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KÄHLERIAN TORSE-FORMING VECTOR FIELDS AND KÄHLERIAN SUBMERSIONS

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Abstract. Let M be a Kählerian manifold and ∇ the Levi-Civita connection of M. In this paper, we consider a linear connection ∇^0 having a certain relation to ∇ and a Kählerian torse-forming vector field on M. The properties of the curvature tensor R^0 of ∇^0 and the Bochner curvature tensors are studied. Also we apply these properties to a Kählerian submersion.

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§1. Introduction

Let M be a real $2m$-dimensional Kählerian manifold with the complex structure J. We denote by ∇ the Levi-Civita connection of M and by $\mathfrak{X}(M)$ the set of all smooth vector fields on M. In [5], S. Yamaguchi introduced the notion of a Kählerian torse-forming vector field on a Kählerian manifold. If, for any $E \in \mathfrak{X}(M)$, a vector field ξ satisfies

$$\nabla_E \xi = aE + bJE + \alpha(E)\xi + \beta(E)J\xi, \tag{1.1}$$

where a and b are functions on M and α and β are 1-forms on M, then we call such a vector field ξ a Kählerian torse-forming vector field. Moreover, if the associated functions a and b satisfy $a^2 + b^2 > 0$ in M, then we call ξ a proper Kählerian torse-forming vector field.

In this paper, we consider the following linear connection ∇^0:

$$\begin{aligned}\nabla^0_E F := \nabla_E F &- \rho(E)F - \rho(F)E + \rho(JE)JF + \rho(JF)JE \\ &- f(E,F)\xi + f(JE,F)J\xi\end{aligned} \tag{1.2}$$

for any $E, F \in \mathfrak{X}(M)$, where ξ is a Kählerian torse-forming vector field, ρ a 1-form on M and f a $(0,2)$-tensor field of M respectively. In [9], S. Yamaguchi and W.N. Yu assumed that there exists a local coordinate system $\{x^h\}$ satisfying

$$\nabla^0_{\partial_i} \partial_j = 0$$

for $1 \leq i, j \leq 2m$ about each point of M, where $\partial_i = \partial/\partial x^i$. They obtained some results on the Bochner curvature tensor, the Ricci tensor, etc. The purpose of this paper is to generalize these results. In §2, we have a relation between the curvature tensor R^0 of ∇^0 and the curvature tensor R of ∇. Moreover, a relation between the Bochner curvature tensor B of ∇ and the Bochner curvature B^0 with respect to ∇^0 is given. In §3, we apply these relations in §2 to a Kählerian submersion.

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§2. A Kählerian torse-forming vector field on Kählerian manifold

Let (M, g, J) be a real $2m$-dimensional Kählerian manifold with the complex structure J and Kählerian metric g. For simplicity, we denote the metric g by (,). We put $|X| := \sqrt{(X, X)}$ for $X \in TM$, where TM is the tangent bundle of M. Hereafter, we assume that ξ is a Kählerian torse-forming vector field satisfying (1.1). Let ρ be a 1-form on M and f a $(0, 2)$-tensor field on M satisfying

$$f(E, F) = f(F, E) \quad \text{and} \quad f(E, JF) = f(F, JE)$$

for any $E, F \in TM$. We define a linear connection ∇^0 by (1.2). Then we can easily obtain

Lemma 2.1. ∇^0 *is a torsion free connection and* $\nabla^0 J = 0$.

The curvature tensor field R^0 and R are defined by

$$R^0(E, F)G := \nabla^0_E \nabla^0_F G - \nabla^0_F \nabla^0_E G - \nabla^0_{[E,F]} G,$$
$$R(E, F)G := \nabla_E \nabla_F G - \nabla_F \nabla_E G - \nabla_{[E,F]} G$$

for any $E, F, G \in \mathfrak{X}(M)$ respectively. Using (1.1) and (1.2), by a straightforward but rather complicated computations, we have

$$\begin{aligned} (2.1) \qquad & R^0(E, F)G - R(E, F)G \\ &= -\{\mu(E, F) - \mu(F, E)\}G - \mu(E, G)F \\ &+ \mu(F, G)E + \mu(E, JG)JF - \mu(F, JG)JE \\ &+ \{\mu(E, JF) - \mu(F, JE)\}JG - \nu(E, F, G)\xi \\ &+ \nu(E, F, JG)J\xi, \end{aligned}$$

where

$$\begin{aligned}(2.2)\qquad &\mu(E,F)\\ &:= (\nabla_E\rho)(F)+\rho(E)\rho(F)-\rho(JE)\rho(JF)\\ &\quad+\{\rho(\xi)-a\}f(E,F)-\{\rho(J\xi)+b\}f(E,JF),\end{aligned}$$

$$\begin{aligned}(2.3)\qquad &\nu(E,F,G)\\ &:= f(F,G)\{\alpha(E)-f(E,\xi)\}-f(E,G)\{\alpha(F)-f(F,\xi)\}\\ &+f(JF,G)\{\beta(E)+f(JE,\xi)\}+(\nabla_E f)(F,G)\\ &-f(JE,G)\{\beta(F)+f(JF,\xi)\}-(\nabla_F f)(E,G).\end{aligned}$$

From (2.3), we see that

$$\nu(E,F,G)+\nu(F,G,E)+\nu(G,E,F)=0, \tag{2.4}$$

$$\nu(E,F,JG)+\nu(F,G,JE)+\nu(G,E,JF)=0. \tag{2.5}$$

Hereafter, we assume the following equation:

$$(R^0(E,F)G,H)+(R^0(E,F)H,G)=0 \tag{2.6}$$

for every $E,F,G,H\in TM$. Then, from (2.1), it is equivalent to

$$\begin{aligned}(2.7)\qquad &-2\{\mu(E,F)-\mu(F,E)\}(G,H)\\ &-\mu(E,G)(F,H)+\mu(F,G)(E,H)-\mu(E,H)(F,G)\\ &+\mu(F,H)(E,G)+\mu(E,JG)(JF,H)-\mu(F,JG)(JE,H)\\ &+\mu(E,JH)(JF,G)-\mu(F,JH)(JE,G)\\ &-\nu(E,F,G)(\xi,H)+\nu(E,F,JG)(J\xi,H)\\ &-\nu(E,F,H)(\xi,G)+\nu(E,F,JH)(J\xi,G)=0.\end{aligned}$$

It can be proved from (2.7) that

$$(m+1)\{\mu(E,F)-\mu(F,E)\}+\nu(E,F,\xi)=0 \tag{2.8}$$

for any $E,F\in TM$. Now we prove

Lemma 2.2. *If ξ is everywhere non-zero and dim$M=2m\geq 6$, then we get*

$$\mu(E,F)=\mu(F,E), \tag{2.9}$$

$$\nu(E,F,\xi)=0, \tag{2.10}$$

$$\nu(\xi,E,F)=\nu(\xi,F,E) \tag{2.11}$$

for every $E, F \in TM$.

