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Weighted inequalities for the two-dimensional Laplace Transform

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Abstract. Necessary conditions and sufficient conditions are given so that the two-dimensional Laplace Transform is bounded between two Lebesgue spaces with weights. Such a boundedness is characterized for a large class of weights.

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§1. Introduction

The two-dimensional Laplace Transform is defined as

$$(\mathcal{L}f)(x_1,x_2) = \int_0^\infty \int_0^\infty f(y_1,y_2)\exp[-(x_1y_1+x_2y_2)]dy_1dy_2, \qquad 0<x_1,x_2<\infty.$$

Throughout this paper it is assumed that

$$1<p\le q<\infty, \qquad p' = \frac{p}{p-1}, \qquad q' = \frac{q}{q-1}$$

and

$$u(.,.) \qquad \text{and} \qquad v(.,.) \qquad \text{are weights on }]0,\infty[^2$$

in the sense that they are measurable positive and finite almost everywhere. And for simplicity, it is supposed that

$$\int_0^{R_1}\int_0^{R_2} v^{1-p'}(x_1,x_2)dx_1dx_2<\infty, \qquad \int_0^{R_1}\int_0^{R_2} u(x_1,x_2)dx_1dx_2<\infty$$

$$\int_{R_1^{-1}}^\infty \int_0^{R_2} v^{1-p'}(x_1,x_2)\exp[-p'x_1R_1]dx_1dx_2<\infty,$$

$$\int_0^{R_1}\int_{R_2^{-1}}^\infty v^{1-p'}(x_1,x_2)\exp[-p'x_2R_2]dx_1dx_2<\infty,$$

and
$$\int_{R_1^{-1}}^{\infty}\int_{R_2^{-1}}^{\infty} u(x_1,x_2)\exp[-q(x_1R_1+x_2R_2)]dx_1dx_2<\infty$$
for all R_1, $R_2>0$.

Our purpose is to derive necessary conditions and sufficient conditions on weights $u(.,.)$ and $v(.,.)$ for which $\mathcal{L}$ is bounded from the Lebesgue space $L^p(]0,\infty[^2, v(x_1,x_2)dx_1dx_2)$ into $L^q(]0,\infty[^2, u(x_1,x_2)dx_1dx_2)$. That is for some constant $C>0$

$$(1.1)\quad \Big(\int_0^{\infty}\int_0^{\infty}(\mathcal{L}f)^q(x_1,x_2)u(x_1,x_2)dx_1dx_2\Big)^{\frac{1}{q}} \leq C\Big(\int_0^{\infty}\int_0^{\infty}f^p(x_1,x_2)v(x_1,x_2)dx_1dx_2\Big)^{\frac{1}{p}}$$

for all $f(.,.)\geq 0$. For convenience this boundedness is also denoted by $\mathcal{L}: L^p_v\to L^q_u$ or by $\mathcal{L}: L^p(v(x_1,x_2)dx_1dx_2)\to L^q(u(x_1,x_2)dx_1dx_2)$ when precisions are needed.

The weighted inequalities for $\mathcal{L}$ deserve interest since this operator is helpful to solve some partial differential equations related to many physical problems as in dynamics of structures, thermodynamics of solids, heat conductions, rotating fluids,........

The one-dimensional version of (1.1) has been investigated by many authors [An-Hg], [Hz], [Bm], [An], [Ra]. As far as the author knows, there is just one paper due to S. Emara [Ea] (Theorem 5, p.210) which treats a particular case of (1.1) and this problem remains open in full generality.

Our present contribution is first to provide some natural necessary conditions for the boundedness $\mathcal{L}: L^p_v\to L^q_u$ (see Theorem 2.1). And the second is to propose some sufficient conditions which are close to the preceding necessary conditions (see Theorem 2.2). So this boundedness is characterized for a large class of weights (see Propositions 2.5 and 2.6). Finally some examples, showing the simplicity and applicability of the criteria proposed in this paper, are given (see Corollaries 2.7 and 2.8).

The key for proving $\mathcal{L}: L^p_v\to L^q_u$ (inspired from [Ra]) is its equivalence with boundedness of four auxiliary operators denoted by $\mathbf{H}$, $\mathbf{L}$, $\mathbf{L}_1$ and $\mathbf{L}_2$ (see Theorem 3.1). Here $\mathbf{H}$ is a two-dimensional version of the classical Hardy operator whose boundedness on weighted Lebesgue spaces was studied in [Sr]. The other operators are two-dimensional versions of an integral operator investigated in [An-Hg], [Hz]. So our boundedness results for $\mathbf{L}$, $\mathbf{L}_1$ and $\mathbf{L}_2$ seem new and can be seen as a first approach to some two-dimensional integral operators (see Theorems 3.4, 3.5 and 3.6).

The main results of this paper are presented in section 2. They depend on basic results which are stated in section 3. Proofs of the main results are given in section 4. And those for the basic results are performed in the last section 5.

§2. The Main Results

First we state some natural necessary conditions related to $\mathcal{L} : L^p_v \to L^q_u$.

Theorem 2.1. *Suppose that $\mathcal{L} : L^p_v \to L^q_u$. Then there is a constant $A > 0$ such that for all R_1, $R_2 > 0$*

(2.1)
$$\Big(\int_0^{R_1^{-1}} \int_0^{R_2^{-1}} u(y_1, y_2) dy_1 dy_2\Big)^{\frac{1}{q}} \Big(\int_0^{R_1} \int_0^{R_2} v^{1-p'}(x_1, x_2) dx_1 dx_2\Big)^{\frac{1}{p'}} \leq A$$

(2.2)
$$\Big(\int_{R_1^{-1}}^{\infty} \int_{R_2^{-1}}^{\infty} \Big[\int_0^{x_1^{-1}} \int_0^{x_2^{-1}} v^{1-p'}(y_1, y_2) dy_1 dy_2\Big]^q u(x_1, x_2) dx_1 dx_2\Big)^{\frac{1}{q}} \leq A\Big(\int_0^{R_1} \int_0^{R_2} v^{1-p'}(x_1, x_2) dx_1 dx_2\Big)^{\frac{1}{p}}$$

(2.3)
$$\Big(\int_{R_1}^{\infty} \int_{R_2}^{\infty} \Big[\int_0^{x_1^{-1}} \int_0^{x_2^{-1}} u(y_1, y_2) dy_1 dy_2\Big]^{p'} v^{1-p'}(x_1, x_2) dx_1 dx_2\Big)^{\frac{1}{p'}} \leq A\Big(\int_0^{R_1^{-1}} \int_0^{R_2^{-1}} u(x_1, x_2) dx_1 dx_2\Big)^{\frac{1}{q'}}$$

(2.4)
$$\Big(\int_{2R_1^{-1}}^{\infty} \int_{2R_2^{-1}}^{\infty} u(x_1, x_2) \exp[-q(x_1 R_1 + x_2 R_2)] dx_1 dx_2\Big)^{\frac{1}{q}} \Big(\int_{2^{-1}R_1}^{R_1} \int_{2^{-1}R_2}^{R_2} v^{1-p'}(x_1, x_2) dx_1 dx_2\Big)^{\frac{1}{p'}} \leq A$$

(2.5)
$$\Big(\int_{2^{-1}R_1}^{R_1} \int_{R_2^{-1}}^{2R_2^{-1}} u(x_1, x_2) dx_1 dx_2\Big)^{\frac{1}{q}} \Big(\int_{2R_1^{-1}}^{\infty} \int_0^{2^{-1}R_2} v^{1-p'}(x_1, x_2) \exp[-p'(x_1 R_1)] dx_1 dx_2\Big)^{\frac{1}{p'}} \leq A$$

and

(2.6)
$$\Big(\int_{R_1^{-1}}^{2R_1^{-1}} \int_{2^{-1}R_2}^{R_2} u(x_1, x_2) dx_1 dx_2\Big)^{\frac{1}{q}} \Big(\int_0^{2^{-1}R_1} \int_{2R_2^{-1}}^{\infty} v^{1-p'}(x_1, x_2) \exp[-p'(x_2 R_2)] dx_1 dx_2\Big)^{\frac{1}{p'}} \leq A.$$

Conditions (2.1), (2.2) and (2.3) are involved in a characterization for the boundedness of some two-dimensional Hardy operator on weighted Lebesgue spaces. Sometimes (2.1) and (2.2) will be referred as Muckenhoupt condition and Sawyer condition respectively.

Cases for which some of these conditions (2.1) to (2.6) do overlap will be examined below.

It is not known whether all of the above six conditions do imply $\mathcal{L} : L^p_v \to L^q_u$. Here we are able to derive this boundedness from variant conditions close to the first ones.

It is just the aim of our second main result.

