],[e_\alpha,e_\beta]).$$

Hence the scalar curvature of (P,h) is described by

$$\begin{aligned} S &= S^*\circ\pi + (\varepsilon K^2\mathcal{S}_0(g,\omega))\circ\pi + \frac{\varepsilon c_0}{\bar{K}^2} \\ &\quad +2l\frac{(\Delta K)\circ\pi}{\bar{K}} - l(l-1)\frac{g(\mathrm{grad}K,\mathrm{grad}K)\circ\pi}{\bar{K}^2}. \end{aligned}$$

We define a map $\mathcal{L} : \mathcal{M}(M)\times\mathcal{C}(P)\times C^\infty(M)^+ \longrightarrow C^\infty$ (M) by

$$\begin{aligned} \mathcal{L}(g,\omega,K) :&= \{S^* + \varepsilon K^2\mathcal{S}_0(g,\omega) + \frac{\varepsilon c_0}{K^2} + 2l\frac{(\Delta K)}{K} - l(l-1)\frac{g(\mathrm{grad}K,\mathrm{grad}K)}{K^2}\}K^l \\ &= S_*K^l, \end{aligned}$$

where

$$S_* := S_*(g,\omega,K) := S^* + \varepsilon K^2 \mathcal{S}_0(g,\omega) + \frac{\varepsilon c_0}{K^2} + 2l\frac{(\Delta K)}{K} - l(l-1)\frac{g(\mathrm{grad}K, \mathrm{grad}K)}{K^2}$$

and $C^\infty(M)^+$ is the set of all positive functions on M. The notation $U \subset\subset M$ means that U is an open subset of M with compact closure. The volume element relative to a metric g is denoted by μ_g. The variational problems for the integral of the scalar curvature of h reduces to those for the integral of $\mathcal{L}(g,\omega,K)$ since $S\mu_h = SK^l\pi^*\mu_g \wedge \mu_{\tilde{k_0}}$, where $\mu_{\tilde{k_0}}$ is the volume element induced from k_0.

At first, we consider variations of the metric. Let $S^2(M)$ be the set of all symmetric tensors on M. For $g \in \mathcal{M}(M)$, $u \in S^2(M)$, and $t \in \boldsymbol{R}$, we set $g(t) := g + tu$. For small $t \in \boldsymbol{R}$, $g(t)$ is in $\mathcal{M}(M)$. Then we denote the curvature tensor, gradient and Laplacian relative to $g(t)$ by $R^*(t)$, grad(t) and $\Delta(t)$, respectively. We set $g_{ij}(t) := g(t)(\partial_i, \partial_j)$ and define $R^{*i}{}_{jkl}(t)$ by the components of the curvature tensor of $g(t)$. We write $g_{ij} = g_{ij}(0)$, $R^{*i}{}_{jkl} = R^{*i}{}_{jkl}(0)$, etc. The indices are raised and lowered by the initial metric g. For $f \in C^\infty(M)$, we have

$$\int_U f g^{ij} R^{*k}{}_{ijk}{}'(0)\mu_g = \int_U \{(f_{,k;i}) + (\Delta f)g_{ik}\}u^{ik}\mu_g, \tag{4.1}$$

$$\frac{d}{dt}(\Delta(t)f)\,|_{t=0} = ((u^{ki})(f_{,k}))_{;i} - \frac{1}{2}((u^j_j)(f_{,k}))^{;k} - \frac{1}{2}u^j_j\Delta f \tag{4.2}$$

and

$$\frac{d}{dt}g(t)(\mathrm{grad}(t)f, \mathrm{grad}(t)f)\,|_{t=0} = u^{ij}(f_{,i})(f_{,j}), \tag{4.3}$$

where a prime denotes the derivative with respect to the parameter t.

Using equations above, we obtain the following theorem.

Theorem 4.1. *(Einstein field equation).* *For all $U \subset\subset M$ and all $u \in S^2(M)$ with support in U, the equation*

$$\frac{d}{dt}\int_U \mathcal{L}(g+tu,\omega,K)\mu_{g(t)} = 0 \quad \text{at } t=0$$

holds if and only if

$$R^*{}_{ij} - \frac{1}{2}S^*g_{ij} = \frac{1}{2}\varepsilon K^2 k_0(\Omega_{hi},\Omega_{mj})g^{hm} + \frac{1}{2}\varepsilon K^2\mathcal{S}_0(g,\omega)g_{ij} + \frac{1}{2}\frac{\varepsilon c_0}{K^2}g_{ij} + \frac{l}{K}(K_{,i;j} + \Delta K g_{ij}) - \frac{1}{2}l(l-1)\frac{g(\mathrm{grad}K,\mathrm{grad}K)}{K^2}g_{ij}, \tag{4.4}$$

where R^*_{ij} are the components of the Ricci curvature of (M,g).