Proof. For $p \in M$, $\mathrm{Span}\{\xi_p, J\xi_p\}$ denotes the 2-dimensional subspace spanned by ξ_p and $J\xi_p$. We take two vectors $Y, Z \in (\mathrm{Span}\{\xi_p, J\xi_p\})^{\perp}$ such that

$$(Y,Y) = (Z,Z) = 1, \quad (Y,Z) = (JY,Z) = 0,$$

where $(\mathrm{Span}\{\xi_p, J\xi_p\})^{\perp}$ means the orthogonal complement. Then, it is easily seen from (2.7) that

$$\mu(E,Z) = (Y,E)\mu(Y,Z) + (Z,E)\mu(Y,Y) + (JY,E)\mu(Y,JZ) + (JZ,E)\mu(Y,JY), \tag{2.12}$$

$$\mu(E,JZ) = (Y,E)\mu(JY,Z) + (Z,E)\mu(JY,Y) + (JY,E)\mu(JY,JZ) + (JZ,E)\mu(JY,JY), \tag{2.13}$$

$$\mu(E,Y) = (Y,E)\mu(Z,Z) + (Z,E)\mu(Z,Y) + (JY,E)\mu(Z,JZ) + (JZ,E)\mu(Z,JY), \tag{2.14}$$

$$\mu(E,JY) = (Y,E)\mu(JZ,Z) + (Z,E)\mu(JZ,Y) + (JY,E)\mu(JZ,JZ) + (JZ,E)\mu(JZ,JY), \tag{2.15}$$

$$\mu(E,F) - \mu(F,E) = -\mu(E,Y)(F,Y) + \mu(F,Y)(E,Y) + \mu(E,JY)(JF,Y) - \mu(F,JY)(JE,Y), \tag{2.16}$$

$$\mu(E,F) - \mu(F,E) = -\mu(E,Z)(F,Z) + \mu(F,Z)(E,Z) + \mu(E,JZ)(JF,Z) - \mu(F,JZ)(JE,Z) \tag{2.17}$$

hold for any $E, F \in TM$. By virtue of (2.14), (2.15) and (2.16), we get

$$\begin{aligned} &\mu(E,F) - \mu(F,E) \\ &+ \{(Z,E)(Y,F) - (Z,F)(Y,E)\}\mu(Z,Y) \\ &+ \{(JY,E)(Y,F) - (JY,F)(Y,E)\}\mu(Z,JZ) \\ &- \{(JY,E)(Y,F) - (JY,F)(Y,E)\}\mu(JZ,Z) \\ &+ \{(JZ,E)(Y,F) - (JZ,F)(Y,E)\}\mu(Z,JY) \\ &+ \{(Z,E)(JY,F) - (Z,F)(JY,E)\}\mu(JZ,Y) \\ &+ \{(JZ,E)(JY,F) - (JZ,F)(JY,E)\}\mu(JZ,JY) = 0 \end{aligned} \tag{2.18}$$

for any $E, F \in TM$. Also, from (2.12), (2.13) and (2.17), we find

$$\begin{aligned}(2.19)\qquad &\mu(E,F) - \mu(F,E)\\ &+ \{(Z,F)(Y,E) - (Z,E)(Y,F)\}\mu(Y,Z)\\ &+ \{(JY,E)(Z,F) - (JY,F)(Z,E)\}\mu(Y,JZ)\\ &+ \{(JZ,E)(Z,F) - (JZ,F)(Z,E)\}\mu(Y,JY)\\ &- \{(JZ,E)(Z,F) - (JZ,F)(Z,E)\}\mu(JY,Y)\\ &+ \{(Y,E)(JZ,F) - (Y,F)(JZ,E)\}\mu(JY,Z)\\ &+ \{(JY,E)(JZ,F) - (JY,F)(JZ,E)\}\mu(JY,JZ) = 0\end{aligned}$$

for any $E, F \in TM$. It follows from (2.18) and (2.19) that

$$(2.20)\qquad \begin{cases} \mu(Z,Y) + \mu(Y,Z) = 0, \quad \mu(Z,JZ) - \mu(JZ,Z) = 0,\\ \mu(Z,JY) + \mu(JY,Z) = 0, \quad \mu(JZ,Y) + \mu(Y,JZ) = 0,\\ \mu(JZ,JY) + \mu(JY,JZ) = 0, \quad \mu(Y,JY) - \mu(JY,Y) = 0. \end{cases}$$

From (2.18), (2.19) and (2.20), we have

$$(2.21)\qquad \begin{cases} \mu(Z,Y) = \mu(Y,Z) = \mu(Z,JZ) = \mu(JZ,Z) = 0\\ \mu(Z,JY) = \mu(JY,Z) = \mu(JZ,Y) = \mu(Y,JZ) = 0\\ \mu(JZ,JY) = \mu(JY,JZ) = \mu(Y,JY) = \mu(JY,Y) = 0. \end{cases}$$

Hence, by means of (2.18) and (2.21), we get (2.9). Moreover, it follows from (2.3), (2.8) and (2.9) that (2.10) and (2.11) hold. □

Since the first Bianchi equation of R^0 holds, from Lemma 2.1, we conclude that

$$(2.22)\qquad R^0(E,F)J = JR^0(E,F) \quad \text{and} \quad R^0(JE,JF) = R^0(E,F)$$

for any $E, F \in TM$. Moreover, making use of (2.22), we find

$$\begin{aligned}(2.23)\qquad &Ric^0(JE,JF) = Ric^0(E,F) = Ric^0(F,E) \text{ and}\\ &Ric^0(E,F) = \frac{1}{2}(\text{Trace of } JR^0(E,JF))\end{aligned}$$

where $Ric^0(E,F) := \sum_{i=1}^{2m}(R^0(e_i,E)F,e_i)$ and (Trace of $JR^0(E,JF)$) $:= \sum_{i=1}^{2m}(JR^0(E,JF)e_i,e_i)$ for an orthonormal frame $\{e_1,\cdots,e_{2m}\}$ of TM.