Theorem 2.2. *The boundedness $\mathcal{L} : L^p_v \to L^q_u$ does hold whenever conditions (2.1), (2.2), (2.3) are satisfied and if there are nonnegative constants A and ε with $\varepsilon \leq 1$ and a nonnegative sequence $(\tau_j)_j$ with $\sum_{j=0}^{\infty} \tau_j < \infty$ such that*

$$(2.7)\quad \Big(\int_{R_1^{-1}}^{\infty}\int_{R_2^{-1}}^{\infty} u(x_1,x_2)\exp[-C_\varepsilon q(x_1R_1+x_2R_2)]dx_1dx_2\Big)^{\frac{1}{q}} \Big(\int_{2^{-1}R_1}^{R_1}\int_{2^{-1}R_2}^{R_2} v^{1-p'}(x_1,x_2)dx_1dx_2\Big)^{\frac{1}{p'}} \leq A$$

$$(2.8)\quad \Big(\int_{2^{-1}R_1}^{R_1}\int_{R_2^{-1}}^{2R_2^{-1}} u(x_1,x_2)dx_1dx_2\Big)^{\frac{1}{q}} \Big(\int_{R_1^{-1}}^{\infty}\int_{2^{-(4+j_2)}R_2}^{2^{-j_2}R_2} v^{1-p'}(x_1,x_2)\exp[-C_\varepsilon p'(x_1R_1)]dx_1dx_2\Big)^{\frac{1}{p'}} \leq A\tau_{j_2}$$

and

$$(2.9)\quad \Big(\int_{R_1^{-1}}^{2R_1^{-1}}\int_{2^{-1}R_2}^{R_2} u(x_1,x_2)dx_1dx_2\Big)^{\frac{1}{q}} \Big(\int_{2^{-(4+j_1)}R_1}^{2^{-j_1}R_1}\int_{R_2^{-1}}^{\infty} v^{1-p'}(x_1,x_2)\exp[-C_\varepsilon p'(x_2R_2)]dx_1dx_2\Big)^{\frac{1}{p'}} \leq A\tau_{j_1}$$

for all R_1, $R_2 > 0$ and j_1, $j_2 \in \{0, 1, \dots, \infty\}$, here $C_\varepsilon = 4^{-1}(1-\varepsilon)$.

This result is inspired from the one-dimensional version one recently found by the author [Ra], where the introduction of C_ε leads to an improvement of results previously proved by S. Blom [Bm]. In the one-dimensional setting, conditions (2.8) and (2.9) do not appear.

The nonnegative sequence $(\tau_j)_j$ involved in (2.8) and (2.9) should be viewed as something playing a balance role in these conditions. Its introduction can be explained by the fact that the exponential is just about one of the variables.

Using results in [Sr], it can be shown that conditions (2.1), (2.2) and (2.3) do not overlap in general, and the Muckenhoupt condition (2.1) alone is not sufficient to get the Sawyer type conditions (2.2) and (2.3). Moreover these last conditions are in general more difficult to check than (2.1). Indeed, first there are more integrations to do in (2.2) (or (2.3)) than in (2.1). Next, it is often easy to get an upper bound for expressions (as the left member of (2.1)), but most of the time a lower bound or an exact value is not always available (as for the right member of (2.2)). So the question of deriving (2.2) and (2.3) from (2.1), but with more assumptions on the weights, arises naturally.

Consequently it is useful to introduce the condition $w(.,.) \in D$ which means that for some constant $c > 0$

$$\int_0^{2R_1}\int_0^{2R_2} w(y_1,y_2)dy_1dy_2 \le c\int_0^{R_1}\int_0^{R_2} w(y_1,y_2)dy_1dy_2 \tag{2.10}$$

for all $R_1, R_2 > 0$. If moreover

$$\int_0^{R_1}\int_0^{R_2} w(y_1,y_2)dy_1dy_2 \le c\int_{2^{-1}R_1}^{R_1}\int_{2^{-1}R_2}^{R_2} w(y_1,y_2)dy_1dy_2 \tag{2.11}$$

then it is written that $w(.,.) \in \widetilde{D}$. Also remind that a weight $w(.,.)$ is said to be *of product type* whenever $w(x_1,x_2) = w_1(x_1)w_2(x_2)$ for some weights $w_1(.)$ and $w_2(.)$ on $]0,\infty[$.

Lemma 2.3. *The Sawyer conditions* (2.2) *and* (2.3) *are implied by the Muckenhoupt condition* (2.1) *whenever one of the following is satisfied*

$$v(.,.) \quad \text{and} \quad u(.,.) \quad \text{are of product type,} \tag{2.12}$$

$$v^{1-p'}(.,.) \in D\cap\widetilde{D} \quad \text{and} \quad u(.,.) \in D\cap\widetilde{D}. \tag{2.13}$$

To deal with the connections of conditions (2.7), (2.8) and (2.9) with the Muckenhoupt condition (2.1), it is helpful to introduce the condition $w(.,.) \in RD$ which means that for some constants ρ, $c > 0$

$$\int_0^{t_1R_1}\int_0^{t_2R_2} w(y_1,y_2)dy_1dy_2 \le ct_1^\rho t_2^\rho\int_0^{R_1}\int_0^{R_2} w(y_1,y_2)dy_1dy_2 \tag{2.14}$$

for all $R_1, R_2 > 0$ and $0 < t_1, t_2 \le 1$.

Lemma 2.4. *Condition* (2.7) *is implied by the Muckenhoupt condition* (2.1) *whenever* $v^{1-p'}(.,.) \in D$ *or* $u(.,.) \in D$.

Conditions (2.8) *and* (2.9) *are also implied by* (2.1) *whenever* $v^{1-p'}(.,.) \in D \cap RD$ *or* $u(.,.) \in D \cap RD$.

As for the one-dimensional case [Ra], it is also possible to derive condition (2.7) from (2.1) whenever $u(.,.)$ is a decreasing function for each variable. But we do not investigate in this direction since the notion of monotone weights is not well appropriated for the setting of two variables functions.

Now these Lemmas can be used to derive a characterization for the boundedness $\mathcal{L} : L^p_v \to L^q_u$, for a large class of weights, as we have announced in the introduction.

Proposition 2.5. *The boundedness* $\mathcal{L} : L^p_v \to L^q_u$ *is equivalent to both the Muckenhoupt condition* (2.1) *and the Sawyer conditions* (2.2) *and* (2.3) *whenever* $v^{1-p'}(.,.) \in D \cap RD$ *or* $u(.,.) \in D \cap RD$.

Proposition 2.6. *Suppose that* $v^{1-p'}(.,.) \in D \cap RD$ *or* $u(.,.) \in D \cap RD$. *The boundedness* $\mathcal{L} : L^p_v \to L^q_u$ *is equivalent to the Muckenhoupt condition* (2.1) *whenever*

$$v(.,.) \qquad \text{and} \qquad u(.,.) \qquad \text{are of product type,}$$

or

$$v^{1-p'}(.,.) \in \widetilde{D} \qquad \text{and} \qquad u(.,.) \in \widetilde{D}.$$

The boundedness criteria for $\mathcal{L} : L^p_v \to L^q_u$ stated above are really simple and efficient to treat concrete examples. The papers [An-Hg], [Hz] and [Bm] dealing with the one-dimensional case just gave examples for the power weights. To go beyond this class of weight functions, it is useful to introduce nondecreasing and positive functions $\varphi(.)$ defined on $]0, \infty[$ such that

$$\varphi(2t) \leq c\varphi(t) \qquad \text{for all } t > 0 \tag{2.15}$$

for some fixed constant $c > 0$. These assumptions on $\varphi(.)$ are summarized by $\varphi(.) \in \triangle_2$. For instance the growth condition (2.15) is satisfied for the function $\varphi(t) = t^\delta \ln^\beta(1+t)$ where $\delta \in]-\infty, \infty[$ and $\beta \geq 0$.

Corollary 2.7. *Let* $\varphi_1(.)$, $\varphi_2(.) \in \triangle_2$. *The boundedness* $\mathcal{L} : L^p_v \to L^q_u$ *does hold whenever* $v(.,.)$ *and* $u(.,.)$ *are defined by*

$$v(x_1,x_2) = v_1(x_1)v_2(x_2) \qquad \text{and} \qquad u(x_1,x_2) = u_1(x_1)u_2(x_2) \tag{2.16}$$

where

$$v_1(x_1) = \varphi_1^{1-p}(x_1), \qquad v_2(x_2) = \varphi_2^{1-p}(x_2)$$

and

$$u_1(x_1) = x_1^{\frac{q}{p'}-1}\varphi_1^{-\frac{q}{p'}}(x_1^{-1}), \qquad u_2(x_2) = x_2^{\frac{q}{p'}-1}\varphi_2^{-\frac{q}{p'}}(x_2^{-1}).$$

This example deals with weights of product types. But the arguments could be modified to treat more general weights.