Proof. At first, about the first term, we have

$$\begin{aligned} &\frac{d}{dt}\int_U K^l S^*(t)\mu_{g(t)}\,|_{t=0} \\ = \;& \int_U K^l(-R^*{}_{ij} + \frac{1}{2}S^* g_{ij})u^{ij}\mu_g + \int_U K^l g^{ij} R^{*k}_{ijk}{}'(0)\mu_g. \end{aligned}$$

From

$$(K^l)_{,i;j} = K^l\{l(l-1)\frac{(K_{,i})(K_{,j})}{K^2} + l\frac{K_{,i;j}}{K}\},$$

$$\Delta(K^l) = K^l\{l\frac{\Delta K}{K} - l(l-1)\frac{g(\mathrm{grad}K, \mathrm{grad}K)}{K^2}\}$$

and (4.1), we have

$$\begin{aligned} \int_U K^l g^{ij} R^{*k}_{ijk}{}'(0)\mu_g \;=\; & \int_U K^l\{l(l-1)\frac{(K_{,i})(K_{,j})}{K^2} + l\frac{K_{,i;j}}{K} + l\frac{\Delta K}{K}g_{ij} \\ & -l(l-1)\frac{g(\mathrm{grad}K, \mathrm{grad}K)}{K^2}g_{ij}\}u^{ij}\mu_g. \end{aligned}$$

For the second and third term, by similar calculations in 9.3.3 Theorem in [1], we get

$$\begin{aligned} &\frac{d}{dt}\int_U \varepsilon K^{l+2}\mathcal{S}_0(g(t),\omega)\mu_{g(t)}\,|_{t=0} \\ = \;& \int_U \{K^l(\frac{1}{2}\varepsilon K^2 g^{hm}k_0(\Omega_{hi},\Omega_{mj}) + \frac{1}{2}\varepsilon K^2\mathcal{S}_0(g,\omega)g_{ij})u^{ij}\}\mu_g \end{aligned}$$

and

$$\frac{d}{dt}\int_U K^l\frac{\varepsilon c_0}{K^2}\mu_{g(t)}\,|_{t=0} = \int_U K^l(\frac{1}{2}\frac{\varepsilon c_0}{K^2}g_{ij})u^{ij}\mu_g.$$

For the fourth term, from (4.2), it follows that

$$\begin{aligned} &\frac{d}{dt}\int_U K^{l-1}((\Delta(t))K)\mu_{g(t)}\,|_{t=0} \\ = \;& \int_U K^{l-1}\{((u^{ki})(K_{,k}))_{;i} - \frac{1}{2}((u^j_j)(K_{,k}))^{;k} - \frac{1}{2}u^j_j(\Delta K)\}\mu_g \\ & + \int_U K^{l-1}(\Delta K)(\frac{1}{2}u^i_i)\mu_g \\ = \;& \int_U K^l\{-(l-1)\frac{(K_{,i})(K_{,j})}{K^2} + \frac{1}{2}(l-1)\frac{g(\mathrm{grad}K, \mathrm{grad}K)}{K^2}g_{ij}\}u^{ij}\mu_g. \end{aligned}$$

For last term, by (4.3), we have

$$\begin{aligned} &\frac{d}{dt}\int_U K^l\frac{g(t)(\mathrm{grad}(t)K, \mathrm{grad}(t)K)}{K^2}\mu_{g(t)}\,|_{t=0} \\ = \;& \int_U K^l\{-\frac{1}{K^2}(K_{,i})(K_{,j}) + \frac{1}{2}\frac{g(\mathrm{grad}K, \mathrm{grad}K)}{K^2}g_{ij}\}u^{ij}\mu_g. \end{aligned}$$

Piecing these results together, we see that (4.4) holds if and only if g is stationary relative to $\mathcal{L}$ for fixed ω and K. Q.E.D.

