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THE 2-GENERATORS FOR CERTAIN SIMPLE PERMUTATION GROUPS OF SMALL DEGREE

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Abstract. The 2-generators of A_7, A_8, A_9, M_{11}, M_{12} and M_{22} are computed up to equivalence of their automorphism groups and the automorphism group of a free group of rank 2. The actions of the automorphism group of the free group on their defining subgroups are also computed.

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§1. Introduction

Let a group G be generated by two ordered pairs $(x_1,\ x_2)$ and $(y_1,\ y_2)$. Then these satisfy the same defining relations for G if and only if there exists an element α in the automorphism group $Aut(G)$ so that $(x_1^\alpha,\ x_2^\alpha) = (y_1,\ y_2)$. Suppose $(y_1,\ y_2) = (x_1,\ x_1x_2)$ or $(y_1,\ y_2) = (x_2,\ x_1)$, then substituting $x_1 = y_1,\ x_2 = y_1^{-1}y_2$, or $x_1 = y_2,\ x_2 = y_1$, in the relations with respect to (x_1, x_2) respectively, we have new relations for G with respect to (y_1, y_2). Let $V(G)$ be the set of all generating pairs (x_1, x_2) of G and let $\widehat{V}(G)$ be the set of the orbits of $Aut(G)$ on $V(G)$. Then the transformations $(x_1,\ x_2) \to (x_1,\ x_1x_2)$ and $(x_1,\ x_2) \to (x_2,\ x_1)$ induce permutations on $V(G)$ and they also induce permutations on $\widehat{V}(G)$, since every orbit of $Aut(G)$ on $V(G)$ is characterized as the subset of $V(G)$ having same defining relations. In the present paper for $G \cong A_7,\ A_8,\ A_9,\ M_{11},\ M_{12}$ and M_{22}, one of the alternating groups of degree 7, 8, 9 or the Mathieu groups of degree 11, 12 and 22, we compute $\widehat{V}(G)$ and next compute how the permutation group generated by the transformations acts on each of its orbits on $\widehat{V}(G)$. In [4] and [6] the same was acheived for A_5

and in [10] for A_6 and $PSL(2,7)$. Some related results for A_6 and A_7 are seen in [7] and [8]. The authors note with thanks that computations were carried out with the group algorithm programming system GAP[9].

Let F be a free group generated by u and v. A normal subgroup N of F is said to be a G-defining subgroup if $G \cong F/N$. Then the elements of $Aut(F)$ given by $(u,\ v) \to (u,\ u^{-1}v)$ and $(u,\ v) \to (v,\ u)$ generate an equivalent permutation group on the set of G-defining subgroups of F as a permutation group on $\widehat{V}(G)$, by a result of [6]. If G is a finite non-abelian simple group, the action of the automorphism group of a free group is likely to be a symmetric or an alternatng group on each of its orbits on the G-defining subgroups in most cases, and it is intended to prove this fact under some general conditions in some references (cf. [2], [3]). For the above groups we computed how a free group with 2 generators acts on the G-defining subgroups. The results are shown in tables at the end of the present paper and the summary is as follows.

Theorem 1.1. *Let F be a free group generated by two elements and suppose $G \cong A_7$, A_8, A_9, M_{11}, M_{12} or M_{22}. Then the action of $Aut(F)$ on each of its orbits on the G-defining subgroups of F is one of the types S_m, A_m, $2 \wr S_m$, $2^{m-1}{:}S_m$, $2^{m-1}A_m$, $S_m \wr S_3$, $A_m \wr S_3$, $A_m^3 S_3$, $A_m^3 S_3^{\dagger}$, $A_m^3 2S_3$, $A_m^3 2^2 S_3$, $A_m^3 2^2 S_3^{\dagger}$, $S_m 2$, $A_m 2$, $2^{m-1}{:}S_m 2$, $2^{m-1}A_m 2$, $A_m^3 2^2 S_3 2$ and $A_m^3 2^2 S_3^{\dagger} 2$. In particular m is even if the action has a normal 2-subgroup.*

Here we explain some of the notation in Theorem 1.1. First $2 \wr S_n$ is a wreath product of a cyclic group of order 2 by a symmetric group S_n of degree n and, as a permutation group, is of degree $2n$ with n blocks of length 2 and quotient action as S_n. $2^{n-1}{:}S_n$, a non-split extension of 2^{n-1} by S_n, is a subgroup of $2 \wr S_n$ with the kernel consisting of all products of an even number of 2-cycles on the blocks. Similarly $2^{n-1}A_n$, a semidirect product of 2^{n-1} by A_n, is also a subgroup of $2 \wr S_n$. Next $2^{n-1}A_n 2$ is of degree two times that of $2^{n-1}A_n$, so of degree $4n$, and the right-most symbol 2 means that a cyclic group of order 2 interchanges the two orbits of $2^{n-1}A_n$, which is in fact induced by the transformation $(x_1,\ x_2) \to (x_2,\ x_1)$. Also between two families of wreath products $A_n \wr S_3$ and $S_n \wr S_3$ of degree $3n$ (with 3 blocks of length n) there exist two families of incomplete wreath products which are denoted by $A_n^3 2S_3$ and $A_n^3 2^2 S_3$ respectively, and finally there are two similar subgroups of $S_n \wr S_3$ which are denoted by $A_n^3 S_3^{\dagger}$ and $A_n^3 2^2 S_3^{\dagger}$. In these cases there are no subgroups isomorphic to S_3 whose transpositions centralize one component of A_n^3 and interchange the remaining two components.