Corollary 2.8. *Let $\varphi(.) \in \triangle_2$. The boundedness $\mathcal{L} : L^p_v \to L^q_u$ does hold whenever $v(.,.)$ and $u(.,.)$ are defined by*

(2.17)
$$v(x_1, x_2) = \varphi^{1-p}(x_1 + x_2), \qquad u(x_1, x_2) = x_1^{\frac{q}{p'}-1} x_2^{\frac{q}{p'}-1} \varphi^{-\frac{q}{p'}}(x_1^{-1} + x_2^{-1}).$$

The boundedness problem for $\mathcal{L}$ from L^p_v into L^q_u in the range $q < p$ still remains unsolved. And it is also an interesting question to ask about the boundedeness for the n-dimensional Laplace operator

$$(\mathcal{L}_n f)(x_1, \dots, x_n) = \int_0^\infty \dots \int_0^\infty f(y_1, \dots, y_n) \exp[-(x_1 y_1 + \dots + x_n y_n)] dy_1 \dots dy_n$$

at least for some reasonable weight functions.

§3. Basic Results

To deal with (1.1), it is convenient to introduce four auxiliary operators as

$$(\mathbf{H}f)(x_1, x_2) = \int_0^{x_1} \int_0^{x_2} f(y_1, y_2) dy_1 dy_2$$

$$(\mathbf{L}f)(x_1, x_2) = \int_0^{x_1} \int_0^{x_2} f(y_1, y_2) \exp[-(y_1^{-1} x_1 + y_2^{-1} x_2)] dy_1 dy_2$$

$$(\mathbf{L}_1 f)(x_1, x_2) = \int_0^{x_1} \int_0^{x_2} f(y_1, y_2) \exp[-(y_1^{-1} x_1)] dy_1 dy_2$$

$$(\mathbf{L}_2 f)(x_1, x_2) = \int_0^{x_1} \int_0^{x_2} f(y_1, y_2) \exp[-(y_2^{-1} x_2)] dy_1 dy_2.$$

The operator $\mathbf{H}$ is known as a two-dimensional Hardy-operator. And the other operators can be seen as two-dimensional versions of an integral operator introduced and studied by Andersen and Heinig [An-Hg].

To make easy comparisons of the various conditions which we will introduce below, it could be useful to note that

$$(\mathbf{L}f)(.,.) \le (\mathbf{L}_1 f)(.,.); (\mathbf{L}_2 f)(.,.) \le (\mathbf{H}f)(.,) \qquad \text{for all } f(.,.) \ge 0.$$

Our main results stated in §2 are based on the following result.

Theorem 3.1. *The boundedness $\mathcal{L} : L^p(v(x_1, x_2) dx_1 dx_2)) \to L^q(u(x_1, x_2) dx_1 dx_2)$ is equivalent to the following four boundednesses*

$$\mathbf{H} : L^p(v(x_1, x_2) dx_1 dx_2) \to L^q(w(x_1, x_2) dx_1 dx_2)$$

$$\mathbf{L} : L^{q'}(w^{1-q'}(x_1,x_2)dx_1dx_2) \to L^{p'}(v^{1-p'}(x_1,x_2)dx_1dx_2)$$

$$\mathbf{L}_1 : L^{p}(v_1(x_1,x_2)dx_1dx_2) \to L^{q}(u_1(x_1,x_2)dx_1dx_2)$$

and

$$\mathbf{L}_2 : L^{p}(v_2(x_1,x_2)dx_1dx_2) \to L^{q}(u_2(x_1,x_2)dx_1dx_2)$$

where

$$w(x_1,x_2) = x_1^{-2}x_2^{-2}u(x_1^{-1},x_2^{-1}) \tag{3.1}$$

$$v_1(x_1,x_2) = x_1^{2p-2}v(x_1^{-1},x_2), \qquad u_1(x_1,x_2) = x_2^{-2}u(x_1,x_2^{-1}) \tag{3.2}$$

and

$$v_2(x_1,x_2) = x_2^{2p-2}v(x_1,x_2^{-1}), \qquad u_2(x_1,x_2) = x_1^{-2}u(x_1^{-1},x_2). \tag{3.3}$$

Consequently our real task is reduced to find conditions guaranteeing these boundednesses. We first begin by the case of the Hardy operator $\mathbf{H}$.

Theorem 3.2. *The weighted Hardy inequality* $\mathbf{H} : L^p(v(x_1,x_2)dx_1dx_2)) \to L^q(w(x_1,x_2)dx_1dx_2)$ *does hold if and only if there is a constant* $A > 0$ *such that for all* $R_1, R_2 > 0$

$$\left(\int_{R_1}^{\infty}\int_{R_2}^{\infty} w(x_1,x_2)dx_1dx_2\right)^{\frac{1}{q}}\left(\int_0^{R_1}\int_0^{R_2} v^{1-p'}(x_1,x_2)dx_1dx_2\right)^{\frac{1}{p'}} \le A \tag{3.4}$$

$$\left(\int_0^{R_1}\int_0^{R_2}\Big[\int_0^{x_1}\int_0^{x_2} v^{1-p'}(y_1,y_2)dy_1dy_2\Big]^q w(x_1,x_2)dx_1dx_2\right)^{\frac{1}{q}} \le A\left(\int_0^{R_1}\int_0^{R_2} v^{1-p'}(x_1,x_2)dx_1dx_2\right)^{\frac{1}{p}} \tag{3.5}$$

and

$$\left(\int_{R_1}^{\infty}\int_{R_2}^{\infty}\Big[\int_{x_1}^{\infty}\int_{x_2}^{\infty} w(y_1,y_2)dy_1dy_2\Big]^{p'} v^{1-p'}(x_1,x_2)dx_1dx_2\right)^{\frac{1}{p'}} \le A\left(\int_{R_1}^{\infty}\int_{R_2}^{\infty} w(x_1,x_2)dx_1dx_2\right)^{\frac{1}{q'}}. \tag{3.6}$$

This result is due E. Sawyer [Sr]. Moreover he showed that the Muckenhoupt condition (3.4) alone is not sufficient to get $\mathbf{H} : L^p_v \to L^q_w$. Observe that condition (3.5) (or (3.6)) is in general more difficult to check than (3.4). So the question of deriving $\mathbf{H} : L^p_v \to L^q_w$ from (3.4), but with more assumptions on the weights, arises naturally. For the one-dimensional case, both conditions (3.4), (3.5) and (3.6) are equivalent. For instance the Sawyer condition (3.5) is implied by the Muckenhoupt condition (3.4) because of the identity

$$\Big(\int_0^x v^{1-p'}(z)dz\Big)^q = q\int_0^x \Big[\int_0^y v^{1-p'}(z)dz\Big]^{q-1} v^{1-p'}(y)dy. \tag{3.7}$$

And similarly (3.6) can be derived from (3.4) and

$$\Big(\int_x^\infty w(z)dz\Big)^{p'} = p'\int_x^\infty \Big[\int_y^\infty w(z)dz\Big]^{p'-1} w(y)dy. \tag{3.8}$$

One of the reasons why condition (3.4) is not sufficient to get $\mathbf{H} : L^p_v \to L^q_w$ can be explained by the fact that the similar two-dimensional versions of (3.7) and (3.8) are not true for all weights.

Lemma 3.3. *The Sawyer conditions* (3.5) *and* (3.6) *are implied by the Muckenhoupt condition* (3.4) *whenever*

$$v(.,.) \quad \text{and} \quad w(.,.) \quad \text{are of product type}$$

or

$$v^{1-p'}(.,.) \in D\cap \widetilde{D} \qquad \text{and} \qquad x_1^{-2}x_2^{-2}w(x_1^{-1},x_2^{-1}) \in D\cap\widetilde{D}.$$

Now our aim is to study $T : L^p_v \to L^q_u$ when T is either of $\mathbf{L}$, $\mathbf{L}_1$ and $\mathbf{L}_2$. Although a characterization result for this boundedness remains an open problem, here we are able to derive sufficient conditions which are not too far from some suitable necessary conditions.

Theorem 3.4. *Suppose that* $\mathbf{L} : L^p_v \to L^q_u$. *Then there is a constant* $A > 0$ *such that for all* $R_1,\ R_2 > 0$

$$\Big(\int_{2^{-1}R_1}^{R_1}\int_{2^{-1}R_2}^{R_2} u(y_1,y_2)dy_1dy_2\Big)^{\frac{1}{q}}\times$$

(3.9)

$$\Big(\int_0^{2^{-1}R_1}\int_0^{2^{-1}R_2} v^{1-p'}(x_1,x_2)\exp[-p'(x_1^{-1}R_1+x_2^{-1}R_2)]dx_1dx_2\Big)^{\frac{1}{p'}} \le A.$$

Conversely $\mathbf{L} : L^p_v \to L^q_u$ *whenever for some* $A > 0$ *and for* $0 < \varepsilon \le 1$

$$\Big(\int_{2^{-1}R_1}^{R_1}\int_{2^{-1}R_2}^{R_2} u(y_1,y_2)dy_1dy_2\Big)^{\frac{1}{q}}\times$$

(3.10)

$$\Big(\int_0^{R_1}\int_0^{R_2} v^{1-p'}(x_1,x_2)\exp[-C_\varepsilon p'(x_1^{-1}R_1+x_2^{-1}R_2)]dx_1dx_2\Big)^{\frac{1}{p'}} \le A$$

for all R_1, $R_2 > 0$ *where* $C_\varepsilon = 4^{-1}(1-\varepsilon)$.