Next, we consider variations of the connection. For $\omega \in \mathcal{C}(P)$, $\tau \in \bar{\Lambda}^1(P,\mathcal{G})$, and $t \in \boldsymbol{R}$, we set $\omega(t) := \omega + t\tau$. Then $\omega(t)$ is in $\mathcal{C}(P)$ for all $t \in \boldsymbol{R}$. Let $\Omega(t)$ be the curvature form of $\omega(t)$. Let $U \subset\subset M$, and suppose that $\alpha \in \bar{\Lambda}^k(P,\mathcal{G})$, while $\beta \in \bar{\Lambda}^{k+1}(P,\mathcal{G})$. Assume that the projected support of α is contained in U. Then

$$\int_U (\bar{g}k_0)(D^\omega \alpha, \beta)\mu_g = \int_U (\bar{g}k_0)(\alpha, \delta^\omega \beta)\mu_g. \tag{4.5}$$

For the curvature form of $\omega(t) = \omega + t\tau$, from the structure equation, we have

$$\frac{d}{dt}\Omega(t)\,|_{t=0} = d\tau + [\omega, \tau] = D^\omega \tau. \tag{4.6}$$

Theorem 4.2. *(Yang-Mills equation). For all $U \subset\subset M$ and all $\tau \in \bar{\Lambda}^1(P,\mathcal{G})$ with projected support in U, the equation*

$$\frac{d}{dt}\int_U \mathcal{L}(g, \omega + t\tau, K)\mu_g = 0 \quad \text{at } t = 0$$

holds if and only if

$$\delta^\omega(\bar{K}^{l+2}\Omega) = 0, \tag{4.7}$$

or equivalently

$$\delta^\omega \Omega = \frac{l+2}{\bar{K}}\Omega(\mathrm{grad}\bar{K},\ \cdot\ ). \tag{4.7$'$}$$

Proof. From (4.5) and (4.6), it follows that

$$\begin{aligned} \frac{d}{dt}\int_U \mathcal{L}(g, \omega + t\tau, K)\mu_g\,|_{t=0} &= -\int_U \varepsilon K^{l+2}(\bar{g}k_0)(\frac{d}{dt}\Omega(t)\,|_{t=0}, \Omega)\mu_g \\ = \ -\int_U \varepsilon K^{l+2}(\bar{g}k_0)(D^\omega\tau, \Omega)\mu_g &= -\int_U (\bar{g}k_0)(\tau, \delta^\omega(\varepsilon\bar{K}^{l+2}\Omega))\mu_g. \end{aligned}$$

Hence we see that the equation (4.7) holds if and only if ω is stationary relative to $\mathcal{L}$ for fixed g and K. Q.E.D.

Finally, we consider variations of the positive function. We start with the following lemma.

Lemma 4.3. *For all $U \subset\subset M$ and all $L \in C^\infty(M)$ with support in U, the equation*

$$\frac{d}{dt}\int_U K^l S_*(g,\omega,K+tL)\mu_g = 0 \quad \text{at } t=0 \tag{4.8}$$

holds if and only if

$$\varepsilon K^{l+1}\mathcal{S}_0(g,\omega) - \varepsilon K^{l-3}c_0 - lK^{l-2}(\Delta K) + l(l-1)K^{l-3}g(\mathrm{grad}K,\mathrm{grad}K) = 0.$$

Proof. For $K \in C^\infty(M)^+$ and $L \in C^\infty(M)$, from a straightforward calculation, it follows that

$$\frac{d}{dt}\varepsilon(K+tL)^2\mathcal{S}_0(g,\omega)\,|_{t=0} = 2\varepsilon KL\mathcal{S}_0(g,\omega),$$

$$\frac{d}{dt}\frac{c_0}{(K+tL)^2}\,|_{t=0} = -\frac{2c_0L}{K^3},$$

$$\frac{d}{dt}\frac{\Delta(K+tL)}{K+tL}\,|_{t=0} = \frac{\Delta L}{K} - \frac{(\Delta K)}{K^2}L$$

and

$$\frac{d}{dt}\frac{g(\mathrm{grad}(K+tL),\mathrm{grad}(K+tL))}{(K+tL)^2}\,|_{t=0} = \frac{2g(\mathrm{grad}K,\mathrm{grad}L)}{K^2} - \frac{2g(\mathrm{grad}K,\mathrm{grad}K)L}{K^3}.$$