As $PSL(2,q)$ is a permutation group of degree $q+1$, our program may also work for $PSL(2,q)$ when q is a small prime power. But we may be able to treat it as a matrix group with finite field elements in GAP because it will save memory space for storing $\widehat{V}(G)$. So in the present paper we restrict our attension to above permutation groups of small degree.

§2. Computation and observations

$Aut(F)$ is generated by Nielsen transformations (see e.g. [5] chapter 3). They permute the G-defining subgroups of F and the action is equivalent to the following permutations on $V(G)$ by [6]: (i) $(x,y) \to (x, x^{-1}y)$; (ii) $(x,y) \to (x, yx^{-1})$; (iii) $(x,y) \to (y,x)$; (iv) $(x,y) \to (x, y^{-1})$. The permutations (i) and (ii) are conjugate under $Inn(G)$, so they act on $\widehat{V}(G)$ as a same permutation. Let the permutations (i) and (iii) be denoted by σ^{-1} and τ, respectively. Then σ is $(x,y) \to (x,xy)$. The permutation (iv) is obtained as a conjugate of $\sigma\tau\sigma^{-1}\tau\sigma\tau$. Hence the permutation group on $\widehat{V}(G)$ induced by $Aut(F)$ is generated by the action of σ and τ on $\widehat{V}(G)$. In this way the action of $Aut(F)$ on $\widehat{V}(G)$ is obtained.

The automorphism groups of alternating groups are symmetric groups, except in the case A_6. So in our cases $Aut(G) \cong S_n$ if $G \cong A_n$. $Aut(M_{11}) \cong M_{11}$, but $Aut(M_{12}) \cong M_{12}2$ and we construct it as a permutation group of degree 24 interchanging two orbits of M_{12}. Finally $Aut(M_{22}) \cong M_{22}2$ is of degree 22 (see e.g. [1]).

We compute $Aut(G)$-conjugacy classes of pairs $(x_1,\ x_2)$ with $x_1,\ x_2 \in G$ and enumerate the representatives of the conjugacy classes such that x_1 and x_2 generate G, which are $\widehat{V}(G)$. Next for each pair $(x_1,\ x_2)$ in $\widehat{V}(G)$ we see what elements in $\widehat{V}(G)$ are $Aut(G)$-conjugate to $(x_1,\ x_1x_2)$ and $(x_2,\ x_1)$ respectively. This determines the action of σ and τ on $\widehat{V}(G)$. We list the representatives and lengths of the orbits of the permutation group on $\widehat{V}(G)$ induced by $Aut(F)$. Finally we will determine how this permutation group acts on each of its orbits.

Consider any one of the orbits of $Aut(F)$ on $\widehat{V}(G)$ and let it be denoted by Ω. Let a and b be permutations obtained by restricting the action of σ and τ to Ω respectively and let $H = \langle a,\ b\rangle$. We ask whether H is primitive by using the GAP-command `IsPrimitive`. If it is, then we seek an element of H satisfying the following.

Theorem 2.1. *(Jordan, see [11]) If a primitive group of degree n contains a cycle of prime degree p with $n - p \geq 3$, then the group is either alternating or symmetric.*

Then computing the signs of permutation a and b, we can determine whether H is alternating or symmetric.

Here we note that we may use the MAGMA-command `IsAlternating` or `IsSymmetric`. However it was sufficient to use the above theorem to determine alternating or symmetric in the GAP-system.

If H is not primitive, then the GAP-command `Blocks` gives blocks of imprimitivity and in our cases the length of a block was 2 or one third of the length of Ω. Let $\overline{\Omega}$ be the set of the blocks, let $\overline{H}$, $\overline{a}$ and $\overline{b}$ be the images of H, a and b on $\overline{\Omega}$ respectively, and let K be the kernel of $H \to \overline{H}$. In either case we compute $\overline{a}$ and $\overline{b}$ and if the degree of $\overline{H}$ is rather large, we apply the above method in order to see if $\overline{H}$ is alternating or symmetric, and if not, we compute it directly. Then we use the following lemmas.

Lemma 2.2. *Let $|\Omega| = 2m$ and suppose that H has a block of length 2 and that $A_m \subseteq \overline{H}$. If there exists an element $h \in H$ such that its order $|h|$ is different from $|\overline{h}|$ and that $\overline{h}$ has a fixed point on $\overline{\Omega}$, then K contains all products of an even number of 2-cycles on blocks, hence $2^{m-1} \leq |K|$, and H contains a subgroup isomorphic to A_m. Furthermore if H contains an element k with $sign(k) = -1$ and $sign(\overline{k}) = 1$, then $H \cong 2 \wr \overline{H}$.*