This is a two-dimensional version of a result proved in [Ra]. Observe that (3.10) is stronger than (3.9) in the sense that (3.10) $\Longrightarrow$ (3.9). But as it will be seen below, for many usual weights, condition (3.10) is not too far from (3.9).

Now we consider the boundedness problem for the operator $\mathbf{L}_1$.

Theorem 3.5. *Suppose that* $\mathbf{L}_1 : L^p_v \to L^q_u$. *Then there is a constant* $A > 0$ *such that for all* R_1, $R_2 > 0$

$$\Big(\int_{2^{-1}R_1}^{R_1}\int_{2^{-1}R_2}^{R_2} u(y_1,y_2)dy_1dy_2\Big)^{\frac{1}{q}}\times \tag{3.11}$$
$$\Big(\int_0^{2^{-1}R_1}\int_0^{2^{-1}R_2} v^{1-p'}(x_1,x_2)\exp[-p'(x_1^{-1}R_1)]dx_1dx_2\Big)^{\frac{1}{p'}} \leq A.$$

Conversely $\mathbf{L}_1 : L^p_v \to L^q_u$ *whenever for some* $A > 0$ *and for* $0 < \varepsilon \leq 1$

$$\Big(\int_{2^{-1}R_1}^{R_1}\int_{2^{-1}R_2}^{R_2} u(y_1,y_2)dy_1dy_2\Big)^{\frac{1}{q}}\times \tag{3.12}$$
$$\Big(\int_0^{R_1}\int_{2^{-(j_2+4)}R_2}^{2^{-j_2}R_2} v^{1-p'}(x_1,x_2)\exp[-C_\varepsilon p'(x_1^{-1}R_1)]dx_1dx_2\Big)^{\frac{1}{p'}} \leq A\tau_{j_2}$$

for all R_1, $R_2 > 0$ *and* $j_2 \in \{0,1,\dots,\infty\}$. *Here* $C_\varepsilon = 4^{-1}(1-\varepsilon)$ *and* $\sum_{j=0}^{\infty}\tau_j < \infty$.

Our proof does not allow to get the boundedness $\mathbf{L}_1 : L^p_v \to L^q_u$ just from the condition

$$\Big(\int_{2^{-1}R_1}^{R_1}\int_{2^{-1}R_2}^{R_2} u(y_1,y_2)dy_1dy_2\Big)^{\frac{1}{q}}\times$$
$$\Big(\int_0^{R_1}\int_0^{R_2} v^{1-p'}(x_1,x_2)\exp[-p'(x_1^{-1}R_1)]dx_1dx_2\Big)^{\frac{1}{p'}} \leq A.$$

The difficulty is that the exponential is just about the first variable x_1. And the sequence $(\tau_j)_j$ is introduced to make a sort of balance. It means that in (3.12) a more hypothesis for one of the weights is needed, compared to the case for condition (3.10). The gap between Theorems 3.4 and 3.5 can be also understood by the fact that the operator $\mathbf{L}_1$ is bigger than $\mathbf{L}$ and consequently its boundedness requires more conditions.

The situation for $\mathbf{L}_2 : L^p_v \to L^q_u$ is similar to the one for $\mathbf{L}_1 : L^p_v \to L^q_u$. For convenience in the sequel, we give the full statement.

Theorem 3.6. *Suppose that* $\mathbf{L}_2 : L^p_v \to L^q_u$. *Then there is a constant* $A > 0$ *such that for all* R_1, $R_2 > 0$

$$\Big(\int_{2^{-1}R_1}^{R_1}\int_{2^{-1}R_2}^{R_2} u(y_1,y_2)dy_1dy_2\Big)^{\frac{1}{q}}\times \tag{3.13}$$
$$\Big(\int_0^{2^{-1}R_1}\int_0^{2^{-1}R_2} v^{1-p'}(x_1,x_2)\exp[-p'(x_2^{-1}R_2)]dx_1dx_2\Big)^{\frac{1}{p'}} \le A.$$

Conversely $\mathbf{L}_2 : L^p_v \to L^q_u$ *whenever for some* $A > 0$ *and for* $0 < \varepsilon \le 1$

$$\Big(\int_{2^{-1}R_1}^{R_1}\int_{2^{-1}R_2}^{R_2} u(y_1,y_2)dy_1dy_2\Big)^{\frac{1}{q}}\times$$
$$\Big(\int_{2^{-(j_1+4)}R_1}^{2^{-j_1}R_1}\int_0^{R_2} v^{1-p'}(x_1,x_2)\exp[-C_\varepsilon p'(x_2^{-1}R_2)]dx_1dx_2\Big)^{\frac{1}{p'}} \tag{3.14}$$
$$\le A\tau_{j_1}$$

for all R_1, $R_2 > 0$ *and* $j_1 \in \{0, 1, \ldots, \infty\}$. *Here* C_ε *and* $(\tau_j)_j$ *are as in Theorem 3.5.*

§4. Proofs of Main Results

Proof of Theorem 2.1

Suppose that $\mathcal{L} : L^p_v \to L^q_u$. Then, by Theorem 3.1, the Hardy inequality $\mathbf{H} : L^p_v \to L^q_w$ does hold with $w(.,.)$ defined as in (3.1). And by Theorem 3.2, this last boundedness implies (3.4), (3.5) and (3.6) which are nothing else than conditions (2.1), (2.2) and (2.3) respectively.

Also by Theorem 3.1, it arises that $\mathbf{L} : L^{q'}_{w^{1-q'}} \to L^{p'}_{v^{1-p'}}$. Then, using the first part of Theorem 3.4, it is necessary that

$$\Big(\int_0^{2^{-1}R_1}\int_0^{2^{-1}R_2} w(y_1,y_2)\exp[-q(y_1^{-1}R_1 + y_2^{-1}R_2)]dy_1dy_2\Big)^{\frac{1}{q}}$$
$$\Big(\int_{2^{-1}R_1}^{R_1}\int_{2^{-1}R_2}^{R_2} v^{1-p'}(x_1,x_2)dx_1dx_2\Big)^{\frac{1}{p'}} \le A.$$

With the definition of $w(.,.)$, this last condition is the same as (2.4).

Again by Theorem 3.1, it can be assumed that $\mathbf{L}_1 : L^p_{v_1} \to L^q_{u_1}$ with $u_1(.,.)$ and $v_1(.,.)$ defined as in (3.2). Then, by the first part of Theorem 3.5,

$$\Big(\int_{2^{-1}R_1}^{R_1}\int_{2^{-1}R_2}^{R_2} u_1(y_1,y_2)dy_1dy_2\Big)^{\frac{1}{q}}$$
$$\Big(\int_0^{2^{-1}R_1}\int_0^{2^{-1}R_2} v_1^{1-p'}(z_1,z_2)\exp[-p'(z_1^{-1}R_1)]dz_1dz_2\Big)^{\frac{1}{p'}} \le A$$

which is the same as (2.5) after using the definition of $u_1(.,.)$ and $v_1(.,.)$.

Condition (2.6) can be obtained similarly as above by using the boundedness $\mathbf{L}_2 : L^p_{v_2} \to L^q_{u_2}$, the first part of Theorem 3.6 and the definition of $u_2(.,.)$ and $v_2(.,.)$ as in (3.3).

Proof of Theorem 2.2

To derive the boundedness $\mathcal{L} : L^p_v \to L^q_u$, the task is reduced to check the four boundednesses as defined in Theorem 3.1.

In view of Theorem 3.2 and the definition of $w(.,.)$ in (3.1), the Hardy inequality $\mathbf{H} : L^p_v \to L^q_w$ does hold under conditions (2.1), (2.2) and (2.3).

The boundedness $\mathbf{L} : L^{q'}_{w^{1-q'}} \to L^{p'}_{v^{1-p'}}$ will be deduced from the second part of Theorem 3.4 whenever

$$\Big(\int_0^{R_1}\int_0^{R_2} w(y_1,y_2)\exp[-C_\varepsilon q(y_1^{-1}R_1 + y_2^{-1}R_2)]dy_1dy_2\Big)^{\frac{1}{q}} \Big(\int_{2^{-1}R_1}^{R_1}\int_{2^{-1}R_2}^{R_2} v^{1-p'}(x_1,x_2)dx_1dx_2\Big)^{\frac{1}{p'}} \leq A$$

for all R_1, $R_2 > 0$. This last condition is the same as (2.7) because of the definition of $w(.,.)$.