Hereafter, in this section, we assume that ξ is everywhere non-zero and $m \geq 3$. Next we calculate the difference between the Ricci tensors. It is clear from (2.1) and (2.9) that

$$
\begin{aligned}
(2.24)\qquad & Ric^0(E,F) - Ric(E,F) \\
&= \sum_{i=1}^{2m}(R^0(e_i,E)F,e_i) - \sum_{i=1}^{2m}(R(e_i,E)F,e_i) \\
&= \sum_{i=1}^{2m}(R^0(E,e_i)e_i,F) - \sum_{i=1}^{2m}(R(E,e_i)e_i,F) \\
&= \sum_{i=1}^{2m}\mu(e_i,e_i)(E,F) + \mu(E,F) + \mu(JE,JF) \\
&- \sum_{i=1}^{2m}\mu(e_i,Je_i)(JE,F) - \sum_{i=1}^{2m}\nu(E,e_i,e_i)(\xi,F) \\
&+ \sum_{i=1}^{2m}\nu(E,e_i,Je_i)(J\xi,F)
\end{aligned}
$$

for any $E, F \in TM$. Since (2.23) holds, subtracting (2.24) from the equation obtained by changing E(resp. F) into JE(resp. JF) in (2.24), it follows that

$$
\begin{aligned}
(2.25)\qquad & \sum_{i=1}^{2m}\nu(E,e_i,e_i)(\xi,F) \\
&= \sum_{i=1}^{2m}\nu(E,e_i,Je_i)(J\xi,F) + \sum_{i=1}^{2m}\nu(JE,e_i,e_i)(\xi,JF) \\
&- \sum_{i=1}^{2m}\nu(JE,e_i,Je_i)(J\xi,JF).
\end{aligned}
$$

If we put $F = \xi$ in (2.25), then

$$
(2.26)\qquad \sum_{i=1}^{2m}\nu(E,e_i,e_i) = -\sum_{i=1}^{2m}\nu(JE,e_i,Je_i)
$$

holds for any $E \in TM$. If we subtract (2.24) from the equation obtained by interchanging E and F in (2.24), then we obtain

$$
(2.27)\qquad 2\sum_{i=1}^{2m}\mu(e_i,Je_i)(JE,F)
$$

$$+\sum_{i=1}^{2m}\nu(E,e_i,e_i)(\xi,F)-\sum_{i=1}^{2m}\nu(F,e_i,e_i)(\xi,E)$$
$$=\sum_{i=1}^{2m}\nu(E,e_i,Je_i)(J\xi,F)-\sum_{i=1}^{2m}\nu(F,e_i,Je_i)(J\xi,E).$$

Putting $F=\xi$ in (2.27), we get

$$\begin{aligned}&2\sum_{i=1}^{2m}\mu(e_i,Je_i)(JE,\xi)\\&+\sum_{i=1}^{2m}\nu(E,e_i,e_i)|\xi|^2-\sum_{i=1}^{2m}\nu(\xi,e_i,e_i)(\xi,E)\\&=-\sum_{i=1}^{2m}\nu(\xi,e_i,Je_i)(J\xi,E).\end{aligned}\tag{2.28}$$

If we replace e_i by Je_i in (2.28) and use (2.11), then we have

$$\begin{aligned}&-2\sum_{i=1}^{2m}\mu(e_i,Je_i)(JE,\xi)\\&+\sum_{i=1}^{2m}\nu(E,Je_i,Je_i)|\xi|^2-\sum_{i=1}^{2m}\nu(\xi,Je_i,Je_i)(\xi,E)\\&=\sum_{i=1}^{2m}\nu(\xi,e_i,Je_i)(J\xi,E).\end{aligned}\tag{2.29}$$

Since

$$\sum_{i=1}^{2m}\nu(E,e_i,e_i)|\xi|^2=\sum_{i=1}^{2m}\nu(E,Je_i,Je_i)|\xi|^2$$

and

$$\sum_{i=1}^{2m}\nu(\xi,e_i,e_i)(\xi,E)=\sum_{i=1}^{2m}\nu(\xi,Je_i,Je_i)(\xi,E)$$

hold, from (2.28) and (2.29), we find

$$2\sum_{i=1}^{2m}\mu(e_i,Je_i)=\sum_{i=1}^{2m}\nu(\xi,e_i,Je_i),\tag{2.30}$$

$$\sum_{i=1}^{2m}\nu(E,e_i,e_i)=\lambda(\xi,E)\tag{2.31}$$

for any $E \in TM$, where we put $|\xi|^2\lambda = \sum_{i=1}^{2m} \nu(\xi, e_i, e_i)$. It follows from (2.26), (2.30) and (2.31) that

$$\sum_{i=1}^{2m} \mu(e_i, Je_i) = 0 \tag{2.32}$$

and

$$\sum_{i=1}^{2m} \nu(E, e_i, Je_i) = -\lambda(J\xi, E) \tag{2.33}$$

hold for any $E \in TM$. Using (2.1), (2.9) and (2.23), we get

$$\begin{aligned} &Ric^0(E,F) - Ric(E,F) \\ &= -\sum_{i=1}^{2m} \frac{1}{2}(R^0(E, JF)e_i, Je_i) \\ &\quad + \sum_{i=1}^{2m} \frac{1}{2}(R(E, JF)e_i, Je_i) \\ &= (m+1)(\mu(E,F) + \mu(JE, JF)) - \nu(E, JF, J\xi). \end{aligned} \tag{2.34}$$

Also making use of (2.1) and (2.9), we obtain

$$\begin{aligned} &Ric^0(E,F) - Ric(E,F) \\ &= \sum_{i=1}(R^0(e_i, E)F, e_i) - \sum_{i=1}(R(e_i, E)F, e_i) \\ &= 2m\mu(E,F) + 2\mu(JE, JF) - \nu(\xi, E, F) \\ &+ \nu(J\xi, E, JF). \end{aligned} \tag{2.35}$$