Proof. The condition on h implies that $K \neq 1$. Then we have K contains all products of an even number of 2-cycles on blocks, since $A_m \subseteq \overline{H}$ and since the length of the block is 2. Let the j-th block be $\overline{j} = \{j, j'\}$. If we take a 3-cycle $\overline{s}$ and an odd-cycle $\overline{t}$ of $\overline{H}$ with $\overline{s} = (\overline{1}, \overline{2}, \overline{3})$ and $\overline{t} = (\overline{3}, \overline{4}, .., \overline{2l+1})$, then we may assume that s is a product of two 3-cycles and so is t and we may set $\{j, j'\}$ so that $s = (1,2,3)(1',2',3')$ and $t = (3,4,..,2l+1)(3',4',..,(2l+1)')$. Hence we have $\langle s,t \rangle \cong A_{2l+1}$. If $m = 2l+2$, then we may have either $r_1 = (2l, 2l+1, 2l+2)((2l)', (2l+1)', (2l+2)')$ or $r_2 = (2l, (2l+1)', 2l+2)((2l)', 2l+1, (2l+2)')$. In the latter case $r_2 w = r_1$ with $w = (2l+1, (2l+1)')(2l+2, (2l+2)') \in K$. So in either case we have $\langle s, t, r_1 \rangle \cong A_{2l+2}$ as a subgroup of H.

The condition on k implies that there exists k' in the above alternating subgroup with $\overline{k} = \overline{k'}$ and that $k^{-1}k'$ is a product of an odd number of transpositions. Hence K contains a transposition (j, j') for every $\overline{j}$. If $\overline{H}$ contains $\overline{r} = (\overline{1}, \overline{2})$, then we may assume $r = (1,2,1',2')$, $(1,2',1',2)$, $(1,2')(1',2)$ or $(1,2)(1',2')$. The last case clearly gives a subgroup isomorphic to S_m. For the remaining cases, $r(1,1')$, $r(2,2')$ and $(1,1')r(1,1')$ are equal to $(1,2)(1',2')$ respectively. Thus the last assertion holds. □

The existence of k in Lemma 2.2 depends only on the signs of a, b, $\overline{a}$ and $\overline{b}$ as $sign(k) = -1$ and $sign(\overline{k}) = 1$. We note that the case for $2^{m-1}{:}S_m$ occurs when an odd permutation of $\overline{H}$ is also odd in H in Theorem 1.1.

If H has 3 blocks, we have $\overline{H} = S_3$. In this case we compute some stabilizing elements of the blocks whose normal closure generates the stabilizer and check whether one component of the stabilizer is primitive and so on. Thus as above we find whether K acts on a block as an alternating or a symmetric group. Next we find an element not acting with the same order on all blocks. This gives $A_m^3 \subseteq K$, where $|\Omega| = 3m$. Then computing the signs of the stabilizing elements on each block of Ω, we determie K to be one of A_m^3, $A_m^3 2$, $A_m^3 2^2$ or S_m^3.

Lemma 2.3. *If $K \cong A_m^3$ or $A_m^3 2^2$ and if any transposition of $\overline{H}$ is an odd or even permutation on Ω according as m is even or odd, then H does not contain a subgroup isomorphic to a split extension of K by S_3. Otherwise H contains a subgroup isomorphic to such a split extension.*

Proof. Let $\Omega_1 = \{1, 2, \cdots, m\}$, $\Omega_2 = \{1', 2', \cdots, m'\}$ and $\Omega_3 = \{1", 2", \cdots, m"\}$ be the blocks and let $\bar{t}$ be a transposition of $\overline{H}$ interchanging Ω_1 and Ω_2. If t itself is an involution and fixes every point of Ω_3, then t does not satisfy the condition of the Lemma. In this case t and one of its conjugate interchanging Ω_2 and Ω_3 generate a subgroup isomorphic to S_3, any of whose involutions centralizes one component of K. Then this subgroup and K generate a subgroup of H isomorphic to a split extension of K by S_3.

Let t_2 be the constituent of t^2 on Ω_2 and let t_3 be the constituent of t on Ω_3. Since $A_m^3 \subseteq K$, by taking a product of t and an element of K, we can choose t_2 either an identity or a transposition, and so does t_3. By the above argument we may assume that both of t_2 and t_3 are not identities. If t_2 is a transposition, then we may set $t = (1, 1', 2, 2') \cdots$ and $t^2 = (1, 2)(1', 2')$. Then we have $A_m^3 2^2 \subseteq K$, since $t^2 \in K$. By taking a product of t and $(1', 2')(1'', 2'')$ which is a conjugate of t^2, we may assume for the Lemma that t_2 is an identity and t_3 is a transposition. Hence there exist an element $(1, 1')(2, 2') \cdots (m, m')(1'', 2'')$. Then this element together with its conjugate $(1, 2)(1', 1'')(2', 2'') \cdots (m', m'')$ generates a subgroup isomorphic to S_3 but its involution does not centalizes a component of K. If $K \cong A_m^3 2$, then $(1, 2)(1', 2')(1'', 2'') \in K$, and the product of this element and the last involution becomes an involution interchanging Ω_2 and Ω_3 and stabilizing all the points of Ω_1. So in this case as in the first paragraph we have a subgroup of H isomorphic to a split extension of K by S_3. Now it is easy to see that these two types of subgroups isomorphic to S_3 are distinguished by the signs in the Lemma. □

In fact we first see whether the subgroup $\langle a,\ a^b \rangle$ of H is transitive on Ω. If it is transitive, then the method mentioned above gives the structure of H. If not, we see that it has two orbits on Ω which are interchanged by b and how the subgroup acts on one of its orbits by the above argument. This completes the computation.