Similarly to get $\mathbf{L}_1 : L^p_{v_1} \to L^q_{u_1}$, with $u_1(.,.)$ and $v_1(.,.)$ defined as in (3.2), by the second part of Theorem 3.5 it is sufficient that

$$\Big(\int_{2^{-1}R_1}^{R_1}\int_{2^{-1}R_2}^{R_2} u_1(y_1,y_2)dy_1dy_2\Big)^{\frac{1}{q}} \Big(\int_0^{R_1}\int_{2^{-(j_2+4)}R_2}^{2^{-j_2}R_2} v_1^{1-p'}(z_1,z_2)\exp[-C_\varepsilon p'(z_1^{-1}R_1)]dz_1dz_2\Big)^{\frac{1}{p'}} \leq A\tau_{j_2}$$

for all R_1, $R_2 > 0$. This condition is the same as (2.8) by the definition of $u_1(.,.)$ and $v_1(.,.)$.

The boundedness $\mathbf{L}_2 : L^p_{v_2} \to L^q_{u_2}$ can be proved as $\mathbf{L}_1 : L^p_{v_1} \to L^q_{u_1}$ by using the second part of Theorem 3.6 and the definition of $u_2(.,.)$ and $v_2(.,.)$ as in (3.3).

Proof of Lemma 2.3

Note that conditions (2.1), (2.2) and (2.3) are respectively the same as (3.4), (3.5) and (3.6) with the weight $w(.,.)$ defined as in (3.1). So the problem is reduced to get (3.4) $\Longrightarrow$ (3.5) and (3.4) $\Longrightarrow$ (3.6). These implications arise immediately from Lemma 3.3.

Proof of Lemma 2.4

One of the keys is to observe that if $w(.,.) \in D$ then for some constants c, $\sigma > 0$

$$\int_0^{2^{k_1}R_1}\int_0^{2^{k_2}R_2} w(y_1,y_2)dy_1dy_2 \leq c2^{(k_1+k_2)\sigma}\int_0^{R_1}\int_0^{R_2} w(y_1,y_2)dy_1dy_2$$

for all $R_1, R_2 > 0$ and $k_1, k_2 \in \{0, 1, \dots, \infty\}$. This inequality is just obtained by iterating condition (2.10).

To get (2.7) from the Muckenhoupt condition (2.1) it is assumed that $u(.,.) \in D$. And for shortness the constant $C_\varepsilon q$ is just denoted as C. The conclusion appears as follows

$$\begin{aligned}
&\int_{R_1^{-1}}^{\infty}\int_{R_2^{-1}}^{\infty} u(y_1,y_1)\exp[-C(y_1R_1+y_2R_2)]dy_1dy_2 \\
&\qquad\qquad \times\left(\int_{2^{-1}R_1}^{R_1}\int_{2^{-1}R_2}^{R_2} v^{1-p'}(z_1,z_2)dz_1dz_2\right)^{\frac{q}{p'}} \\
&= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\int_{2^{k_1}R_1^{-1}}^{2^{(k_1+1)}R_1^{-1}}\int_{2^{k_2}R_2^{-1}}^{2^{(k_2+1)}R_2^{-1}} u(y_1,y_2)\exp[-C(y_1R_1+y_2R_2)]dy_1dy_2 \\
&\qquad\qquad \times\left(\int_{2^{-1}R_1}^{R_1}\int_{2^{-1}R_2}^{R_2} v^{1-p'}(z_1,z_2)dz_1dz_2\right)^{\frac{q}{p'}} \\
&\le \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\exp[-C(2^{k_1}+2^{k_2})]\int_{2^{k_1}R_1^{-1}}^{2^{(k_1+1)}R_1^{-1}}\int_{2^{k_2}R_2^{-1}}^{2^{(k_2+1)}R_2^{-1}} u(y_1,y_2)dy_1dy_2 \\
&\qquad\qquad \times\left(\int_{2^{-1}R_1}^{R_1}\int_{2^{-1}R_2}^{R_2} v^{1-p'}(z_1,z_2)dz_1dz_2\right)^{\frac{q}{p'}} \\
&\le c_1\sum_{k_1=0}^{\infty}2^{\sigma k_1}\exp[-C2^{k_1}]\sum_{k_2=0}^{\infty}2^{\sigma k_2}\exp[-C2^{k_2}]\int_0^{R_1^{-1}}\int_0^{R_2^{-1}} u(y_1,y_2)dy_1dy_2 \\
&\qquad\qquad \times\left(\int_{2^{-1}R_1}^{R_1}\int_{2^{-1}R_2}^{R_2} v^{1-p'}(z_1,z_2)dz_1dz_2\right)^{\frac{q}{p'}} \\
&\le c_2\left(\int_0^{R_1^{-1}}\int_0^{R_2^{-1}} u(y_1,y_2)dy_1dy_2\right)\left(\int_0^{R_1}\int_0^{R_2} v^{1-p'}(z_1,z_2)dz_1dz_2\right)^{\frac{q}{p'}} \\
&\quad \le c_2A^q.
\end{aligned}$$

The implication (2.1) $\implies$ (2.7) remains also true whenever $v^{1-p'}(.,.) \in D$ since in this case, for the above chain of inequalities, we have to use

$$\begin{aligned}
&\int_0^{R_1}\int_0^{R_2} v^{1-p'}(y_1,y_2)dy_1dy_2 \\
&\qquad\qquad \le c2^{(k_1+k_2)\sigma}\int_0^{2^{-(k_1+1)}R_1}\int_0^{2^{-(k_2+1)}R_2} v^{1-p'}(y_1,y_2)dy_1dy_2
\end{aligned}$$

for all $R_1, R_2 > 0$ and $k_1, k_2 \in \{0, 1, \dots, \infty\}$.

Remind that for each $w(.,.) \in RD$ then for some constants c, $\rho > 0$

$$\int_0^{2^{-j_1}R_1}\int_0^{2^{-j_2}R_2} w(y_1,y_2)dy_1dy_2 \le c2^{-(j_1+j_2)\rho}\int_0^{R_1}\int_0^{R_2} w(y_1,y_2)dy_1dy_2$$

for all $R_1, R_2 > 0$ and $j_1, j_2 \in \{0, 1, \dots, \infty\}$.

To derive (2.8) from the Muckenhoupt condition (2.1) it is assumed that $u(.,.) \in D \cap RD$. For simplicity the constant $C_\varepsilon p'$ is just denoted by C. Now the implication is obtained as follows

$$\begin{aligned}
&\Big(\int_{2^{-1}R_1}^{R_1}\int_{R_2^{-1}}^{2R_2^{-1}} u(y_1,y_2)dy_1dy_2\Big)^{\frac{p'}{q}} \\
&\qquad\times\Big(\int_{R_1^{-1}}^{\infty}\int_{2^{-(4+j_2)}R_2}^{2^{-j_2}R_2} v^{1-p'}(z_1,z_2)\exp[-Cz_1R_1]dz_1dz_2\Big) \\
&\le c_3\sum_{k_1=0}^{\infty}\Big(\int_0^{R_1}\int_0^{2R_2^{-1}} u(y_1,y_2)dy_1dy_2\Big)^{\frac{p'}{q}} \\
&\qquad\times\Big(\int_{2^{k_1}R_1^{-1}}^{2^{(k_1+1)}R_1^{-1}}\int_{2^{-(4+j_2)}R_2}^{2^{-j_2}R_2} v^{1-p'}(z_1,z_2)\exp[-Cz_1R_1]dz_1dz_2\Big) \\
&\le c_3\sum_{k_1=0}^{\infty}\exp[-C2^{k_1}]\Big(\int_0^{R_1}\int_0^{2R_2^{-1}} u(y_1,y_2)dy_1dy_2\Big)^{\frac{p'}{q}} \\
&\qquad\times\Big(\int_{2^{k_1}R_1^{-1}}^{2^{(k_1+1)}R_1^{-1}}\int_{2^{-(4+j_2)}R_2}^{2^{-j_2}R_2} v^{1-p'}(z_1,z_2)dz_1dz_2\Big) \\
&\le c_4\sum_{k_1=0}^{\infty}2^{k_1\sigma\frac{p'}{q}}\exp[-C2^{k_1}]\Big(\int_0^{2^{-(k_1+1)}R_1}\int_0^{2R_2^{-1}} u(y_1,y_2)dy_1dy_2\Big)^{\frac{p'}{q}} \\
&\qquad\times\Big(\int_{2^{k_1}R_1^{-1}}^{2^{(k_1+1)}R_1^{-1}}\int_{2^{-(4+j_2)}R_2}^{2^{-j_2}R_2} v^{1-p'}(z_1,z_2)dz_1dz_2\Big) \\
&\le c_5 2^{-j_2\rho\frac{p'}{q}}\sum_{k_1=0}^{\infty}2^{k_1\sigma\frac{p'}{q}}\exp[-C2^{k_1}]\Big(\int_0^{2^{-(k_1+1)}R_1}\int_0^{2^{j_2}R_2^{-1}} u(y_1,y_2)dy_1dy_2\Big)^{\frac{p'}{q}} \\
&\qquad\times\Big(\int_0^{2^{(k_1+1)}R_1^{-1}}\int_0^{2^{-j_2}R_2} v^{1-p'}(z_1,z_2)dz_1dz_2\Big) \\
&\le c_5 A^{p'}2^{-j_2\rho\frac{p'}{q}}\sum_{k_1=0}^{\infty}2^{k_1\sigma\frac{p'}{q}}\exp[-C2^{k_1}] = c_6A^{p'}\tau_{j_2}^{p'}
\end{aligned}$$

where $\tau_{j_2} = 2^{-j_2\rho\frac{1}{q}}$, and such that $\sum_{j_2=0}^{\infty}\tau_{j_2} < \infty$. The implication (2.1) $\Longrightarrow$ (2.8) remains also true whenever $v^{1-p'}(.,.) \in D \cap RD$.

