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$\mathfrak{L}_p$-EQUIVALENCE OF IMPULSE DIFFERENTIAL EQUATIONS

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Abstract. By the Schauder's fixed point theorem the $\mathfrak{L}_p$-equivalence between two impulse differential equations is proved.

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1. Introduction

Impulse differential equations constitute a useful mathematical apparatus for the investigation of evolutionary processes in physics, chemistry, control theory and robotics which are subject to the action of short-time forces in the form of impulses. The work of Mil'man and Myshkis [1] marked the beginning of the mathematical theory of these equations.

In the present paper an $\mathfrak{L}_p$-equivalence between two arbitrary impulse differential equations is proved. That means that for every solution of the first equation there exists a solution of the second equation such that the difference of these solutions lies in the space $\mathfrak{L}_p$, and vice versa. Similar problems under other conditions are considered in [2], [3].

2. Statement of the problem

Let $X = \mathbb{R}^n$ be the n -dimensional Eucledean space with identity operator I and norm $\| \cdot \|$. By $T = \{t_n\}$ we denote a sequence of points $0 = t_0 < t_1 < t_2 < ...$ satisfying the condition $\lim_{n\to\infty} t_n = \infty$.

Consider the impulse equation

$$\frac{dx}{dt} = F(t,x) \qquad \text{for } \ t \neq t_n \tag{1}$$

$$x(t_n^+) = Q_n \, x(t_n) \qquad \text{for } \ t = t_n \tag{2}$$

and

$$\frac{dy}{dt} = G(t,y) \qquad \text{for } \ t \neq t_n \tag{3}$$

$$y(t_n^+) = D_n \, y(t_n) \qquad \text{for } \ t = t_n \ , \tag{4}$$

where $F(t,x), G(t,y) : \mathbb{R}_+ \times X \to X$ $(\mathbb{R}_+ = [0,\infty))$ are continuous functions and $Q_n, D_n : X \to X$ $(n = 1, 2, ...)$. Moreover, we assume that all considered functions are continuous from the left.

Definition 1. *We shall say that the function $\psi(t)$ $(t \geq 0)$ is a solution of the equation $(1) - (2)$ $((3) - (4))$ if for $t \neq t_n$, it satisfies equation (1) $((3))$ and for $t = t_n$ the condition of "jump" (2) $((4))$.*

Let $1 \leq p < \infty$. By B_r we denote the closed ball in the space X with a center at zero and radius r.

Let $\Omega \subset \mathbb{R}_+$. By $\mathfrak{L}_p\,(\Omega, X)$ we denote the space of all functions $x : \Omega \to X$ for which $\int\limits_{\Omega} ||x(t)||^p dt < \infty$. When $X = \mathbb{R}$ we shall write $\mathfrak{L}_p\,(\Omega)$.

Definition 2. *The equation $(3) - (4)$ is called $\mathfrak{L}_p$-equivalent to the equation $(1) - (2)$ in the ball B_r if there exists $\rho > 0$ such that for any solution $x(t)$ of $(1) - (2)$ lying in B_r there exists a solution $y(t)$ of $(3) - (4)$ lying in the ball $B_{r+\rho}$ and satisfying the relation $y(t) - x(t) \in \mathfrak{L}_p\,(\mathbb{R}_+, X)$. If equation $(3) - (4)$ is $\mathfrak{L}_p$-equivalent to equation $(1) - (2)$ in the ball B_r and vice versa, we shall say that equations $(1) - (2)$ and $(3) - (4)$ are $\mathfrak{L}_p$-equivalent in the ball B_r.*

3. Main results

3.1. Equivalent equations

Let

$$w(t,s) = \prod_{j=n(t)}^{n(s)+1} Q_j \qquad (0 \leq s < t) \tag{5}$$

and

$$\tilde{w}(t,s) = \prod_{i=n(t)}^{n(s)+1} D_i \qquad (0 \leq s < t) \ , \tag{6}$$

where $n(\tau) = \max\{n : t_n < \tau\}$.