We give an observation on a case where H has blocks of length 2.

Proposition 2.4. *If an element (x, y) in $V(G)$ is not conjugate to (x^{-1}, y^{-1}) in $Aut(G)$ then they make a block of length 2 in an orbit of $Aut(F)$ on $\widehat{V}(G)$*

Proof. First we note that the transformation $(u, v) \to (u^{-1}, v^{-1})$ is contained in $Aut(F)$. So such a mutually inverse pair of elements of $V(G)$ as in Proposition 2.4 represents a pair of elements in a same orbit of $Aut(F)$ on $\widehat{V}(G)$. Now (x, y) goes to (x, xy) and (x^{-1}, y^{-1}) goes to $(x^{-1}, x^{-1}y^{-1})$ under a. The latter is conjugate to $(x^{-1}, (xy)^{-1})$ by x^{-1}. Thus a mutually inverse pair of elements in $V(G)$ goes to another such a pair under a *modulo* $Aut(G)$. Clearly the same statement holds for b. Hence such a pair makes a block of an orbit of $Aut(F)$ on $\widehat{V}(G)$. □

In all cases in Theorem 1.1 that have blocks of length 2 the blocks are in fact made up of these mutually inverse pairs. In the remaining cases (x, y) is conjugate to (x^{-1}, y^{-1}) in $Aut(G)$, in which case they represent a same element of $\widehat{V}(G)$.

TABLES

In tables below 'degree' column shows the lengths of the orbits of $Aut(F)$ on $\widehat{V}(G)$ and '$Aut(F)$' column gives the actions of $Aut(F)$ on its orbits on $\widehat{V}(G)$. The 2-generators are representatives *modulo* $Aut(G)$ of the orbits of $Aut(F)$ on $\widehat{V}(G)$. In cases for M_{11}, M_{12} and M_{22}, the 2-generators are given as products of specific generators a and b below each table.

No.	2-generators of A_7		degree	$Aut(F)$
1	$(1,2)(3,4)$,	$(1,3,5,2,4,6,7)$	16	A_{16}
2	$(1,2,3)(4,5)(6,7)$,	$(2,3,4)(5,6,7)$	21	A_{21}
3	$(1,2,3)(4,5)(6,7)$,	$(2,4,3,6,7)$	21	S_{21}
4	$(1,2)(3,4)$,	$(2,5,3,6)(4,7)$	24	$A_8^3 2S_3$
5	$(1,2)(3,4)$,	$(2,3)(4,5,6,7)$	30	$S_{10} \wr S_3$
6	$(1,2)(3,4)$,	$(1,3,2,4,5,6,7)$	36	S_{36}
7	$(1,2)(3,4)$,	$(1,3)(2,5)(4,6,7)$	36	$A_{12}^3 2S_3$
8	$(1,2)(3,4)$,	$(1,3,6,4,7,2,5)$	36	A_{36}
9	$(1,2)(3,4)$,	$(2,3,5,6,7)$	40	S_{40}
10	$(1,2,3)(4,5,6)$,	$(2,3,4,6,7)$	48	$2^{23}A_{24}$
11	$(1,2,3,4)(5,6)$,	$(2,3)(4,5,7,6)$	56	$2^{27}{:}S_{28}$
12	$(1,2)(3,4)$,	$(2,5,4,6,7)$	72	A_{72}
13	$(1,2,3)(4,5)(6,7)$,	$(2,3,4,5,6)$	84	$2 \wr S_{42}$
14	$(1,2,3)(4,5)(6,7)$,	$(2,4)(3,6,5,7)$	84	$2 \wr S_{42}$
15	$(1,2,3)$,	$(2,4)(3,5,6,7)$	120	$2^{59}A_{60}$
16	$(1,2,3)(4,5)(6,7)$,	$(2,4)(3,5,6,7)$	192	$2^{47}A_{48}2$

No.	2-generators of A_8		degree	$Aut(F)$
1	$(1,2,3)(4,5,6)$,	$(1,4)(2,6,8,3,5,7)$	15	$S_5 \wr S_3$
2	$(1,2,3)(4,5)(6,7)$,	$(3,4)(5,6,8,7)$	42	$A_{14}^3 2S_3$
3	$(1,2,3)(4,5)(6,7)$,	$(1,4)(2,6,5,7,3,8)$	45	$S_{15} \wr S_3$
4	$(1,2)(3,4)$,	$(2,5,7,3,4,6,8)$	90	S_{90}
5	$(1,2)(3,4)$,	$(1,3,5)(2,4,6,7,8)$	96	S_{96}
6	$(1,2)(3,4)(5,6)(7,8)$,	$(2,3)(4,5)(6,7,8)$	96	$A_{32} \wr S_3$
7	$(1,2)(3,4)$,	$(1,3,2,5,7)(4,6,8)$	198	S_{198}
8	$(1,2)(3,4)$,	$(2,5,3,6,8,4,7)$	252	A_{252}
9	$(1,2)(3,4)$,	$(2,3,4,5,6,7,8)$	260	$2 \wr S_{130}$
10	$(1,2)(3,4)$,	$(2,3,5,6,7,8,4)$	270	S_{270}
11	$(1,2)(3,4)(5,6)(7,8)$,	$(2,3,4,5,6,7,8)$	384	S_{384}
12	$(1,2)(3,4)$,	$(2,5,3,6,4,7,8)$	432	$2^{215}{:}S_{216}$
13	$(1,2,3)(4,5)(6,7)$,	$(2,3,4,5,6,8,7)$	480	$2^{239}A_{240}$
14	$(1,2,3)(4,5)(6,7)$,	$(2,3,4)(5,6,8)$	540	$2 \wr S_{270}$
15	$(1,2)(3,4)$,	$(2,3,5,7,4,6,8)$	768	$2^{383}A_{384}$
16	$(1,2,3)(4,5)(6,7)$,	$(3,4)(5,6,7,8)$	1092	$2 \wr S_{546}$
17	$(1,2,3)(4,5)(6,7)$,	$(3,4,5,6)(7,8)$	1092	$2 \wr S_{546}$
18	$(1,2,3)(4,5)(6,7)$,	$(2,3,4,5,6,7,8)$	1296	$2^{647}{:}S_{648}$