The proof of (2.1) $\Longrightarrow$ (2.9) follows by similar arguments.

Proof of Proposition 2.5

This result is just an immediate consequence of Theorems 2.1, 2.2 and Lemma 2.4. Indeed, by this last one and the hypothesis on $u(.,.)$ or $v^{1-p'}(.,.)$, conditions (2.7), (2.8) and (2.9) are implied by (2.1).

Proof of Proposition 2.6

This result is an immediate consequence of Proposition 2.5 and Lemma 2.3. Indeed, by this last one and the hypothesis on $u(.,.)$ or $v^{1-p'}(.,.)$, the Sawyer conditions (2.2) and (2.3) are implied by (2.1). And the conclusion follows by Proposition 2.5.

Proof of Corollary 2.7

The idea is to apply Proposition 2.6. So the real task remains to prove $v^{1-p'}(.,.) \in D \cap RD$ and the Muckenhoupt condition (2.1), since $v^{1-p'}(.,.)$ and $u(.,.)$ are of product type.

By the definition of $v_1(.)$ and since $\varphi_1(.) \in \triangle_2$ then

$$\begin{aligned}\int_0^{2R_1} v_1^{1-p'}(t)dt = &\int_0^{2R_1} \varphi_1(t)dt = 2\int_0^{R_1} \varphi_1(2s)ds \\ \leq &c_1 \int_0^{R_1} \varphi_1(s)ds = c_1 \int_0^{R_1} v_1^{1-p'}(t)dt.\end{aligned}$$

And using the fact that $\varphi_1(.)$ is an increasing function then

$$\begin{aligned}\int_0^{2^{-j_1}R_1} v_1^{1-p'}(t)dt = &\int_0^{2^{-j_1}R_1} \varphi_1(t)dt = 2^{-j_1}\int_0^{R_1} \varphi_1(2^{-j_1}s)ds \\ \leq &2^{-j_1} \int_0^{R_1} \varphi_1(s)ds = 2^{-j_1} \int_0^{R_1} v_1^{1-p'}(t)dt.\end{aligned}$$

Similar inequalities are also true for the weight $v_2(.)$. Consequently $v^{1-p'}(.,.) \in D \cap RD$ since $v(x_1,x_2) = v_1(x_1)v_2(x_2)$.

On the other hand

$$\begin{aligned}&\left(\int_0^{R_1^{-1}} u_1(y)dy\right)^{\frac{1}{q}} \left(\int_0^{R_1} v_1^{1-p'}(x)dx\right)^{\frac{1}{p'}} \\ &\qquad = \left(\int_{R_1}^{\infty} u_1(t^{-1})t^{-2}dt\right)^{\frac{1}{q}} \left(\int_0^{R_1} \varphi_1(s)ds\right)^{\frac{1}{p'}} \\ &\qquad \leq \left(\int_{R_1}^{\infty} t^{-(\frac{q}{p'}+1)}\varphi_1^{-\frac{q}{p'}}(t)dt\right)^{\frac{1}{q}} \Big(R_1\varphi_1(R_1)\Big)^{\frac{1}{p'}} \\ &\qquad \leq \left(\varphi_1^{-\frac{q}{p'}}(R_1)\int_{R_1}^{\infty} t^{-(\frac{q}{p'}+1)}dt\right)^{\frac{1}{q}} \Big(R_1\varphi_1(R_1)\Big)^{\frac{1}{p'}} \\ &\qquad \leq c_2\Big(R_1^{-\frac{q}{p'}}\varphi_1^{-\frac{q}{p'}}(R_1)\Big)^{\frac{1}{q}} \Big(R_1\varphi_1(R_1)\Big)^{\frac{1}{p'}} = c_2.\end{aligned}$$

An analogous chain of inequalities is also true for the weights $v_2(.)$ and $u_2(.)$. Consequently the Muckenhoupt condition (2.1) is satisfied since the weights are of product type.

Proof of Corollary 2.8

Again on the basis of Proposition 2.6, the task remains to prove: $v^{1-p'}(.,.) \in D \cap \widetilde{D} \cap RD$, $u(.,.) \in \widetilde{D}$ and the Muckenhoupt condition (2.1).

The fact that $v^{1-p'}(.,.) \in D$ follows from $\varphi(.) \in \triangle_2$ since

$$\int_0^{2R_1}\int_0^{2R_2} v^{1-p'}(t_1,t_2)dt_1dt_2 = \int_0^{2R_1}\int_0^{2R_2} \varphi(t_1+t_2)dt_1dt_2$$
$$= 4\int_0^{R_1}\int_0^{R_2} \varphi(2(s_1+s_2))ds_1ds_2 \le c_1\int_0^{R_1}\int_0^{R_2} \varphi(s_1+s_2)ds_1ds_2$$
$$= c_1\int_0^{R_1}\int_0^{R_2} v^{1-p'}(t_1,t_2)dt_1dt_2.$$

Moreover it is true that $v^{1-p'}(.,.) \in \widetilde{D}$ since

$$\int_0^{R_1}\int_0^{R_2} v^{1-p'}(t_1,t_2)dt_1dt_2 \le R_1R_2\varphi(R_1+R_2) \le c_2R_1R_2\varphi[2^{-1}(R_1+R_2)]$$
$$\le c_3\int_{2^{-1}R_1}^{R_1}\int_{2^{-1}R_2}^{R_2} \varphi((s_1+s_2))ds_1ds_2 = c_3\int_{2^{-1}R_1}^{R_1}\int_{2^{-1}R_2}^{R_2} v^{1-p'}(t_1,t_2)dt_1dt_2.$$

The condition $v^{1-p'}(.,.) \in RD$ is satisfied because for $0 < a_1, a_2 \le 1$

$$\int_0^{a_1R_1}\int_0^{a_2R_2} v^{1-p'}(t_1,t_2)dt_1dt_2 = \int_0^{a_1R_1}\int_0^{a_2R_2} \varphi(t_1+t_2)dt_1dt_2$$
$$= a_1a_2\int_0^{R_1}\int_0^{R_2} \varphi(a_1s_1+a_2s_2)ds_1ds_2$$
$$\le a_1a_2\int_0^{R_1}\int_0^{R_2} \varphi(s_1+s_2)ds_1ds_2$$
$$= a_1a_2\int_0^{R_1}\int_0^{R_2} v^{1-p'}(t_1,t_2)dt_1dt_2.$$

Necessarily $u(.,.) \in \widetilde{D}_\infty$ because

$$\int_0^{R_1}\int_0^{R_2} u(t_1,t_2)dt_1dt_2 = \int_{R_1^{-1}}^{\infty}\int_{R_2^{-1}}^{\infty} t_1^{-(\frac{q}{p'}+1)}t_2^{-(\frac{q}{p'}+1)}\varphi^{-\frac{q}{p'}}(t_1+t_2)dt_1dt_2$$
$$\le c_4\varphi^{-\frac{q}{p'}}[2(R_1^{-1}+R_2^{-1})]R_1^{\frac{q}{p'}}R_2^{\frac{q}{p'}}$$
$$\le c_5\int_{R_1^{-1}}^{2R_1^{-1}}\int_{R_2^{-1}}^{2R_2^{-1}} t_1^{-(\frac{q}{p'}+1)}t_2^{-(\frac{q}{p'}+1)}\varphi^{-\frac{q}{p'}}(t_1+t_2)dt_1dt_2$$

$$= c_5 \int_{2^{-1}R_1}^{R_1} \int_{2^{-1}R_2}^{R_2} u(t_1,t_2)dt_1dt_2.$$

And finally the Muckenhoupt condition (2.1) does hold since

$$\Big(\int_0^{R_1^{-1}} \int_0^{R_2^{-1}} u(y_1,y_2)dy_1dy_2\Big)^{\frac{1}{q}} \Big(\int_0^{R_1} \int_0^{R_2} v^{1-p'}(x_1,x_2)dx_1dx_2\Big)^{\frac{1}{p'}}$$
$$= \Big(\int_{R_1}^{\infty} \int_{R_2}^{\infty} t_1^{-(\frac{q}{p'}+1)} t_2^{-(\frac{q}{p'}+1)} \varphi^{-\frac{q}{p'}}(t_1+t_2)dt_1dt_2\Big)^{\frac{1}{q}} \times$$
$$\Big(\int_0^{R_1} \int_0^{R_2} \varphi(x_1+x_2)dx_1dx_2\Big)^{\frac{1}{p'}}$$
$$\le \Big(\varphi^{-\frac{q}{p'}}(R_1+R_2) \int_{R_1}^{\infty} \int_{R_2}^{\infty} t_1^{-(\frac{q}{p'}+1)} t_2^{-(\frac{q}{p'}+1)} dt_1dt_2\Big)^{\frac{1}{q}} \Big(R_1R_2\varphi(R_1+R_2)\Big)^{\frac{1}{p'}}$$
$$\le c_6\Big(R_1^{-\frac{q}{p'}} R_2^{-\frac{q}{p'}} \varphi^{-\frac{q}{p'}}(R_1+R_2)\Big)^{\frac{1}{q}} \Big(R_1R_2\varphi(R_1+R_2)\Big)^{\frac{1}{p'}} = c_6.$$