Moreover by Green's theorem, we obtain

$$\int_U K^{l-1}(\Delta L)\mu_g = \int_U g(\mathrm{grad}(K^{l-1}),\mathrm{grad}L)\mu_g = \int_U \Delta(K^{l-1})L\mu_g$$
$$= \int_U \{(l-1)K^{l-2}\Delta K - (l-1)(l-2)K^{l-3}g(\mathrm{grad}K,\mathrm{grad}K)\}L\mu_g$$

and

$$\int_U K^{l-2}g(\mathrm{grad}K,\mathrm{grad}L)\mu_g = \int_U g(K^{l-2}\mathrm{grad}K,\mathrm{grad}L)\mu_g$$
$$= -\int_U \mathrm{div}(K^{l-2}\mathrm{grad}K)L\mu_g = -\int_U \{g(\mathrm{grad}K^{l-2},\mathrm{grad}K) + K^{l-2}\mathrm{div}(\mathrm{grad}K)\}L\mu_g$$
$$= \int_U \{-(l-2)K^{l-3}g(\mathrm{grad}K,\mathrm{grad}K) + K^{l-2}(\Delta K)\}L\mu_g.$$

Hence, from these equations, we see that the equation (4.8) holds if and only if

$$\varepsilon K^{l+1}\mathcal{S}_0(g,\omega) - \varepsilon K^{l-3}c_0 - lK^{l-2}(\Delta K) + l(l-1)K^{l-3}g(\mathrm{grad}K,\mathrm{grad}K) = 0.$$

Q.E.D.

Theorem 4.4. *(Brans-Dicke type wave equation). For all $U \subset\subset M$ and all $L \in C^\infty(M)$ with support in U, the equation*

$$\frac{d}{dt}\int_U \mathcal{L}(g,\omega,K+tL)\mu_g = 0 \quad \text{at } t=0$$

holds if and only if

$$(4.9) \qquad lK^2S^* + \varepsilon(l+2)K^4\mathcal{S}_0(g,\omega) + \varepsilon(l-2)c_0 + 2l(l-1)K(\Delta K) \\ -l(l-1)(l-2)g(\mathrm{grad}K,\mathrm{grad}K) = 0.$$

Proof. By Lemma 4.3, K is stationary relative to $\mathcal{L}$ for fixed g and ω if and only if

$$\begin{aligned} 0 \;=\; & lK^{l-1}\{S^* + \varepsilon K^2\mathcal{S}_0(g,\omega) + \frac{\varepsilon c_0}{K^2} + 2l\frac{(\Delta K)}{K} - l(l-1)\frac{g(\mathrm{grad}K,\mathrm{grad}K)}{K^2}\} \\ & +2\varepsilon K^{l+1}\mathcal{S}_0(g,\omega) - 2\varepsilon K^{l-3}c_0 + \{2l(l-1) - 2l - 2l(l-1)\}K^{l-2}(\Delta K) \\ & +\{-2l(l-1)(l-2) + 2l(l-1)(l-2) + 2l(l-1)\}K^{l-3}g(\mathrm{grad}K,\mathrm{grad}K) \end{aligned}$$

holds. Then we have

$$\begin{aligned} lK^2S^* + \varepsilon(l+2)K^4\mathcal{S}_0(g,\omega) \;&+\; \varepsilon(l-2)c_0 + 2l(l-1)K(\Delta K) \\ &-\; l(l-1)(l-2)g(\mathrm{grad}K,\mathrm{grad}K) = 0. \end{aligned}$$

Q.E.D.

By Theorems 4.1 and 4.4, we have the following corollary.

Corollary 4.5. *If the equation (4.4) and (4.9) hold, then*

$$(4.10) \quad (n+l-2)\{\varepsilon K^4\mathcal{S}_0(g,\omega) - \varepsilon c_0 - lK\Delta K + l(l-1)g(\mathrm{grad}K,\mathrm{grad}K)\} = 0.$$

If $n+l>2$, then the equation (4.10) reduces to

$$(4.11) \qquad \varepsilon K^4\mathcal{S}_0(g,\omega) - \varepsilon c_0 - lK\Delta K + l(l-1)g(\mathrm{grad}K,\mathrm{grad}K) = 0.$$

Proof. Contracting the equation (4.4) by g, we have

$$\begin{aligned} &(1-\frac{1}{2}n)S^* + \varepsilon(2-\frac{n}{2})K^2\mathcal{S}_0(g,\omega) - \frac{1}{2}n\frac{\varepsilon c_0}{K^2} \\ &-\frac{l}{K}(-\Delta K + n\Delta K) + \frac{1}{2}nl(l-1)\frac{g(\mathrm{grad}K,\mathrm{grad}K)}{K^2} = 0. \end{aligned}$$

From this equation and (4.9), it follows that

$$(n+l-2)\{\varepsilon K^4\mathcal{S}_0(g,\omega) - \varepsilon c_0 - lK\Delta K + l(l-1)g(\mathrm{grad}K,\mathrm{grad}K)\} = 0.$$

Q.E.D.