If we subtract (2.34) from (2.35) and use (2.3) and (2.10), then we have

$$(m-1)\{\mu(E,F) - \mu(JE,JF)\} - \nu(\xi, E, F) - \nu(JF, J\xi, E) = 0,$$

which yields that

$$\mu(E, \xi) = \mu(JE, J\xi) \tag{2.36}$$

for any $E \in TM$. Hence, from (2.7), we get for $E, F, G, H \in TM$

$$\begin{aligned} &2(m-1)\mu(E,F) \\ &= \{2(m-1)\bar{a} + (\lambda + \epsilon)|\xi|^2\}(E,F) \\ &- (\lambda + \epsilon)\Big\{(E,\xi)(F,\xi) + (E, J\xi)(F, J\xi)\Big\}, \end{aligned} \tag{2.37}$$

$$
\begin{aligned}
(2.38)\qquad & 2(m-1)|\xi|^2\nu(E,F,G)\\
&= -(\lambda+\epsilon)|\xi|^2\Big\{(E,JG)(F,J\xi)-(F,JG)(E,J\xi)\\
&+2(J\xi,G)(JF,E)+(G,E)(F,\xi)-(G,F)(E,\xi)\Big\}\\
&-2\{2\lambda+(m+1)\epsilon\}(J\xi,G)\Big\{(J\xi,F)(\xi,E)\\
&-(J\xi,E)(\xi,F)\Big\},
\end{aligned}
$$

where

$$
\bar{a} := \frac{1}{|\xi|^2}\mu(\xi,\xi), \quad \lambda := \frac{1}{|\xi|^2}\sum_{i=1}^{2m}\nu(\xi,e_i,e_i), \quad \epsilon := -\frac{1}{|\xi|^4}\nu(\xi,J\xi,J\xi).
$$

Therefore we get the following theorem.

Theorem 2.3. *Suppose that M is a Kählerian manifold with the complex structure J, $\dim M \geq 6$ and there exists an everywhere non-zero Kählerian torse-forming vector field ξ. If the curvature tensor R^0 satisfies* (2.6)*, then we have*

$$
\begin{aligned}
(2.39)\quad & (R^0(E,F)G,H)-(R(E,F)G,H)\\
&= -\Big\{\bar{a}+\frac{\lambda+\epsilon}{2(m-1)}|\xi|^2\Big\}\Big\{(E,G)(F,H)-(F,G)(E,H)\\
&-(E,JG)(JF,H)+(F,JG)(JE,H)+2(JE,F)(JG,H)\Big\}\\
&+\frac{\lambda+\epsilon}{2(m-1)}\Bigg[\Big\{(E,JG)(J\xi,F)-(F,JG)(J\xi,E)\\
&+2(E,JF)(J\xi,G)-(F,G)(\xi,E)+(E,G)(\xi,F)\Big\}(\xi,H)\\
&-\Big\{(E,JG)(\xi,F)-(F,JG)(\xi,E)+2(E,JF)(\xi,G)\\
&+(F,G)(J\xi,E)-(E,G)(J\xi,F)\Big\}(J\xi,H)+(\xi,G)\Big\{(\xi,E)(F,H)\\
&-(\xi,F)(E,H)-(J\xi,E)(JF,H)+(J\xi,F)(JE,H)\Big\}\\
&+(J\xi,G)\Big\{(J\xi,E)(F,H)-(J\xi,F)(E,H)+(\xi,E)(JF,H)\\
&-(\xi,F)(JE,H)\Big\}-2(JG,H)\big\{(J\xi,E)(\xi,F)-(J\xi,F)(\xi,E)\big\}\Bigg]\\
&+\frac{2\lambda+(m+1)\epsilon}{(m-1)|\xi|^2}\Big\{(\xi,E)(J\xi,F)-(\xi,F)(J\xi,E)\Big\}\Big\{(J\xi,G)(\xi,H)\\
&-(J\xi,H)(\xi,G)\Big\}
\end{aligned}
$$

for $E, F, G, H \in TM$.

From Theorem 2.3, we get

$$\begin{aligned}(2.40)\qquad & Ric^0(E,F) - Ric(E,F) \\ & = 2(m+1)\bar{a}(E,F) + \frac{m}{m-1}(\lambda+\epsilon)|\xi|^2(E,F) \\ & - \frac{m\lambda+\epsilon}{m-1}\{(\xi,E)(\xi,F) + (J\xi,E)(J\xi,F)\},\end{aligned}$$

$$(2.41)\qquad r^0 - r = 4m(m+1)\bar{a} + \{2m\lambda + 2(m+1)\epsilon\}|\xi|^2$$

where $r^0 := \sum_{i=1}^{2m} Ric^0(e_i, e_i)$.

For the Levi-Civita connection ∇, the Bochner curvature tensor B [2] is defined by

$$\begin{aligned}(B(E,F)G,H) &:= (R(E,F)G,H) + \frac{1}{2m+4}\Big\{(E,G)Ric(F,H) \\ & - (F,G)Ric(E,H) + (F,H)Ric(E,G) - (E,H)Ric(F,G) \\ & + (JE,G)Ric(JF,H) - (JF,G)Ric(JE,H) + (JF,H)Ric(JE,G) \\ & - (JE,H)Ric(JF,G) + 2(JE,F)Ric(JG,H) + 2(JG,H)Ric(JE,F)\Big\} \\ & - \frac{r}{(2m+4)(2m+2)}\Big\{(E,G)(F,H) - (F,G)(E,H) + (JE,G)(JF,H) \\ & - (JF,G)(JE,H) + 2(JE,F)(JG,H)\Big\}\end{aligned}$$

for any $E, F, G, H \in TM$. Similarly, we define the following tensor B^0 by

$$\begin{aligned}(B^0(E,F)G,H) &:= (R^0(E,F)G,H) + \frac{1}{2m+4}\Big\{(E,G)Ric^0(F,H) \\ & - (F,G)Ric^0(E,H) + (F,H)Ric^0(E,G) - (E,H)Ric^0(F,G) \\ & + (JE,G)Ric^0(JF,H) - (JF,G)Ric^0(JE,H) + (JF,H)Ric^0(JE,G) \\ & - (JE,H)Ric^0(JF,G) + 2(JE,F)Ric^0(JG,H) + 2(JG,H)Ric^0(JE,F)\Big\} \\ & - \frac{r^0}{(2m+4)(2m+2)}\Big\{(E,G)(F,H) - (F,G)(E,H) + (JE,G)(JF,H) \\ & - (JF,G)(JE,H) + 2(JE,F)(JG,H)\Big\}\end{aligned}$$

for any $E, F, G, H \in TM$. We call B^0 the Bochner curvature tensor with respect to the linear connection ∇^0. Then, by virtue of (2.39), (2.40) and (2.41), we get the following Theorem.