Lemma 1. *Each solution $x(t)$ of equation $(1)-(2)$ which lies in the ball B_r is a solution of the nonlinear integral equation*

$$x(t) = w(t,0)x(0) + \sum_{i=0}^{n(t)-1} w(t,t_i^+) \int_{t_i}^{t_{i+1}} F(s,x(s))ds + \int_{t_{n(t)}}^{t} F(s,x(s))ds \tag{7}$$

and each solution $y(t)$ of equation $(3)-(4)$ which lies in the ball $B_{r+\rho}$ is a solution of the nonlinear integral equation

$$y(t) = \tilde{w}(t,0)y(0) + \sum_{i=0}^{n(t)-1} \tilde{w}(t,t_i^+) \int_{t_i}^{t_{i+1}} G(s,y(s))ds + \int_{t_{n(t)}}^{t} G(s,y(s))ds \tag{8}$$

Lemma 1 is proved by straightforward verification.

Set

$$z(t) = y(t) - x(t). \tag{9}$$

Then the function $z(t)$ is a solution of the nonlinear integral equation

$$\begin{aligned} z(t) &= \tilde{w}(t,0)y(0) - w(t,0)x(0)+ \\ &+ \sum_{i=0}^{n(t)-1} \{\tilde{w}(t,t_i^+) \int_{t_i}^{t_{i+1}} G(s,x(s)+z(s))ds - w(t,t_i^+) \int_{t_i}^{t_{i+1}} F(s,x(s))ds\}+ \\ &+ \int_{t_{n(t)}}^{t} \{G(s,x(s)+z(s)) - F(s,x(s))\}ds. \end{aligned} \tag{10}$$

Let

$$\begin{aligned} H(x,z)(t) &= \tilde{w}(t,0)y(0) - w(t,0)x(0)+ \\ &+ \sum_{i=0}^{n(t)-1} \{\tilde{w}(t,t_i^+) \int_{t_i}^{t_{i+1}} G(s,x(s)+z(s))ds - w(t,t_i^+) \int_{t_i}^{t_{i+1}} F(s,x(s))ds\}+ \\ &+ \int_{t_{n(t)}}^{t} \{G(s,x(s)+z(s)) - F(s,x(s))\}ds. \end{aligned} \tag{11}$$

Then

$$z(t) = H(x,z)(t). \tag{12}$$

From Definition 2 it follows that to establish the $\mathfrak{L}_p$ - equivalence of equation $(3)-(4)$ to equation $(1)-(2)$ it suffices to show that for each solution $x(t)$ of equation $(1)-(2)$ lying in the ball B_r the operator equation (12) has a fixed point $z(t)$ such that $x(t)+z(t) \in B_{r+\rho}$ for some $\rho > 0$ and which lies in $\mathfrak{L}_p(\mathbb{R}_+, X)$.

Let $S(\mathbb{R}_+, X)$ be the space of all functions which are continuous for $t \neq t_n$ $(n = 1, 2, ...)$, have at the points t_n limits on the left and right and are left continuous. The space $S(\mathbb{R}_+, X)$ is linear and localy convex. A metric can be introduced by

$$\rho(x,y) = \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{\max\limits_{t\in(t_n,t_{n+1}]} \|x(t)-y(t)\|}{1+\max\limits_{t\in(t_n,t_{n+1}]} \|x(t)-y(t)\|}.$$

The convergence with respect to this metric coincides with the uniform convergence on each bounded interval. For this space an analog of Arzella-Ascoli's theorem is valid.

Lemma 2. [2] *The set $M \subset S(\mathbb{R}_+, X)$ is relatively compact if and only if M is equicontinuous on each interval $(t_{n-1}, t_n]$ $(n = 1, 2, ...)$.*

Proof. We apply the theorem of Arzella-Ascoli on each interval $(t_{n-1}, t_n]$ $(n = 1, 2, ...)$ and constitute diagonal line sequence, which is converging on each from them.□

In the proof of the existence of a fixed point of the operator H from the equation (12) we use a modification of Schauder's classical principle.

Lemma 3. [2] *Let the operator H transform the set*

$$C(r) = \{x \in S(\mathbb{R}_+, X) : x(t) \in B_r,\ t \geq 0\}$$

into inself and be continuous and compact.