§5. Cosmology

In this section, we assume that M is a warped product and satisfies the equations in the previous section. By using these equations, we will consider cosmology. Let M_S be an m- dimensional semi-Riemannian manifold. Let $f > 0$ be a smooth function on an interval I in $\boldsymbol{R}^1_1$. Assume that M is the product manifold $I \times M_S$. Let p_I (resp. p_S) be the projection of M onto I (resp. M_S). The metric on M is defined by $g := p_I^*\sigma_I + (f \circ p_I)^2 p_S^* \sigma_S$, where σ_I and σ_S are the metric tensors on I and M_S, respectively. Especially, M is called a Robertson-Walker spacetime, if M_S is a connected 3-dimensional Riemannian manifold of constant curvature $\kappa = -1,\ 0$ *or* 1, see [5], for example.

Let $(x^0,\ x^1, ..., x^m)$ be a coordinate system on $\mathcal{U} \subset M = I \times M_S$. We assume that the function K depend only on x^0. We compute the curvatures of (M, g). The indices $A,\ B, ...$(resp. $i,\ j, ...$) range from 1 to m (resp. from 0 to m). Then we have

$$g_{00} = -1,\ g_{AB} = f^2\sigma_{AB} \text{ and } g^{00} = -1,\ g^{AB} = \frac{\sigma^{AB}}{f^2}$$

and

$$R^*_{00} = -m\frac{\ddot{f}}{f},\ R^*_{0A} = 0 \text{ and } R^*_{AB} = \{f\ddot{f} + (m-1)\dot{f}^2\}\sigma_{AB} + \bar{R}_{AB},$$

where $\bar{R}_{AB}$ are the components of the Ricci curvature of (M_S, σ_S) and $\sigma_{AB} = (\sigma_S)_{AB}$. Putting $\bar{S}$ the scalar curvature of σ_S, the scalar curvature S^* is described by

$$S^* = 2m\frac{\ddot{f}}{f} + m(m-1)\frac{\dot{f}^2}{f^2} + \frac{\bar{S}}{f^2}.$$

Since the function K depends only on x^0, we get

$$g(\text{grad}K, \text{grad}K) = -\dot{K}^2,\ \Delta K = \ddot{K} + m\frac{\dot{f}}{f}\dot{K}$$

and

$$K_{,i;j} = \begin{cases} \ddot{K} & (i = j = 0) \\ 0 & (i = 0,\ j = A) \\ -f\dot{f}\dot{K}\sigma_{AB} & (i = A,\ j = B) \end{cases}.$$

For the self-action, we have

$$\mathcal{S}_0(g, \omega) = -\frac{1}{4}g^{hj}g^{im}k_0(\Omega_{hi}, \Omega_{jm}) = \frac{1}{2}\frac{1}{f^2}a - \frac{1}{4}\frac{1}{f^4}b,$$

where $a := \sigma^{AB}k_0(\Omega_{0A}, \Omega_{0B})$ and $b := \sigma^{AB}\sigma^{CD}k_0(\Omega_{AC}, \Omega_{BD})$.

From Theorem 4.1, we have

Proposition 5.1. *The following equations hold.*

$$m(m-1)\frac{\dot f^2}{f^2}+\frac{\bar S}{f^2}=\frac{\varepsilon}{2}\frac{K^2}{f^2}a+\frac{\varepsilon}{4}\frac{K^2}{f^4}b-\frac{\varepsilon c_0}{K^2} -2ml\frac{\dot f}{f}\frac{\dot K}{K}-l(l-1)\frac{\dot K^2}{K^2}, \tag{5.1}$$

$$k_0(\Omega_{B0},\Omega_{CA})\sigma^{BC}=0 \tag{5.2}$$

and

$$(1-m)f\ddot f\sigma_{AB}+(1-\frac{m}{2})(m-1)\dot f^2\sigma_{AB}+\bar R_{AB}-\frac{1}{2}\bar S\sigma_{AB} \tag{5.3}$$
$$=-\frac{1}{2}\varepsilon K^2k_0(\Omega_{0A},\Omega_{0B})+\frac{1}{2}\varepsilon\frac{K^2}{f^2}k_0(\Omega_{CA},\Omega_{DB})\sigma^{CD}+\frac{1}{4}\varepsilon K^2a\sigma_{AB}-\frac{1}{8}\varepsilon\frac{K^2}{f^2}b\sigma_{AB} +\frac{1}{2}\frac{\varepsilon c_0}{K^2}f^2\sigma_{AB}+lf^2\frac{\ddot K}{K}\sigma_{AB}+l(m-1)f\dot f\frac{\dot K}{K}\sigma_{AB}+\frac{1}{2}l(l-1)\frac{\dot K^2}{K^2}f^2\sigma_{AB}.$$