§5. Proofs of Basic Results

Proof of Theorem 3.1

First observe that

$$(\mathcal{L}f)(x_1,x_2) = (\mathcal{L}_1f)(x_1,x_2) + (\mathcal{L}_2f)(x_1,x_2) + (\mathcal{L}_3f)(x_1,x_2) + (\mathcal{L}_4f)(x_1,x_2)$$

where

$$(\mathcal{L}_1f)(x_1,x_2) = \int_0^{x_1^{-1}} \int_0^{x_2^{-1}} f(y_1,y_2)\exp[-(x_1y_1+x_2y_2)]dy_1dy_2$$

$$(\mathcal{L}_2f)(x_1,x_2) = \int_{x_1^{-1}}^{\infty} \int_0^{x_2^{-1}} f(y_1,y_2)\exp[-(x_1y_1+x_2y_2)]dy_1dy_2$$

$$(\mathcal{L}_3f)(x_1,x_2) = \int_0^{x_1^{-1}} \int_{x_2^{-1}}^{\infty} f(y_1,y_2)\exp[-(x_1y_1+x_2y_2)]dy_1dy_2$$

and

$$(\mathcal{L}_4f)(x_1,x_2) = \int_{x_1^{-1}}^{\infty} \int_{x_2^{-1}}^{\infty} f(y_1,y_2)\exp[-(x_1y_1+x_2y_2)]dy_1dy_2.$$

So the boundedness $\mathcal{L} : L^p_v \to L^q_u$ is equivalent both to $\mathcal{L}_i : L^p_v \to L^q_u$ for $i \in \{1,\dots,4\}$.

The boundedness $\mathcal{L}_1 : L^p_v \to L^q_u$

Since $-2 < -(x_1y_1 + x_2y_2) < 0$ for $0 < y_1 < x_1^{-1}$ and $0 < y_2 < x_2^{-1}$, then

$$(\mathcal{L}_1 f)(x_1, x_2) \approx \int_0^{x_1^{-1}} \int_0^{x_2^{-1}} f(y_1, y_2) dy_1 dy_2 = (Hf)(x_1^{-1}, x_2^{-1}).$$

Consequently the boundedness $\mathcal{L}_1 : L_v^p \to L_u^q$ is equivalent to

$$\mathbf{H} : L^p(v(x_1, x_2)dx_1dx_2) \to L^q(w(x_1, x_2)dx_1dx_2) \tag{5.1}$$

with $w(x_1, x_2) = x_1^{-2}x_2^{-2}u(x_1^{-1}, x_2^{-1})$. Remind that by $A(x_1, x_2) \approx B(x_1, x_2)$ we mean that for some fixed constants $c_1, c_2 > 0$ then $c_1A(x_1, x_2) \le B(x_1, x_2) \le c_2A(x_1, x_2)$ for all x_1, $x_2 > 0$.

The boundedness $\mathcal{L}_4 : L_v^p \to L_u^q$

Observe that

$$\int_0^\infty \int_0^\infty \Big[\int_{x_1^{-1}}^\infty \int_{x_2^{-1}}^\infty f(y_1, y_2) \exp[-(x_1y_1 + x_2y_2)] dy_1dy_2\Big]^q u(x_1, x_2)dx_1dx_2$$
$$= \int_0^\infty \int_0^\infty \Big[\int_{x_1}^\infty \int_{x_2}^\infty f(y_1, y_2) \exp[-(x_1^{-1}y_1 + x_2^{-1}y_2)] dy_1dy_2\Big]^q \times$$
$$x_1^{-2}x_1^{-2}u(x_1^{-1}, x_2^{-1})dx_1dx_2$$
$$= \int_0^\infty \int_0^\infty (L^*f)^q(x_1, x_2)w(x_1, x_2)dx_1dx_2$$

where $(\mathbf{L}^*f)(x_1, x_2) = \displaystyle\int_{x_1}^\infty \int_{x_2}^\infty f(y_1, y_2) \exp[-(x_1^{-1}y_1 + x_2^{-1}y_2)] dy_1dy_2$.
Consequently $\mathcal{L}_4 : L_v^p \to L_u^q$ is the same as $\mathbf{L}^* : L_v^p \to L_w^q$ and, by a duality argument, this last boundedness is equivalent to

$$\mathbf{L} : L^{q'}(w^{1-q'}(x_1, x_2)dx_1dx_2) \to L^{p'}(v^{1-p'}(x_1, x_2)dx_1dx_2). \tag{5.2}$$

The boundedness $\mathcal{L}_2 : L_v^p \to L_u^q$

Since

$$(\mathcal{L}_2 f)(x_1, x_2) \approx \int_{x_1^{-1}}^\infty \int_0^{x_2^{-1}} f(y_1, y_2) \exp[-x_1y_1] dy_1dy_2,$$

then

$$\int_0^\infty \int_0^\infty (\mathcal{L}_2 f)^q(x_1, x_2)u(x_1, x_2)dx_1dx_2 \tag{5.3}$$
$$\approx \int_0^\infty \int_0^\infty \Big[\int_{x_1^{-1}}^\infty \int_0^{x_2^{-1}} f(y_1, y_2) \exp[-x_1y_1] dy_1dy_2\Big]^q u(x_1, x_2)dx_1dx_2$$

$$= \int_0^\infty \int_0^\infty \Big[\int_{x_1}^\infty \int_0^{x_2} f(y_1, y_2) \exp[-x_1^{-1} y_1] dy_1 dy_2\Big]^q w(x_1, x_2) dx_1 dx_2$$

$$= \int_0^\infty \int_0^\infty \Big[\int_0^{x_1^{-1}} \int_0^{x_2} f(y_1^{-1}, y_2) y_1^{-2} \exp[-x_1^{-1} y_1^{-1}] dy_1 dy_2\Big]^q \times w(x_1, x_2) dx_1 dx_2$$

$$= \int_0^\infty \int_0^\infty \Big[\int_0^{x_1} \int_0^{x_2} f(y_1^{-1}, y_2) y_1^{-2} \exp[-x_1 y_1^{-1}] dy_1 dy_2\Big]^q \times x_1^{-2} w(x_1^{-1}, x_2) dx_1 dx_2$$

$$= \int_0^\infty \int_0^\infty (\mathbf{L}_1 g)^q (x_1, x_2) u_1(x_1, x_2) dx_1 dx_2.$$

with

$$g(x_1, x_2) = f(x_1^{-1}, x_2) x_1^{-2} \quad \text{and} \quad u_1(x_1, x_2) = x_1^{-2} w(x_1^{-1}, x_2) = x_2^{-2} u(x_1, x_2^{-1}).$$

On the other hand for

$$v_1(x_1, x_2) = x_1^{2p-2} v(x_1^{-1}, x_2)$$

then

$$\int_0^\infty \int_0^\infty g^p(y_1, y_2) v_1(y_1, y_2) dy_1 dy_2 = \int_0^\infty \int_0^\infty f^p(y_1^{-1}, y_2) v(y_1^{-1}, y_2) y_1^{-2} dy_1 dy_2$$

$$= \int_0^\infty \int_0^\infty f^p(x_1, x_2) v(x_1, x_2) dx_1 dx_2. \tag{5.4}$$

With (5.3) and (5.4), it appears that the boundedness $\mathcal{L}_2 : L_v^p \to L_u^q$ is equivalent to

$$\mathbf{L}_1 : L^p(v_1(x_1, x_2) dx_1 dx_2) \to L^q(u_1(x_1, x_2) dx_1 dx_2). \tag{5.5.}$$

The boundedness $\mathcal{L}_3 : L_v^p \to L_u^q$

Adapting line by line the above arguments for $\mathcal{L}_2 : L_v^p \to L_u^q$, it is also true that the boundedness $\mathcal{L}_3 : L_v^p \to L_u^q$ is equivalent to

$$\mathbf{L}_2 : L^p(v_2(x_1, x_2) dx_1 dx_2) \to L^q(u_2(x_1, x_2) dx_1 dx_2) \tag{5.6.}$$

where

$$u_2(x_1, x_2) = x_1^{-2} u(x_1^{-1}, x_2) \quad \text{and} \quad v_2(x_1, x_2) = x_2^{2p-2} v(x_1, x_2^{-1}).$$

Now Theorem 3.1 follows from (5.1), (5.2), (5.5) and (5.6).