Then H has a fixed point in $C(r)$.

3.2. Conditions for $\mathfrak{L}_p$-equivalence

Theorem 1. *Let the following conditions are fulfilled.*

1.*The operator-valued functions $w(t,s)$ and $\tilde{w}(t,s)$ satisfy the condition*

$$\|\tilde{w}(t,0)\xi - w(t,0)\eta\| \leq \chi_{r,\rho}(t) \qquad (0 \leq t < \infty)\ , \tag{13}$$

where $\xi \in B_{r+\rho}$, $\eta \in B_r$, $\chi_{r,\rho}(t) \in \mathfrak{L}_p(\mathbb{R}_+)$ and $r, \rho > 0$.

2.*The functions $F(t,x)$, $G(t,y)$ and $w(t,s)$, $\tilde{w}(t,s)$ satisfy the condition*

$$\sup_{\substack{\|u\|\leq r \\ \|v\|\leq r+\rho}} \sum_{i=0}^{n(t)-1} \|\tilde{w}(t,t_i^+) \int_{t_i}^{t_{i+1}} G(s,v)ds - w(t,t_i^+) \int_{t_i}^{t_{i+1}} F(s,u)ds\| \leq \psi_{\,r,\rho}(t)\ , \tag{14}$$

where $\psi_{\,r,\rho}(t) \in \mathfrak{L}_p(\mathbb{R}_+)$ *and*

$$\sup_{\substack{\|u\|\le r\\ \|v\|\le r+\rho}} \int\limits_{t_{n(t)}}^{t} \|G(s,v)-F(s,u)\|ds \le \varphi_{r,\rho}(t)\ , \tag{15}$$

where $\varphi_{r,\rho}(t) \in \mathfrak{L}_p(\mathbb{R}_+)$.

3.*The function* $G(t,y)$ *satisfies the condition*

$$\sup_{\|v\|\le r+\rho} \|G(t,v)\| \le \Phi_{r,\rho}(t)\ , \tag{16}$$

where $\Phi_{r,\rho}(t)$ *is integrable on each interval* $(t_{n-1},t_n]$ $(n=1,2,...)$.

4.*The inequality*

$$\chi_{r,\rho}(t) + \psi_{\,r,\rho}(t) + \varphi_{r,\rho}(t) \le \rho \tag{17}$$

holds for each $t \ge 0$.

Then the equation $(3)-(4)$ *is* $\mathfrak{L}_p$*-equivalent to the equation* $(1)-(2)$ *in the ball* B_r.

Proof. We shall show that for any function $x(t) \in B_r$ $(t \ge 0)$ the operator $H(x,z)$ defined by (11) maps the set

$$C(\rho) = \{z \in S(\mathbb{R}_+, X) :\ z(t) \in B_\rho\ ,\ t \ge 0\}$$

into itself.

Let $x(t) \in B_r$ $(t \ge 0)$ and let $z \in C(\rho)$. Then from (11) we obtain the estimate:

$$\begin{aligned}
&\|H(x,z)(t)\| \le \|\tilde{w}(t,0)y(0) - w(t,0)x(0)\| + \\
&+ \sum_{i=0}^{n(t)-1} \|\tilde{w}(t,t_i^+) \int\limits_{t_i}^{t_{i+1}} G(s,x(s)+z(s))ds - w(t,t_i^+) \int\limits_{t_i}^{t_{i+1}} F(s,x(s))ds\| + \\
&+ \int\limits_{t_{n(t)}}^{t} \|G(s,x(s)+z(s)) - F(s,x(s))\|ds \le \|\tilde{w}(t,0)y(0) - w(t,0)x(0)\| + \\
&+ \sup_{\substack{\|u\|\le r\\ \|v\|\le r+\rho}} \sum_{i=0}^{n(t)-1} \|\tilde{w}(t,t_i^+) \int\limits_{t_i}^{t_{i+1}} G(s,v)ds - w(t,t_i^+) \int\limits_{t_i}^{t_{i+1}} F(s,u)ds\| + \\
&+ \sup_{\substack{\|u\|\le r\\ \|v\|\le r+\rho}} \int\limits_{t_{n(t)}}^{t} \|G(s,v)-F(s,u)\|ds \le \chi_{r,\rho}(t) + \psi_{\,r,\rho}(t) + \varphi_{r,\rho}(t) \le \rho
\end{aligned}$$

for each $t \ge 0$.