Theorem 2.4. *Suppose that M is a Kählerian manifold with the complex structure J, dim$M \geq 6$ and there exists an everywhere non-zero Kählerian torse-forming vector field ξ. If the curvature tensor R^0 satisfies (2.6), then, for $E, F, G, H \in TM$,*

$$(B^0(E,F)G,H) - (B(E,F)G,H) = \{2\lambda + (m+1)\epsilon\}C(E,F,G,H), \tag{2.42}$$

moreover, we have

$$B^0 = B \quad \text{if and only if} \quad 2\lambda + (m+1)\epsilon = 0,$$

where

$$\begin{aligned}
&C(E,F,G,H) := \\
&-\frac{1}{2(m^2-1)(m+2)}|\xi|^2\Big\{(E,G)(F,H) - (F,G)(E,H) - (E,JG)(JF,H) \\
&\qquad + (F,JG)(JE,H) + 2(JE,F)(JG,H)\Big\} \\
&+\frac{1}{2(m-1)(m+2)}\Big[\Big\{(E,JG)(J\xi,F) - (F,JG)(J\xi,E) + 2(E,JF)(J\xi,G) \\
&\quad - (F,G)(\xi,E) + (E,G)(\xi,F)\Big\}(\xi,H) - \Big\{(E,JG)(\xi,F) - (F,JG)(\xi,E) \\
&\quad + 2(E,JF)(\xi,G) + (F,G)(J\xi,E) - (E,G)(J\xi,F)\Big\}(J\xi,H) \\
&\quad + (\xi,G)\Big\{(\xi,E)(F,H) - (\xi,F)(E,H) - (J\xi,E)(JF,H) + (J\xi,F)(JE,H)\Big\} \\
&\quad + (J\xi,G)\Big\{(J\xi,E)(F,H) - (J\xi,F)(E,H) + (\xi,E)(JF,H) - (\xi,F)(JE,H)\Big\} \\
&\quad - 2(JG,H)\Big\{(J\xi,E)(\xi,F) - (J\xi,F)(\xi,E)\Big\}\Big] \\
&+\frac{1}{(m-1)|\xi|^2}\Big\{(\xi,E)(J\xi,F) - (\xi,F)(J\xi,E)\Big\}\Big\{(J\xi,G)(\xi,H) - (J\xi,H)(\xi,G)\Big\}.
\end{aligned}$$

§3. Kählerian submersions and the Bochner curvature tensor

Let (M,g,J) be as in §2 and $(\overline{M}, \bar{g}, \overline{J})$ a real $2n$-dimensional almost complex manifold with the almost complex structure $\overline{J}$ and metric $\bar{g}$. For simplicity, we denote the metric $\bar{g}$ by $(\ ,\ )$. A smooth surjective mapping $\pi : M \to \overline{M}$ is called a *Riemannian submersion* [1] if π has maximal rank and $\pi_*|_{(\mathrm{Ker}\ \pi_*)^\perp}$ is linear isometry, where π_* is the derivative mapping of π. Vectors on M which are in the kernel of π_* are tangent to the fibers $\widehat{M_p}(=\pi^{-1}(p), p \in \overline{M},)$. We call these *vertical* vectors. Vectors which are orthogonal to vertical distribution are said to be *horizontal*. We denote the vertical and horizontal distributions in the tangent bundle of the total space M by $\mathcal{V}(M)$ and $\mathcal{H}(M)$, respectively. Then TM has the orthogonal decomposition: $TM = \mathcal{V}(M) \oplus \mathcal{H}(M)$. The projection mappings are denoted by $\mathcal{V} : TM \to \mathcal{V}(M)$ and $\mathcal{H} : TM \to \mathcal{H}(M)$. Let E and F be arbitrary vector fields on M. The O'Neill configuration tensors [1] of the Riemannian submersion $\pi : M \to \overline{M}$ are given by

$$T_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F, \quad A_E F = \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F + \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F.$$

The properties of T and A are well-known, contained in O'Neill's original paper, and included here only for completeness.

Lemma 3.1 **([1]).** *Let $\pi : M \to \overline{M}$ be a Riemannian submersion. Then*

(a) *at any point $p \in M$, the linear operators T_E and A_E are skew-symmetric,*

(b) $T_E\{\mathcal{H}(M)\} \subset \mathcal{V}(M)$ *and* $T_E\{\mathcal{V}(M)\} \subset \mathcal{H}(M)$,

(c) $A_E\{\mathcal{H}(M)\} \subset \mathcal{V}(M)$ *and* $A_E\{\mathcal{V}(M)\} \subset \mathcal{H}(M)$,

(d) *T is vertical and A is horizontal, i.e., $T_E = T_{\mathcal{V}E}$ and $A_E = A_{\mathcal{H}E}$,*

(e) $T_V W = T_W V$ *for all* $V, W \in \mathcal{V}(M)$,

(f) $A_X Y = A_Y X$ *for all* $X, Y \in \mathcal{H}(M)$.

A Riemannian submersion $\pi : M \to \overline{M}$ is said to be a *Kählerian submersion* if $\pi_* \circ J = \overline{J} \circ \pi_*$. B. Watson [4] proved that the vertical and horizontal distributions are J-invariant. Moreover he showed the following theorem.

Theorem 3.2 **([4]).** *Let $\pi : M \to \overline{M}$ be a Kählerian submersion. Then the base space and each fiber are Kählerian manifolds, and the horizontal distribution is integrable.*

Let $\pi : M \to \overline{M}$ be a Kählerian submersion. Then, from Theorem 3.2, we find $A = 0$. Geometrical features of the fibers will be distinguished by a caret (ˆ). We obtain

Lemma 3.3 **([1], [4]).** *Let X,Y be horizontal vector fields and U,V vertical vector fields. Then*

$$\begin{aligned}
\nabla_U V &= T_U V + \widehat{\nabla}_U V, \\
\nabla_U X &= \mathcal{H}\nabla_U X + T_U X, \\
\nabla_X U &= \mathcal{V}\nabla_X U, \\
\nabla_X Y &= \mathcal{H}\nabla_X Y,
\end{aligned}$$

where $\widehat{\nabla}$ is the family of Levi-Civita connections on fibres.

For vertical vectors V_1, V_2, V_3, V_4 at $p \in M$, let $(\widehat{R}(V_1, V_2)V_3, V_4)$ be the curvature tensor of the fiber $\widehat{M}_{\pi(p)}$ at p. The horizontal lift of the curvature tensor $\overline{R}$ of $\overline{M}$ will also denoted by $\overline{R}$, that is, $\pi_*(\overline{R}(X, Y)Z) = \overline{R}(\pi_* X, \pi_* Y)\pi_* Z$ at each $p \in M$. Then we have the following lemma.