Contracting (5.3) by σ_S and from (5.1), we obtain

Corollary 5.2. *It follows that*

$$(1-m)m\frac{\ddot f}{f}=(\frac{m}{2}-1)\varepsilon\frac{K^2}{f^2}a+\frac{1}{4}\varepsilon\frac{K^2}{f^4}b +\frac{\varepsilon c_0}{K^2}+ml\frac{\dot f}{f}\frac{\dot K}{K}+ml\frac{\ddot K}{K}+l(l-1)\frac{\dot K^2}{K^2}. \tag{5.4}$$

From Theorems 4.2, 4.4 and Corollary 4.5, the following equations hold.

Proposition 5.3. *We have*

$$\delta^\omega\Omega=-(l+2)\frac{\dot K\circ\pi}{\bar K}\Omega(\partial_0,\ \cdot), \tag{5.5}$$

$$2ml\frac{\ddot f}{f}+lm(m-1)\frac{\dot f^2}{f^2}+l\frac{\bar S}{f^2}+\frac{\varepsilon(l+2)}{2}\frac{K^2}{f^2}a-\frac{\varepsilon(l+2)}{4}\frac{K^2}{f^4}b +\varepsilon(l-2)\frac{c_0}{K^2}+2l(l-1)\frac{\ddot K}{K}+2l(l-1)m\frac{\dot f}{f}\frac{\dot K}{K}+l(l-1)(l-2)\frac{\dot K^2}{K^2}=0 \tag{5.6}$$

and

$$\frac{\varepsilon}{2}\frac{K^2}{f^2}a-\frac{\varepsilon}{4}\frac{K^2}{f^4}b-\varepsilon\frac{c_0}{K^2}-l\frac{\ddot K}{K}-ml\frac{\dot f}{f}\frac{\dot K}{K}-l(l-1)\frac{\dot K^2}{K^2}=0. \tag{5.7}$$

The equations (5.4) and (5.7) imply the following corollary.

Corollary 5.4. *It follows that*

$$(m-1)\{m\frac{\ddot{f}}{f}+\frac{\varepsilon}{2}a+l\frac{\ddot{K}}{K}\}=0. \tag{5.8}$$

If $m>1$, then we have

$$m\frac{\ddot{f}}{f}+\frac{\varepsilon}{2}a+l\frac{\ddot{K}}{K}=0. \tag{5.9}$$

We consider the case of $K(x^0)=F(f(x^0))$, where F is a function on the set of all positive real numbers. Then we have the following equation from the (5.9).

Corollary 5.5. *We have*

$$(\frac{m}{f}+l\frac{F'}{F})\ddot{f}+\frac{\varepsilon}{2}a+l\frac{F''}{F}\dot{f}^2=0, \tag{5.10}$$

where $F'=dF/df$.

We assume that σ_S and k_0 are positive definite metrics and $\varepsilon=1$, that is, M and P are Lorentz manifold. Usually, physicists consider this case in standard models. Then we have $a\geq 0$. Moreover, we assume $F'\leq 0$ and $F''\geq 0$. The assumption $F'\leq 0$ means that fibers are contracting when the space M_S is expanding. From Corollary 5.5, we get the following corollary.

Corollary 5.6. *If $\ddot{f}\leq 0$ and $\{\ t\in I \mid \ddot{f}(t)=0\ \}$ has no interior points, then we have*

$$-\frac{m}{l}\leq\frac{fF'}{F}\leq 0.$$

For example, when $K=f^{\alpha}$ $(\alpha\leq 0)$ or $K=\exp(\beta f)$ $(\beta<0)$, this inequality reduces to $-(m/l)\leq\alpha\leq 0$, $-\dfrac{m}{l\beta}\geq f(>0)$, respectively. When we refer to f as the scale factor, Corollary 5.6 indicates the relation among F, $\dim M_S$, $\dim G$ and the scale of the universe.

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