We obtain $\|H(x,z)(t)\| \le \rho$, i.e., $H(x,z) \in C(\rho)$. Hence, for any $x \in C(\rho)$, the set $C(\rho)$ is invariant with respect to $H(x,z)$.

Let be $L = \{u(t) = H(x,z)(t) : \ \|z\| \leq \rho\}$.

First we shall establish that the set L is compact in $S(\mathbb{R}_+, X)$.

We shall show the equicontinuity of the functions of the set L. In fact, for $t', t'' \in (t_{n-1}, t_n]$ following equalities hold:

$$w(t', s) = w(t'', s) = w(t_n, s)$$

$$\tilde{w}(t', s) = \tilde{w}(t'', s) = \tilde{w}(t_n, s)$$

$$n(t') = n(t'') = n - 1$$

For $t', t'' \in (t_{n-1}, t_n]$ we obtain

$$\begin{aligned}
&\|u(t') - u(t'')\| = \\
&= \|(\tilde{w}(t', 0)y(0) - w(t', 0)x(0)) - (\tilde{w}(t'', 0)y(0) - w(t'', 0)x(0)) + \\
&+ \sum_{i=0}^{n(t')-1} \{\tilde{w}(t', t_i^+) \int_{t_i}^{t_{i+1}} G(s, x(s) + z(s))ds - w(t', t_i^+) \int_{t_i}^{t_{i+1}} F(s, x(s))ds\}ds - \\
&- \sum_{i=0}^{n(t'')-1} \{\tilde{w}(t'', t_i^+) \int_{t_i}^{t_{i+1}} G(s, x(s) + z(s))ds - w(t'', t_i^+) \int_{t_i}^{t_{i+1}} F(s, x(s))ds\} + \\
&+ \int_{t_{n(t')}}^{t'} \{G(s, x(s) + z(s)) - F(s, x(s))\}ds - \\
&- \int_{t_{n(t'')}}^{t'} \{G(s, x(s) + z(s)) - F(s, x(s))\}ds\| \leq \\
&\leq | \sup_{\substack{\|u\| \leq r \\ \|v\| \leq r + \rho}} \int_{t''}^{t'} \|G(s, v) - F(s, u)\|ds|
\end{aligned}$$

The equicontinuity of the functions of the set L follows from the last estimate.

From Lemma 2 the compactness of the set L follows.

We shall show that the operator $H(x, z)$ is continuous in $S(\mathbb{R}_+, X)$.

Let the sequence $\{z_n(t)\} \subset C(\rho)$ be convergent in the metric of the space $S(\mathbb{R}_+, X)$ (i.e., uniformly converges on each bounded interval) to the function $z(t) \in C(\rho)$. Then, for $t \in \mathbb{R}_+$ the sequence $G(t, x(t) + z_n(t))$ converges to $G(t, x(t)+z(t))$. From conditions 3 of Theorem 1 it follows that the convergent sequence of functions $G(t, x(t)+z_n(t))$ is majorized by the intergrable function

$\Phi_{r,\rho}(t)$. That's why within the integral in formula

$$H(x,z_n)(t)=\tilde{w}(t,0)y(0)-w(t,0)x(0)+$$
$$+\sum_{i=0}^{n(t)-1}\{\tilde{w}(t,t_i^+)\int_{t_i}^{t_{i+1}}G(s,x(s)+z_n(s))ds-w(t,t_i^+)\int_{t_i}^{t_{i+1}}F(s,x(s))ds\}+$$
$$+\int_{t_{n(t)}}^{t}\{G(s,x(s)+z_n(s))-F(s,x(s))\}ds$$

we may pass to the limit. Hence $H(x,z_n)(t)$ tends to $H(x,z)(t)$ for $t\in\mathbb{R}_+$. Since $H(x,z)$ maps $C(\rho)$ into a compact set, $H(x,z_n)$ tends to $H(x,z)$ in $S(\mathbb{R}_+,X)$ as well.