Lemma 3.4 ([1], [4]). *Let U, V, W, W' be vertical vector fields and X, Y, Z, Z' horizontal vector fields. Then*

$$(R(U,V)W,W') = (\widehat{R}(U,V)W,W') + (T_U W, T_V W') - (T_V W, T_U W'), \tag{3.1}$$

$$(R(U,V)W,X) = ((\nabla_U T)_V W, X) - ((\nabla_V T)_U W, X), \tag{3.2}$$

$$(R(X,U)Y,V) = ((\nabla_X T)_U Y, V) + (T_U X, T_V Y), \tag{3.3}$$

$$(R(U,V)X,Y) = (T_U X, T_V Y) - (T_V X, T_U Y), \tag{3.4}$$

$$(R(X,Y)Z,U) = 0, \tag{3.5}$$

$$(R(X,Y)Z,Z') = (\overline{R}(X,Y)Z,Z'). \tag{3.6}$$

Let ξ be an everywhere non-zero Kählerian torse-forming vector field of M satisfying (1.1). We put

$$\xi^H := \mathcal{H}\xi, \quad \xi^V := \mathcal{V}\xi.$$

Then, by virtue of Lemma 3.3, the following identities hold:

$$\mathcal{H}\nabla_X \xi^H = aX + bJX + \alpha(X)\xi^H + \beta(X)J\xi^H, \tag{3.7}$$

$$\mathcal{V}\nabla_X \xi^V = \alpha(X)\xi^V + \beta(X)\widehat{J}\xi^V, \tag{3.8}$$

$$\mathcal{H}\nabla_U \xi^H + T_U \xi^V = \alpha(U)\xi^H + \beta(U)J\xi^H, \tag{3.9}$$

$$\widehat{\nabla}_U \xi^V + T_U \xi^H = aU + b\widehat{J}U + \alpha(U)\xi^V + \beta(U)\widehat{J}\xi^V, \tag{3.10}$$

where $X \in \mathcal{H}(M)$, $U \in \mathcal{V}(M)$ and $\widehat{J}$ is the induced almost complex structure of each fiber. For $U, V, W, W' \in \mathcal{V}(M)$ and $X, Y, Z, Z' \in \mathcal{H}(M)$, from (2.39), (3.1), (3.5) and (3.6), we get

$$\begin{aligned}
&(\widehat{R}(U,V)W,W') + (T_U W, T_V W') - (T_V W, T_U W') \\
&- (R^0(U,V)W,W') \\
=&\Big\{\bar{a} + \frac{\lambda+\epsilon}{2(m-1)}|\xi|^2\Big\}\Big\{(U,W)(V,W') - (V,W)(U,W') \\
&- (U,\widehat{J}W)(\widehat{J}V,W') + (V,\widehat{J}W)(\widehat{J}U,W') + 2(\widehat{J}U,V)(\widehat{J}W,W')\Big\}
\end{aligned} \tag{3.11}$$

$$
\begin{aligned}
&-\frac{\lambda+\epsilon}{2(m-1)}\Bigg[\Big\{(U,\widehat{J}W)(\widehat{J}\xi^V,V)-(V,\widehat{J}W)(\widehat{J}\xi^V,U)\\
&+2(U,\widehat{J}V)(\widehat{J}\xi^V,W)-(V,W)(\xi^V,U)+(U,W)(\xi^V,V)\Big\}(\xi^V,W')\\
&-\Big\{(U,\widehat{J}W)(\xi^V,V)-(V,\widehat{J}W)(\xi^V,U)+2(U,\widehat{J}V)(\xi^V,W)\\
&+(V,W)(\widehat{J}\xi^V,U)-(U,W)(\widehat{J}\xi^V,V)\Big\}(\widehat{J}\xi^V,W')\\
&+(\xi^V,W)\Big\{(\xi^V,U)(V,W')-(\xi^V,V)(U,W')-(\widehat{J}\xi^V,U)(\widehat{J}V,W')\\
&+(\widehat{J}\xi^V,V)(\widehat{J}U,W')\Big\}+(\widehat{J}\xi^V,W)\Big\{(\widehat{J}\xi^V,U)(V,W')\\
&-(\widehat{J}\xi^V,V)(U,W')+(\xi^V,U)(\widehat{J}V,W')-(\xi^V,V)(\widehat{J}U,W')\Big\}\\
&-2(\widehat{J}W,W')\Big\{(\widehat{J}\xi^V,U)(\xi^V,V)-(\widehat{J}\xi^V,V)(\xi^V,U)\Big\}\Bigg]\\
&-\frac{2\lambda+(m+1)\epsilon}{(m-1)|\xi|^2}\Big\{(\xi^V,U)(\widehat{J}\xi^V,V)-(\xi^V,V)(\widehat{J}\xi^V,U)\Big\}\Big\{\\
&\quad(\widehat{J}\xi^V,W)(\xi^V,W')-(\widehat{J}\xi^V,W')(\xi^V,W)\Big\},
\end{aligned}
$$

$$
\begin{aligned}
(3.12)\quad &2(m-1)|\xi|^2(R^0(X,Y)Z,U)\\
&=(\lambda+\epsilon)|\xi|^2\Bigg[\Big\{(X,JZ)(J\xi^H,Y)-(Y,JZ)(J\xi^H,X)\\
&+2(X,JY)(J\xi^H,Z)-(Y,Z)(\xi^H,X)+(X,Z)(\xi^H,Y)\Big\}(\xi^V,U)\\
&-\Big\{(X,JZ)(\xi^H,Y)-(Y,JZ)(\xi^H,X)+2(X,JY)(\xi^H,Z)\\
&+(Y,Z)(J\xi^H,X)-(X,Z)(J\xi^H,Y)\Big\}(\widehat{J}\xi^V,U)\Bigg]\\
&+\big\{4\lambda+2(m+1)\epsilon\big\}\Big\{(\xi^H,X)(J\xi^H,Y)-(\xi^H,Y)(J\xi^H,X)\\
&\Big\}\Big\{(J\xi^H,Z)(\xi^V,U)-(\widehat{J}\xi^V,U)(\xi^H,Z)\Big\},
\end{aligned}
$$