No.	2-generators of A_9		degree	$Aut(F)$
1	$(1,2)(3,4)(5,6)(7,8)$,	$(1,3,2,4,5,7,9,6,8)$	25	A_{25}
2	$(1,2,3)(4,5)(6,7)$,	$(3,8,4,6,5,7,9)$	36	A_{36}
3	$(1,2)(3,4)$,	$(1,3,5,7,2,4,6,8,9)$	72	A_{72}
4	$(1,2)(3,4)$,	$(1,3,2,4,5,6,7,8,9)$	81	S_{81}
5	$(1,2)(3,4)$,	$(2,3,5,6,7,8,9)$	90	S_{90}
6	$(1,2)(3,4)$,	$(1,5)(2,6,3,7,9)(4,8)$	114	S_{114}
7	$(1,2)(3,4)$,	$(2,3)(4,5,6,7,8,9)$	135	$A_{45}^3 2S_3$
8	$(1,2)(3,4)$,	$(1,5)(2,6,3,7)(4,8,9)$	138	$A_{46} \wr S_3$
9	$(1,2)(3,4)$,	$(1,3,6,4,7,8,9,2,5)$	162	S_{162}
10	$(1,2)(3,4)$,	$(2,5,3,6)(4,7,8,9)$	222	$A_{74}^3 S_3^\dagger$
11	$(1,2)(3,4)(5,6)(7,8)$,	$(2,3)(4,5,7,9,6,8)$	231	$A_{77}^3 2S_3$
12	$(1,2)(3,4)(5,6)(7,8)$,	$(2,3)(4,5,7,9,8,6)$	243	$A_{81}^3 S_3^\dagger$
13	$(1,2,3)(4,5)(6,7)$,	$(3,4,5,8,9,7,6)$	252	A_{252}
14	$(1,2,3)(4,5)(6,7)$,	$(3,4)(5,8,7,9)$	300	$A_{100}^3 2S_3$
15	$(1,2,3)(4,5)(6,7)$,	$(3,4,6)(5,8)(7,9)$	315	S_{315}
16	$(1,2)(3,4)$,	$(2,5,4,6,7,8,9)$	324	S_{324}
17	$(1,2)(3,4)$,	$(1,3)(2,5)(4,6,7,8,9)$	360	$A_{120}^3 2S_3$
18	$(1,2)(3,4)(5,6)(7,8)$,	$(2,3)(4,5,9)(6,7)$	432	S_{432}
19	$(1,2)(3,4)(5,6)(7,8)$,	$(2,3,5)(4,7,9,8,6)$	460	A_{460}
20	$(1,2)(3,4)(5,6)(7,8)$,	$(2,3,5,7,4)(6,9,8)$	486	S_{486}

No.	2-generators of A_9 (continued)		degree	$Aut(F)$
21	$(1,2)(3,4)(5,6)(7,8)$,	$(2,3)(4,5,6,9,8,7)$	567	$A_{189}^3 2^2 S_3^{\dagger}$
22	$(1,2)(3,4)(5,6)(7,8)$,	$(2,3,5,7,4)(6,8,9)$	574	S_{574}
23	$(1,2)(3,4)(5,6)(7,8)$,	$(2,3,4,5,7)(6,9,8)$	612	$2 \wr S_{306}$
24	$(1,2)(3,4)$,	$(1,3,2,5,4,6,7,8,9)$	828	$2 \wr S_{414}$
25	$(1,2,3)(4,5)(6,7)$,	$(2,3,4)(5,6,8,9,7)$	864	A_{864}
26	$(1,2)(3,4)(5,6)(7,8)$,	$(2,3,5,7,9)(4,6,8)$	940	$2 \wr S_{470}$
27	$(1,2)(3,4)$,	$(2,3,5)(4,6,7,8,9)$	2560	$2^{1279} A_{1280}$
28	$(1,2,3)(4,5)(6,7)$,	$(3,4,6,9,5,8,7)$	3120	$2^{1559}{:}S_{1560}$
29	$(1,2)(3,4)$,	$(2,5,3,6,8,9)(4,7)$	3144	$2^{1571} A_{1572}$
30	$(1,2)(3,4)(5,6)(7,8)$,	$(2,3)(4,5,7,6,8,9)$	3304	$2^{1651}{:}S_{1652}$
31	$(1,2)(3,4)(5,6)(7,8)$,	$(2,3,5)(4,7,8,9,6)$	3980	$2 \wr S_{1990}$
32	$(1,2,3)(4,5)(6,7)$,	$(3,4,6,7,9,5,8)$	4320	$2^{2159}{:}S_{2160}$
33	$(1,2,3)(4,5)(6,7)$,	$(2,3,4)(5,6,8,7,9)$	5760	$2^{2879} A_{2880}$
34	$(1,2)(3,4)(5,6)(7,8)$,	$(2,3,4,5,7)(6,8,9)$	5964	$2 \wr S_{2982}$
35	$(1,2,3)(4,5)(6,7)$,	$(3,4,5,8,6,7,9)$	6300	$2 \wr S_{3150}$
36	$(1,2)(3,4)(5,6)(7,8)$,	$(2,3)(4,5)(6,7,9)$	7452	$2 \wr S_{3726}$
37	$(1,2)(3,4)(5,6)(7,8)$,	$(2,3)(4,5,7)(8,9)$	9288	$2^{4643}{:}S_{4644}$
38	$(1,2,3)(4,5)(6,7)$,	$(3,4)(5,6,8)(7,9)$	12960	$2^{6479}{:}S_{6480}$