From Lemma 3 it follows that for any $x\in C(\rho)$ the operator $H(x,z)$ has a fixed point z in $C(\rho)$, i.e. $z=H(x,z)$.

We shall show that this fixed point $z(t)$ lies in $\mathfrak{L}_p(\mathbb{R}_+,X)$.

$$\|z(t)\|\le\|\tilde{w}(t,0)y(0)-w(t,0)x(0)\|+$$
$$+\sup_{\substack{\|u\|\le r\\ \|v\|\le r+\rho}}\sum_{i=0}^{n(t)-1}\|\tilde{w}(t,t_i^+)\int_{t_i}^{t_{i+1}}G(s,v)ds-w(t,t_i^+)\int_{t_i}^{t_{i+1}}F(s,u)ds\|+$$
$$+\sup_{\substack{\|u\|\le r\\ \|v\|\le r+\rho}}\int_{t_{n(t)}}^{t}\|G(s,v)-F(s,u)\|ds\le\chi_{r,\rho}(t)+\psi_{r,\rho}(t)+\varphi_{r,\rho}(t)$$
$$\|z\|_p=(\int_0^\infty\|z(t)\|^p dt)^{\frac{1}{p}}\le(\int_0^\infty|\chi_{r,\rho}(t)+\psi_{r,\rho}(t)+\varphi_{r,\rho}(t)|^p dt)^{\frac{1}{p}}\le$$
$$\le\|\chi_{r,\rho}\|_p+\|\psi_{r,\rho}\|_p+\|\varphi_{r,\rho}\|_p$$

Hence this fixed point belongs to the space $\mathfrak{L}_p(\mathbb{R}_+,X)$, i.e., the equations $(3)-(4)$ are $\mathfrak{L}_p$-equivalent to the equations $(1)-(2)$ in ball B_r.

Theorem 1 is proved. □

Remark 1. *Condition* (13) *means that the "impulse difference" of the two equations belongs in the space* $\mathfrak{L}_p(\mathbb{R}_+)$.
Condition (14) *means that the sum of the "integral differences" of* G *and* F *with weights* $\tilde{w}$ *and* w *on the balls* $B_{r+\rho}$ *and* B_r *respectively on any interval* $[t_i,t_{i+1}]$ *lies in the space* $\mathfrak{L}_p(\mathbb{R}_+)$.
Condition (15) means that the "integral difference" of the ordinary parts on any interval $[t_i,t_{i+1}]$ *lies in the space* $\mathfrak{L}_p(\mathbb{R}_+)$.

Remark 2. *It may be noted that the condition* (18) *in* [2] *is not fulfilled if one of the equations is an ordinary. Let the equation* $(3)-(4)$ *be ordinary i.e.* $D_n=I$. *Then for any solution of the impulse equation there exists a solution* *of the ordinary equation. If we have evidently or numerical representation of the solution of the ordinary equation, then the solution of the impulse equation will be $\mathfrak{L}_p$-near to this solution.*

Corollary 1. *Let the operators Q_n , D_n $(n = 1, 2, ...)$ are linear and the following conditions are fulfilled.*

1. *The operator-valued function $w(t,s)$ and $\tilde{w}(t,s)$ satisfy the conditions*

$$\|\tilde{w}(t,s)\| \leq M \quad (0 \leq s < t < \infty) \ , \tag{18}$$

where M is a positive number and

$$\|\tilde{w}(t,0)\xi - w(t,0)\eta\| \leq \chi_{r,\rho}(t) \ \ (0 \leq t < \infty) \ , \tag{19}$$

where $\xi \in B_{r+\rho}$, $\eta \in B_r$, $\chi_{r,\rho}(t) \in \mathfrak{L}_p(\mathbb{R}_+)$ and $r, \rho > 0$.