$$
\begin{aligned}
(3.13)\quad &(\overline{R}(X,Y)Z,Z')-(R^0(X,Y)Z,Z')\\
&=\Big\{\bar{a}+\frac{\lambda+\epsilon}{2(m-1)}|\xi|^2\Big\}\Big\{(X,Z)(Y,Z')-(Y,Z)(X,Z')\\
&-(X,JZ)(JY,Z')+(Y,JZ)(JX,Z')+2(JX,Y)(JZ,Z')\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{\lambda+\epsilon}{2(m-1)}\Bigg[\Big\{(X,JZ)(J\xi^H,Y)-(Y,JZ)(J\xi^H,X)\\
&+2(X,JY)(J\xi^H,Z)-(Y,Z)(\xi^H,X)+(X,Z)(\xi^H,Y)\Big\}(\xi^H,Z')\\
&-\Big\{(X,JZ)(\xi^H,Y)-(Y,JZ)(\xi^H,X)+2(X,JY)(\xi^H,Z)\\
&+(Y,Z)(J\xi^H,X)-(X,Z)(J\xi^H,Y)\Big\}(J\xi^H,Z')\\
&+(\xi^H,Z)\Big\{(\xi^H,X)(Y,Z')-(\xi^H,Y)(X,Z')-(J\xi^H,X)(JY,Z')\\
&+(J\xi^H,Y)(JX,Z')\Big\}+(J\xi^H,Z)\Big\{(J\xi^H,X)(Y,Z')\\
&-(J\xi^H,Y)(X,Z')+(\xi^H,X)(JY,Z')-(\xi^H,Y)(JX,Z')\Big\}\\
&-2(JZ,Z')\Big\{(J\xi^H,X)(\xi^H,Y)-(J\xi^H,Y)(\xi^H,X)\Big\}\Bigg]\\
&-\frac{2\lambda+(m+1)\epsilon}{(m-1)|\xi|^2}\Big\{(\xi^H,X)(J\xi^H,Y)-(\xi^H,Y)(J\xi^H,X)\Big\}\Big\{\\
&(J\xi^H,Z)(\xi^H,Z')-(J\xi^H,Z')(\xi^H,Z)\Big\},
\end{aligned}
$$

Let $\{X_1,\cdots,X_{2n}\}$ be an orthonormal frame of $\mathcal{H}(M)$ and $\{V_1,\cdots,V_{2s}\}$ an orthonormal frame of $\mathcal{V}(M)$ respectively. In [3], it is known that

$$
\sum_{l=1}^{2n}((\nabla_{X_l}T)_UV,X_l)=0 \tag{3.14}
$$

for every $U,V\in\mathcal{V}(M)$. By virtue of (2.39), (3.3) and (3.14), we get the following lemma.

Lemma 3.5. *For $U,V\in\mathcal{V}(M)$ and $X,Y\in\mathcal{H}(M)$, we have*

$$
\begin{aligned}
&|T|^2\\
&=\sum_{l=1}^{2n}\sum_{\alpha=1}^{2s}(R^0(V_\alpha,X_l)X_l,V_\alpha)+4ns\{\bar a+\frac{\lambda+\epsilon}{2(m-1)}|\xi|^2\}\\
&-\frac{\lambda+\epsilon}{m-1}\{2n|\xi^V|^2+2s|\xi^H|^2\}+\frac{2\{2\lambda+(m+1)\epsilon\}|\xi^H|^2|\xi^V|^2}{(m-1)|\xi|^2},
\end{aligned} \tag{3.15}
$$

where $|T|^2:=\sum\limits_{l=1}^{2n}\sum\limits_{\alpha=1}^{2s}(T_{V_\alpha}X_l,T_{V_\alpha}X_l)$.

We deal with the case where the Bochner curvature tensor B^0 with respect to the linear connection ∇^0 vanishes, namely

$$
(R^0(E,F)G,H) \tag{3.16}
$$

$$+\frac{1}{2m+4}\Big\{(E,G)Ric^0(F,H)-(F,G)Ric^0(E,H)$$
$$+(F,H)Ric^0(E,G)-(E,H)Ric^0(F,G)$$
$$+(JE,G)Ric^0(JF,H)-(JF,G)Ric^0(JE,H)$$
$$+(JF,H)Ric^0(JE,G)-(JE,H)Ric^0(JF,G)$$
$$+2(JE,F)Ric^0(JG,H)+2(JG,H)Ric^0(JE,F)\Big\}$$
$$-\frac{r^0}{(2m+4)(2m+2)}\Big\{(E,G)(F,H)-(F,G)(E,H)$$
$$+(JE,G)(JF,H)-(JF,G)(JE,H)$$
$$+2(JE,F)(JG,H)\Big\}=0$$

for any $E,F,G,H\in TM$. Then we obtain the following lemma.

Lemma 3.6. *Let* $dim\overline{M}=2n\geq 4$. *If* B^0 *vanishes, then, for every* $p\in M$,

$$2\lambda(p)+(m+1)\epsilon(p)=0,\quad \xi^H_p=0\quad \textit{or}\quad \xi^V_p=0.$$

Proof. If we substitute (3.16) into (3.12), then we get

$$(3.17)\qquad -\frac{2(m-1)|\xi|^2}{2m+4}\Big\{(X,Z)Ric^0(Y,U)-(Y,Z)Ric^0(X,U)$$
$$+(JX,Z)Ric^0(JY,U)-(JY,Z)Ric^0(JX,U)$$
$$+2(JX,Y)Ric^0(JZ,U)=(\lambda+\epsilon)|\xi|^2\Bigg[\Big\{$$
$$(X,JZ)(J\xi^H,Y)-(Y,JZ)(J\xi^H,X)+2(X,JY)(J\xi^H,Z)$$
$$-(Y,Z)(\xi^H,X)+(X,Z)(\xi^H,Y)\Big\}(\xi^V,U)$$
$$-\Big\{(X,JZ)(\xi^H,Y)-(Y,JZ)(\xi^H,X)+2(X,JY)(\xi^H,Z)$$
$$+(Y,Z)(J\xi^H,X)-(X,Z)(J\xi^H,Y)\Big\}(\widehat{J}\xi^V,U)\Bigg]$$
$$+\big\{4\lambda+2(m+1)\epsilon\big\}\Big\{(\xi^H,X)(J\xi^H,Y)$$
$$-(\xi^H,Y)(J\xi^H,X)\Big\}\Big\{J\xi^H,Z)(\xi^V,U)-(\widehat{J}\xi^V,U)(\xi^H,Z)\Big\}.$$

We put $Y=Z=X_l$ in (3.17) and take the summation over $l=1,\cdots,2n$, then we have

$$\frac{2(m-1)(n+1)|\xi|^2}{m+2}Ric^0(X,U)$$

$$= 2\Big\{-(\lambda+\epsilon)(n+1)|\xi|^2 + \{2\lambda+(m+1)\epsilon\}|\xi^H|^2\Big\}\Big\{ (X,\xi^H)(U,\xi^V) + (X,J\xi^H)(U,\widehat{J}\xi^V)\Big\}.$$