No.	2-generators of M_{11}		degree	$Aut(F)$
1	$b^2ab^3a^{10}b^2a^5$,	aba^7b^2	66	$A_{33}2$
2	$b^2ab^3a^{10}b^2a^5$,	ab^3a^2bab	96	$A_{48}2$
3	a,	b	288	$2^{143}{:}S_{144}$
4	a,	$a^2ba^2ba^4b^2a^7$	768	$2^{383}A_{384}$
5	a,	$b^3a^4ba^6b^2a^8$	792	$2^{197}{:}S_{198}2$
6	a,	$a^7ba^6ba^{10}b$	1296	$2^{647}{:}S_{648}$
7	a,	$a^7bab^2a^2$	1380	$2 \wr S_{690}$
8	a,	aba^3ba^4b	1792	$2^{447}A_{448}2$

$a = (1,6,9,8,4,3,2,10,7,11,5),\ b = (3,7,8,11)(4,5,9,6).$

No.	2-generators of M_{12}		degree	$Aut(F)$
1	$a^4ba^2b^3a^6b^5$,	$ab^3a^3b^5a^{10}b^7$	18	A_{18}
2	$a^4ba^2b^3a^6b^5$,	$b^7a^8b^3a^9ba$	22	S_{22}
3	$a^4ba^2b^3a^6b^5$,	$a^5b^7a^3b^5a^{10}ba$	22	S_{22}
4	$a^4ba^2b^3a^6b^5$,	$aba^2b^6a^6ba^3$	36	S_{36}
5	$a^4ba^2b^3a^6b^5$,	$a^7b^5a^{10}b^5a^5b^3$	36	$A_{18}2$
6	$a^4ba^2b^3a^6b^5$,	$a^6b^3a^4b^5a^6b$	40	S_{40}
7	$a^4ba^2b^3a^6b^5$,	$ba^7b^3a^8b^3$	48	$A_{16}^3 2S_3$
8	$a^4ba^2b^3a^6b^5$,	$a^2ba^2b^3a^{10}b^3$	48	A_{48}

No.	2-generators of M_{12} (continued)		degree	$Aut(F)$
9	$a^4ba^2b^3a^6b^5$,	$a^7b^3a^9b^7a^5b^5$	60	$S_{20} \wr S_3$
10	$a^4ba^2b^3a^6b^5$,	$a^2b^7a^5b^5a^7b^3a^6$	63	$A_{21} \wr S_3$
11	a,	$a^7b^5a^9b^3a^4$	64	S_{64}
12	a,	$a^2b^6a^6b^3$	64	$2^{31}A_{32}$
13	$b^3a^6b^3a^{10}b^9$,	$a^2ba^7b^3a^8b^3$	90	$S_{30} \wr S_3$
14	a,	$ab^7a^9ba^7b^8a^4$	96	$2^{47}A_{48}$
15	$a^4ba^2b^3a^6b^5$,	$a^5ba^{10}b^5a^5b^5$	117	$A_{39} \wr S_3$
16	$a^4ba^2b^3a^6b^5$,	$b^3a^6b^5ab^9a^4$	120	$A_{40} \wr S_3$
17	$a^4ba^2b^3a^6b^5$,	$a^3ba^7b^8a^{10}b^9a^9$	120	$A_{40}^3 2^2 S_3$
18	a,	$a^2b^7a^3b^5a^{10}b^2$	160	S_{160}
19	$a^4ba^2b^3a^6b^5$,	$ba^4b^7a^8b^5a^6$	162	$S_{54} \wr S_3$
20	a,	$a^3ba^3b^4a^6b^7a^8$	180	S_{180}
21	a,	$a^3b^8a^6b^9$	180	S_{180}
22	$a^4ba^2b^3a^6b^5$,	$a^4ba^6b^5ab$	192	$A_{64} \wr S_3$
23	$a^4ba^2b^3a^6b^5$,	$ab^3a^5b^5a^7ba^9$	198	$A_{66}^3 2S_3$
24	$a^4ba^2b^3a^6b^5$,	$ab^7a^5b^5a^7b^5a^5$	216	$A_{72}^3 2S_3$
25	$a^4ba^2b^3a^6b^5$,	$b^3a^{10}ba^4b^7a^{10}$	264	$A_{88} \wr S_3$
26	$a^4ba^2b^3a^6b^5$,	$ba^8b^3a^9b^3a^6$	288	$A_{96} \wr S_3$
27	a,	$a^3ba^6b^5a^8b^3a^8$	360	A_{360}
28	a,	b	360	A_{360}
29	a,	$ab^2ab^3a^9b^7a^5$	396	A_{396}
30	a,	$a^3b^8a^{10}ba^2ba^7$	396	A_{396}
31	a,	ab^2a^4b	672	$2^{335}A_{336}$
32	a,	$a^6ba^7ba^{10}ba^2$	1080	$2^{539}{:}S_{540}$
33	a,	$a^9b^3a^4ba^2$	1776	$2^{887}A_{888}$
34	a,	$ab^6a^5b^9$	2120	$2^{1059}{:}S_{1060}$
35	a,	$a^2ba^7ba^6b^2$	2288	$2^{1143}A_{1144}$
36	a,	$b^6a^{10}ba^4b^3a$	2592	$2^{1295}A_{1296}$
37	a,	$a^3ba^3b^7a^5b^9a^2$	2808	$2^{1403}{:}S_{1404}$
38	a,	$ab^2ab^9a^6b^5a$	3384	$2^{1691}{:}S_{1692}$
39	a,	$b^6a^6b^5ab^9a^2$	3400	$2^{1699}{:}S_{1700}$
40	a,	$a^2b^9a^5b^5a^7b^8$	4400	$2^{2199}A_{2200}$
41	a,	$b^4ab^3a^6b^3a^2$	9728	$2^{2431}A_{2432}2$