2. *The functions $F(t,x)$ and $G(s,y)$ satisfy the condition*

$$\sup_{\substack{\|u\| \leq r \\ \|v\| \leq r+\rho}} \int_0^t \|\tilde{w}(t,s)G(s,v) - w(t,s)F(s,u)\| ds \leq \psi_{\,r,\rho}(t) \ , \tag{20}$$

where $\psi_{\,r,\rho}(t) \in \mathfrak{L}_p(\mathbb{R}_+)$.

3. *The function $G(t,y)$ satisfies the condition*

$$\sup_{\|v\| \leq r+\rho} \|G(t,v)\| \leq \varphi_{r,\rho}(t) \in \mathfrak{L}_1(\mathbb{R}_+) \tag{21}$$

4. *The inequality*

$$\chi_{r,\rho}(t) + \psi_{\,r,\rho}(t) \leq \rho \tag{22}$$

holds for each $t \geq 0$.

Then the equation $(3) - (4)$ is $\mathfrak{L}_p$-equivalent to the equation $(1) - (2)$ in the ball B_r.

Proof. The corollary follows immediately from the relations

$$x(t) = \tilde{w}(t,0)x(0) + \int_0^t \tilde{w}(t,s)F(s,x(s))ds,$$

$$y(t) = w(t,0)y(0) + \int_0^t w(t,s)G(s,y(s))ds. \ \square$$

Example. Consider the impulse equations

$$\frac{dx}{dt} = F(t,x) \qquad \text{for } \ t \neq n \tag{23}$$

$$x(n^+) = 5^{-n}(2 - \sin x(n)) \qquad \text{for } \ n = 1, 2, ... \tag{24}$$

and

$$\frac{dy}{dt} = G(t,y) \qquad \text{for } \ t \neq n \tag{25}$$

$$y(n^+) = 5^{-n} \sin y(n) \qquad \text{for } \ n = 1, 2, ... \ , \tag{26}$$

where $F(t,x), G(t,y) : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ are continuous functions. Let for some $0 < r < \Delta$ the functions $G(t,y)$ and$F(t,x)$ satisfy the conditions

$$\sup_{\substack{|u|\leq r \\ |v|\leq \Delta}} \int_{[t]}^{t} |G(s,v) - F(s,u)| ds \leq \varphi(t) \in \mathfrak{L}_p(\mathbb{R}_+) \tag{27}$$

$$\sup_{|v|\leq \Delta} |G(t,v)| \leq \Phi(t) \tag{28}$$

The function $\varphi(t)$ satisfies the condition

$$4.5^{-[t]} + 4.t.5^{-[t]} + \varphi(t) \leq \Delta - r \tag{29}$$

The function $\Phi(t)$ is integrable on each interval $(n-1, n]$ $(n = 1, 2, ...)$.
We note that the conditions (27) – (29) are fulfilled for example by

$$F(t,x) = \frac{\ln 5}{4} 5^{-t} \frac{x}{1+x^2}$$

$$G(t,y) = \frac{\ln 5}{4} 5^{-t} y \sin^2 y$$

Indeed in this case we have

$$\sup_{\substack{|u|\leq r \\ |v|\leq \Delta}} \int_{[t]}^{t} \frac{\ln 5}{4} 5^{-s} |v \sin^2 v - \frac{u}{1+u^2}| ds = \frac{1}{4}(5^{-[t]} - 5^{-t}) \sup_{\substack{|u|\leq r \\ |v|\leq \Delta}} |v \sin^2 v - \frac{u}{1+u^2}| \leq$$

$$\leq \frac{1}{2} 5^{-[t]} \left(\sup_{|v|\leq \Delta} |v \sin^2 v| + \sup_{|u|\leq r} \frac{|u|}{1+u^2} \right) \leq \frac{1}{2} 5^{-[t]} (\Delta + r)$$

Set

$$\varphi(t) = \frac{1}{2} 5^{-[t]} (\Delta + r) \in \mathfrak{L}_p(\mathbb{R}_+)$$

The function $\varphi(t)$ satisfies (29)

$$4.5^{-[t]} + 4.t.5^{-[t]} + \frac{1}{2}5^{-[t]}(\Delta + r) < \Delta - r,$$

for each $t \geq 0$.
Otherwise

$$\sup_{|v|\leq\Delta} |G(t,v)| = \sup_{|v|\leq\Delta} |\frac{\ln 5}{4}5^{-t}v\sin^2 v| \leq \frac{\ln 5}{4}\Delta 5^{-t} \ \in \mathfrak{L}_p(\mathbb{R}_+)$$

We shall show that the conditions of Theorem 1 are fulfilled.