In the above equation, putting $X=\xi^H$ and $U=\xi^V$, we obtain

$$\frac{2(m-1)(n+1)|\xi|^2}{m+2}Ric^0(\xi^H,\xi^V) = 2\Big\{-(\lambda+\epsilon)(n+1)|\xi|^2 + \{2\lambda+(m+1)\epsilon\}|\xi^H|^2\Big\}|\xi^H|^2|\xi^V|^2. \tag{3.18}$$

Also, if we put $U=\xi^V$ and $X=\xi^H$ in (3.17), then we obtain

$$\frac{-2(m-1)|\xi|^2}{2m+4}\Big\{(\xi^H,Z)Ric^0(Y,\xi^V) - (Y,Z)Ric^0(\xi^H,\xi^V) + (J\xi^H,Z)Ric^0(JY,\xi^V) - (JY,Z)Ric^0(J\xi^H,\xi^V) + 2(J\xi^H,Y)Ric^0(JZ,\xi^V)\Big\}$$
$$= |\xi^V|^2\Big[(\lambda+\epsilon)|\xi|^2\Big\{(\xi^H,Z)(\xi^H,Y) - (Y,Z)|\xi^H|^2\Big\} + \Big\{2\{2\lambda+(m+1)\epsilon\}|\xi^H|^2 - 3(\lambda+\epsilon)|\xi|^2\Big\}(J\xi^H,Y)(J\xi^H,Z)\Big].$$

Substituting $Y=Z=J\xi^H$ into the above equation, we have

$$\frac{2(m-1)|\xi|^2}{m+2}Ric^0(\xi^H,\xi^V)|\xi^H|^2 = |\xi^V|^2|\xi^H|^4\Big\{\{2\lambda+(m+1)\epsilon\}|\xi^H|^2 - 2(\lambda+\epsilon)|\xi|^2\Big\}. \tag{3.19}$$

It follows from (3.18) and (3.19) that

$$\{2\lambda+(m+1)\epsilon\}|\xi^H|^6|\xi^V|^2 = 0.$$

□

We put

$$U_1 := \{p\in M|\xi^H_p \neq 0\}\cap\{p\in M|\xi^V_p\neq 0\},$$
$$U_2 := \{p\in M|\xi^H_p = 0\},$$
$$U_3 := \{p\in M|\xi^V_p = 0\}.$$

Let U_2°(resp. U_3°) be the set of all interior points of U_2(resp. U_3). We prove

Lemma 3.7. *If ξ is everywhere non-zero proper, then $U_2^\circ = U_3^\circ = \emptyset$.*

Proof. Suppose that $U_2^\circ \neq \emptyset$. From (3.7), we have $aX + bJX = 0$ for every $X \in \mathcal{H}(U_2^\circ)$. Hence we get $a = b = 0$ on U_2. This contradicts the fact that ξ is proper. Therefore, we see that $U_2^\circ = \emptyset$.

Next suppose that $U_3^\circ \neq \emptyset$. From (3.10), we obtain $T_U\xi^H = aU + b\widehat{J}U$ for $U \in \mathcal{V}(U_3^\circ)$. Moreover we find $(\xi^H, T_U U) = -(T_U\xi^H, U) = -a(U,U)$ for every $U \in \mathcal{V}(U_3^\circ)$. Since each fiber is minimal, we see that $a = 0$ on U_3. Also since $JT_U U = T_U\widehat{J}U$, we have $(\xi^H, JT_U U) = (\xi^H, T_U\widehat{J}U) = -(T_U\xi^H, \widehat{J}U) = -b(U,U)$. Hence we conclude that $b = 0$ on U_3. This contradicts the fact that ξ is proper. Therefore, we see that $U_3^\circ = \emptyset$. □

From Theorem 2.4, Lemma 3.6 and Lemma 3.7, we have the following theorem.

Theorem 3.8. *Suppose that $\pi : M \to \overline{M}$ is a Kählerian submersion, dim$M \geq 6$, dim$\overline{M} \geq 4$ and there exists an everywhere non-zero proper Kählerian torse-forming vector field ξ. If the Bochner curvature tensor B^0 with respect to ∇^0 vanishes, then the Bochner curvature tensor B of M vanishes.*

By virtue of Theorem 3.8 and (3.13), we obtain

Corollary 3.9. *Let π, ξ, ρ, f be as in Theorem 3.8. If the Bochner curvature tensor B^0 with respect to ∇^0 vanishes, then the Bochner curvature tensor of $\overline{M}$ vanishes.*

Next we deal with the case where the curvature tensor R^0 of ∇^0 satisfies

$$
\begin{aligned}
(3.20)\qquad &(R^0(E,F)G,H) \\
&= -\frac{r^0}{4m(m+1)}\big\{(E,G)(F,H) - (F,G)(E,H) \\
&- (E,JG)(JF,H) + (F,JG)(JE,H) \\
&+ 2(JE,F)(JG,H)\big\}
\end{aligned}
$$

for any $E, F, G, H \in TM$. We can easily see that B^0 vanishes, because R^0 satisfies (3.20). By the quite same method as the proof of Lemma 3.6, we get the following lemma.

Lemma 3.10. *Let dim$\overline{M} = 2n \geq 4$. If R^0 is a tensor satisfies (3.20), then, for every $p \in M$,*

$$\lambda(p) = \epsilon(p) = 0, \quad \xi_p^H = 0 \quad \textit{or} \quad \xi_p^V = 0.$$

If the curvature tensor R of the Levi-Civita connection ∇ satisfies (3.20), then M is said to be a space of constant holomorphic sectional curvature. From Lemma 3.5, Lemma 3.7 and Lemma 3.10, we have the following theorem.

Theorem 3.11. *Suppose that* $\pi : M \to \overline{M}$ *is a Kählerian submersion,* $\dim M \geq 6$, $\dim \overline{M} \geq 4$ *and there exists an everywhere non-zero proper Kählerian torse-forming vector field* ξ. *If the curvature tensor* R^0 *satisfies* (3.20), *then* M *is of non-positive constant holomorphic sectional curvature.*

By virtue of Theorem 3.11, (3.11) and (3.13), we obtain

Corollary 3.12. *Let* π, ξ, ρ, f *be as in Theorem 3.11, If the curvature tensor* R^0 *satisfies* (3.20), *then* $\overline{M}$ *is of non-positive constant holomorphic sectional curvature and each fiber is an Einstein manifold.*

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