$$a = (2,9,3,12,4,8,11,6,7,10,5),\ b = (1,2)(3,4,10,9,5,6,8,7,12,11).$$

No.	2-generators of M_{22}		degree	$Aut(F)$
1	$a^7b^8ab^5a^6b^{10}$,	$a^7b^4a^7b^{10}a^6b^7$	33	S_{33}
2	$a^7b^8ab^5a^6b^{10}$,	$a^9b^6a^2b^7a^4b^2$	33	A_{33}
3	$a^7b^8ab^5a^6b^{10}$,	$ba^5b^6a^9b^7a^6$	33	S_{33}
4	$a^7b^8ab^5a^6b^{10}$,	$b^7ab^4a^8b^6a^7$	33	A_{33}

No.	2-generators of M_{22} (continued)		degree	$Aut(F)$
5	$a^7b^8ab^5a^6b^{10}$,	$ab^9a^5b^{10}a^2b$	42	$A_{21}2$
6	a,	$b^7a^7b^2a^{10}b^5a^{10}$	48	$A_{16}^3 2^2 S_3$
7	$a^7b^8ab^5a^6b^{10}$,	$a^2b^2a^2b^5a^7b^8$	48	S_{48}
8	$a^7b^8ab^5a^6b^{10}$,	$a^9b^5a^{10}b^3a^5b^7$	48	S_{48}
9	$a^2b^8a^6b^3a^7b^8$,	$ab^3a^2b^{10}ab$	60	$A_{20}^3 2S_3$
10	$a^7b^8ab^5a^6b^{10}$,	$a^{10}b^6a^5b^6a^8$	66	$A_{33}2$
11	$a^2b^8a^6b^3a^7b^8$,	$a^3b^8a^8b^9ab^4$	72	$A_{24}^3 2^2 S_3$
12	$a^2b^8a^6b^3a^7b^8$,	$a^3b^5a^4b^7a^{10}b^6a^5$	84	$A_{14}^3 2^2 S_3^\dagger 2$
13	$a^7b^8ab^5a^6b^{10}$,	$a^3b^4a^7b^6a^6b^7$	84	$S_{42}2$
14	a,	b	90	S_{90}
15	a,	$b^6a^2b^{10}a^5b^8$	180	$2 \wr S_{90}$
16	a,	$a^5b^8a^2b^3a^9b^7$	198	$A_{66}^3 2^2 S_3^\dagger$
17	a,	$ab^9a^4b^6a^2b^9a^9$	216	$A_{72}^3 2S_3$
18	a,	$b^3a^4b^7a^4b^3$	288	A_{288}
19	a,	$b^6a^4b^7a^{10}b^4a^7$	360	$A_{120} \wr S_3$
20	a,	$a^3b^3a^5b^8a^7b^4$	420	$A_{140}^3 2S_3$
21	a,	$a^2b^9a^9b^2a^7b^6$	432	A_{432}
22	a,	$ab^2a^6b^6a^9b^5a^2$	480	$2^{239}{:}S_{240}$
23	a,	$a^8b^9a^5$	480	A_{480}
24	a,	$a^5b^3a^2b^6a^2b^3$	486	S_{486}
25	a,	$b^4ab^8a^4b^4a^3$	504	$A_{84}^3 2^2 S_3 2$
26	a,	$ab^6a^7b^8a^3$	576	$2^{287}{:}S_{288}$
27	a,	$a^3b^5a^6b^5a^7$	868	$S_{434}2$
28	a,	$a^5b^3a^7b^5a^6b$	900	$2 \wr S_{450}$
29	a,	$a^7b^9a^6b^7ab^4$	1056	$2^{527}A_{528}$
30	a,	$a^{10}b^2a^2b^7a^{10}b^9$	1080	$2^{539}A_{540}$
31	a,	$b^2ab^{10}a^6$	1296	$2^{647}A_{648}$
32	a,	$a^7b^2a^9b^6a^4$	1440	$2^{719}{:}S_{720}$
33	a,	$a^6b^6a^6b^{10}a^7b^4$	1512	$2^{755}A_{756}$
34	a,	$a^5b^2a^6b^6a^8b^5a^5$	1800	$2^{899}A_{900}$
35	a,	$a^3b^9a^4b^7a^6b^5$	1920	$2^{959}A_{960}$
36	a,	$b^7ab^8a^8b^6a^{10}$	2232	$2^{1115}A_{1116}$
37	a,	$b^4a^9b^3a^5b^8a^4$	2232	$2^{1115}A_{1116}$
38	a,	$b^{10}a^7b^3a^9b^2a$	2244	$2 \wr S_{1122}$
39	a,	$ab^9a^7b^8a^7$	2244	$2 \wr S_{1122}$
40	a,	$a^4b^8a^6b^{10}ab^3$	2376	$2^{1187}{:}S_{1188}$
41	a,	$ab^{10}ab^8ab^5$	2664	$2^{1331}{:}S_{1332}$
42	a,	$b^5ab^5a^8b^4a^4$	2688	$2^{671}A_{672}2$
43	a,	$a^2b^4a^{10}b^6a^{10}b^3$	2784	$2^{1391}{:}S_{1392}$
44	a,	$a^3b^4a^8b^9$	3036	$2 \wr S_{1518}$
45	a,	$a^2ba^9b^4a^{10}$	3036	$2 \wr S_{1518}$