We have

$$Q_n\, x = 5^{-n}(2 - \sin x)\,, D_n\, y = 5^{-n}\sin y.$$

Then for any $\xi \in B_\Delta$, $\eta \in B_r$ $(0 < r < \Delta)$, $t \in (t_n, t_{n+1}]$ we obtain

$$|\tilde{w}(t,0)\xi - w(t,0)\eta| = |\prod_{i=[t]}^{1} D_i\xi - \prod_{i=[t]}^{1} Q_i\eta| =$$
$$= |D_n\xi_{n-1} - Q_n\eta_{n-1}| = |5^{-[t]}\sin\xi_{n-1} - 5^{-[t]}(2 - \sin\eta_{n-1})| \leq 4.5^{-[t]}\ ,$$

where

$$\xi_{n-1} = D_{n-1}D_{n-2}...D_1\xi\ ,\ \eta_{n-1} = Q_{n-1}Q_{n-2}...Q_1\eta$$

Set $\chi(t) = 4.5^{-[t]}$.

We shall show that $\chi(t) \in \mathfrak{L}_p(\mathbb{R}_+)$

$$\int_0^\infty |\chi_{r,\rho}(t)|^p dt = \int_0^\infty |4.5^{-[t]}|^p dt == 4^p\int_0^\infty 5^{(1-t)p}dt = 20^p\int_0^\infty 5^{-pt}dt < \infty$$

Hence $\chi(t) \in \mathfrak{L}_p(\mathbb{R}_+)$.

We shall show that the condition 2 of Theorem 1 is fulfilled. Let $t \in (t_n, t_{n+1}]$. Then

$$\sup_{\substack{|u|\leq r\\|v|\leq\Delta}} \sum_{i=0}^{[t]-1} |\tilde{w}(t,i^+)\int_i^{i+1} G(s,v)ds - w(t,i^+)\int_i^{i+1} F(s,u)ds| =$$
$$= \sup_{\substack{|u|\leq r\\|v|\leq\Delta}} \sum_{i=0}^{[t]-1} |5^{-[t]}\sin\xi_{n-1,i+1} - 5^{-[t]}(2 - \sin\eta_{n-1,i+1})| \leq$$
$$\leq \sum_{i=0}^{[t]-1} 4.5^{-[t]} \leq 4.t.5^{-[t]}\ ,$$

where

$$\xi_{n-1,i+1} = D_{n-1}\ D_{n-2}...D_{i+1}\int_i^{i+1} G(s,v)ds$$
$$\eta_{n-1,i+1} = Q_{n-1}\ Q_{n-2}...Q_{i+1}\int_i^{i+1} F(s,u)ds$$

Set $\psi(t) = 4.t.5^{-[t]}$.

We shall show that $\psi(t) \in \mathfrak{L}_p(\mathbb{R}_+)$.

$$\int_0^\infty |\psi(t)|^p dt = \int_0^\infty \left|4.t.5^{-[t]}\right|^p dt = 4^p \int_0^\infty t^p.5^{-p[t]} dt \le$$
$$\le 4^p \int_0^\infty t^p.5^{(1-t)p} dt = 20^p \int_0^\infty t^p.5^{-pt} dt < \infty.$$

Hence $\psi(t) \in \mathfrak{L}_p(\mathbb{R}_+)$.

For the condition 4 of Theorem 1 we obtain

$$\chi(t) + \psi(t) + \varphi(t) = 4.5^{-[t]} + 4.t.5^{-[t]} + \varphi(t) \le \Delta - r$$

for each $t \ge 0$. Hence the equation $(25) - (26)$ is $\mathfrak{L}_p$-equivalent to the equation $(23) - (24)$ in the ball B_r $(0 < r < \Delta)$.

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