No.	2-generators of M_{22} (continued)		degree	$Aut(F)$
46	$a,$	$a^6b^4a^4b^6a^5b^4$	3036	$2 \wr S_{1518}$
47	$a,$	$b^3a^7b^9a^6b^4$	3072	$2^{1535}A_{1536}$
48	$a,$	$aba^{10}b^2a^4b^4a^9$	3240	$2^{1619}{:}S_{1620}$
49	$a,$	$b^8a^4b^{10}a^7b^5$	3960	$2^{1979}A_{1980}$
50	$a,$	$a^8b^9ab^{10}a$	3960	$2^{1979}{:}S_{1980}$
51	$a,$	$a^8b^2a^5b^9a^9b^7$	3960	$2^{1979}A_{1980}$
52	$a,$	$a^8b^6a^6b^7a^5$	3960	$2^{1979}{:}S_{1980}$
53	$a,$	$b^8a^9b^4a^6b^8a^4$	4032	$2^{1007}A_{1008}2$
54	$a,$	$a^4b^5a^6b^6a^6b^9$	4032	$2^{2015}A_{2016}$
55	$a,$	$a^6b^9a^6b^5ab^2$	4092	$2 \wr S_{2046}$
56	$a,$	$a^5b^8a^2b^2$	4092	$2 \wr S_{2046}$
57	$a,$	$a^5b^8ab^3a^7b^8$	6000	$2^{2999}{:}S_{3000}$
58	$a,$	$a^5b^5a^{10}b^9a$	6624	$2^{3311}A_{3312}$
59	$a,$	$a^2b^3a^5b^7a^{10}b^7$	6656	$2^{1663}A_{1664}2$
60	$a,$	$a^3b^{10}ab^7a^9b^{10}$	6660	$2 \wr S_{3330}$
61	$a,$	$b^3a^5b^9a^9b^5a^5$	6660	$2 \wr S_{3330}$
62	$a,$	$a^7b^3a^{10}b^7a^2b^9$	6720	$2^{3359}A_{3360}$
63	$a,$	$a^4b^4a^8b^3ab^4$	6840	$2^{3419}A_{3420}$
64	$a,$	$ab^9a^2ba^8b^8a^7$	7680	$2^{3839}{:}S_{3840}$
65	$a,$	$a^2b^{10}a^7b^8a^4$	7896	$2^{1973}{:}S_{1974}2$
66	$a,$	$a^4b^7a^8b^6ab^4$	9216	$2^{2303}A_{2304}2$
67	$a,$	$aba^5b^8a^8b^7$	9216	$2^{2303}A_{2304}2$
68	$a,$	$a^3b^2a^{10}b^7a^4b^6$	9744	$2^{2435}A_{2436}2$
69	$a,$	$a^5b^7a^4b^9a^2b^7$	10304	$2^{2575}A_{2576}2$
70	$a,$	$a^2b^7a^8b^9a^7b^4$	10920	$2^{2729}{:}S_{2730}2$
71	$a,$	$b^2a^7b^4a^6b^8a^8$	10920	$2^{2729}{:}S_{2730}2$

$$a = (1,5,12,21,6,18,4,2,19,14,17)(3,20,8,22,16,7,9,13,15,10,11),$$
$$b = (1,13,18,15,7,14,4,21,3,12,22)(2,11,19,16,5,8,6,20,10,9,17).$$

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