Proof of Lemma 3.3

Since

$$\int_{R_1}^{\infty}\int_{R_2}^{\infty} w(x_1,x_2)dx_1dx_2 = \int_0^{R_1^{-1}}\int_0^{R_2^{-1}} w(y_1^{-1},y_2^{-1})y_1^{-2},y_2^{-2}dy_1dy_2$$

and

$$\begin{aligned}
&\int_{R_1}^{\infty}\int_{R_2}^{\infty}\Big[\int_{x_1}^{\infty}\int_{x_2}^{\infty} w(y_1,y_2)dy_1dy_2\Big]^{p'} v^{1-p'}(x_1,x_2)dx_1dx_2\\
&= \int_0^{R_1^{-1}}\int_0^{R_2^{-1}}\Big[\int_{x_1^{-1}}^{\infty}\int_{x_2^{-1}}^{\infty} w(y_1,y_2)dy_1dy_2\Big]^{p'} x_1^{-2}x_2^{-2}v^{1-p'}(x_1^{-1},x_2^{-1})dx_1dx_2\\
&= \int_0^{R_1^{-1}}\int_0^{R_2^{-1}}\Big[\int_0^{x_1}\int_0^{x_2} w(y_1^{-1},y_2^{-1})y_1^{-2}y_2^{-2}dy_1dy_2\Big]^{p'}\times\\
&\qquad\qquad v^{1-p'}(x_1^{-1},x_2^{-1})x_1^{-2}x_2^{-2}dx_1dx_2,
\end{aligned}$$

the problem is just reduced to prove the implication (3.4) $\Longrightarrow$ (3.5), where the hypothesis is about the pair $\Big(w(.,.);v^{1-p'}(.,.)\Big)$. The implication (3.4) $\Longrightarrow$ (3.6) is rather for the pair of weights $\Big(x_1^{-2}x_2^{-2}v^{1-p'}(x_1^{-1},x_2^{-1});x_1^{-2}x_2^{-2}w(x_1^{-1},x_2^{-1})\Big)$.

Now to derive the implication (3.4) $\Longrightarrow$ (3.5), the main point is to find a constant $c_1 > 0$ such that

$$\begin{aligned}
(5.7)\quad &\Big(\int_0^{R_1}\int_0^{R_2} v^{1-p'}(x_1,x_2)dx_1dx_2\Big)^q\\
&\le c_1\int_0^{R_1}\int_0^{R_2}\Big[\int_0^{y_1}\int_0^{y_2} v^{1-p'}(z_1,z_2)dz_1dz_2\Big]^{q-1} v^{1-p'}(y_1,y_2)dy_1dy_2
\end{aligned}$$

for all R_1, $R_2 > 0$. Indeed if this last inequality is true, then the Sawyer condition (3.5) follows the Muckenhoupt one (3.4) since

$$\begin{aligned}
&\int_0^{R_1}\int_0^{R_2}\Big[\int_0^{x_1}\int_0^{x_2} v^{1-p'}(y_1,y_2)dy_1dy_2\Big]^q w(x_1,x_2)dx_1dx_2\\
&\le c_1\int_0^{R_1}\int_0^{R_2}\quad\int_0^{x_1}\int_0^{x_2}\Big[\int_0^{y_1}\int_0^{y_2} v^{1-p'}(z_1,z_2)dz_1dz_2\Big]^{q-1}\\
&\qquad\qquad\times v^{1-p'}(y_1,y_2)dy_1dy_2 \quad w(x_1,x_2)dx_1dx_2\\
&= c_1\int_0^{R_1}\int_0^{R_2}\quad\Big(\int_{y_1}^{\infty}\int_{y_2}^{\infty} w(x_1,x_2)dx_1dx_2\Big)\\
&\qquad\qquad\times\Big(\int_0^{y_1}\int_0^{y_2} v^{1-p'}(z_1,z_2)dz_1dz_2\Big)^{q-1} v^{1-p'}(y_1,y_2)dy_1dy_2
\end{aligned}$$

$$\leq c_1 A^q \int_0^{R_1}\int_0^{R_2}\Big(\int_0^{y_1}\int_0^{y_2} v^{1-p'}(z_1,z_2)dz_1dz_2\Big)^{\frac{q}{p}-1} v^{1-p'}(y_1,y_2)dy_1dy_2$$
$$\leq c_1 A^q \Big(\int_0^{R_1}\int_0^{R_2} v^{1-p'}(y_1,y_2)dy_1dy_2\Big)^{\frac{q}{p}} \qquad \text{since } \frac{q}{p}-1\geq 0.$$

Now to end with Lemma 3.3, we have to prove the main point (5.7). For a weight $v(.,.)$ of a product type, i.e. $v(x_1,x_2)=v_1(x_1)v_2(x_2)$ this inequality appears by using identities (3.7) for the weights $v_1(.)$ and $v_2(.)$. For more general weights $v(.,.)$, inequality (5.7) follows from the assumption $v^{1-p'}(.,.)\in D\cap\widetilde{D}$. Indeed

$$\Big(\int_0^{R_1}\int_0^{R_2} v^{1-p'}(x_1,x_2)dx_1dx_2\Big)^q$$
$$=c_2\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} 2^{-q\sigma(k_1+k_2)}\Big(\int_0^{R_1}\int_0^{R_2} v^{1-p'}(x_1,x_2)dx_1dx_2\Big)^q$$
$$\leq c_3\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\Big(\int_0^{2^{-k_1}R_1}\int_0^{2^{-k_2}R_2} v^{1-p'}(x_1,x_2)dx_1dx_2\Big)^q$$
$$\leq c_4\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\Big(\int_0^{2^{-(k_1+1)}R_1}\int_0^{2^{-(k_2+1)}R_2} v^{1-p'}(z_1,z_2)dz_1dz_2\Big)^{q-1}$$
$$\times\int_{2^{-(k_1+1)}R_1}^{2^{-k_1}R_1}\int_{2^{-(k_2+1)}R_2}^{2^{-k_2}R_2} v^{1-p'}(x_1,x_2)dx_1dx_2$$
$$\leq c_4\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\int_{2^{-(k_1+1)}R_1}^{2^{-k_1}R_1}\int_{2^{-(k_2+1)}R_2}^{2^{-k_2}R_2}$$
$$\Big[\int_0^{x_1}\int_0^{x_2} v^{1-p'}(z_1,z_2)dz_1dz_2\Big]^{q-1} v^{1-p'}(x_1,x_2)dx_1dx_2$$
$$=c_4\int_0^{\infty}\int_0^{\infty}\Big(\int_0^{x_1}\int_0^{x_2} v^{1-p'}(z_1,z_2)dz_1dz_2\Big)^{q-1} v^{1-p'}(x_1,x_2)dx_1dx_2.$$

Proof of Theorem 3.4
The Necessary Part

To derive condition (3.9) from the boundedness $\mathbf{L}: L^p_v\to L^q_u$, by a duality argument, it can be assumed that for some constant $C>0$

$$\Big(\int_0^{\infty}\int_0^{\infty}\Big[\int_{x_1}^{\infty}\int_{x_2}^{\infty} f(y_1,y_2)u(y_1,y_2)\exp[-(x_1^{-1}y_1+x_2^{-1}y_2)]dy_1dy_2\Big]^{p'}$$
$$v^{1-p'}(x_1,x_2)dx_1dx_2\Big)^{\frac{1}{p'}}\leq C\Big(\int_0^{\infty} f^{q'}(z_1,z_2)u(z_1,z_2)dz_1dz_2\Big)^{\frac{1}{q'}}$$

for all $f(.,.) \geq 0$. Take R_1, $R_2 > 0$ and $f(.,.)$ a nonnegative function whose the support is $]2^{-1}R_1, R_1[\times]2^{-1}R_2, R_2[$. Since for $0 < x_1 < 2^{-1}R_1$ and $0 < x_2 < 2^{-1}R_2$

$$\begin{aligned}
\int_{x_1}^{\infty}\int_{x_2}^{\infty} & f(y_1,y_2)u(y_1,y_2)\exp[-(x_1^{-1}y_1 + x_2^{-1}y_2)]dy_1dy_2 \\
&= \int_{2^{-1}R_1}^{R_1}\int_{2^{-1}R_2}^{R_2} f(y_1,y_2)u(y_1,y_2)\exp[-(x_1^{-1}y_1 + x_2^{-1}y_2)]dy_1dy_2 \\
&\geq \Big(\int_{2^{-1}R_1}^{R_1}\int_{2^{-1}R_2}^{R_2} f(y_1,y_2)u(y_1,y_2)dy_1dy_2\Big)\exp[-(R_1x_1^{-1} + R_2x_2^{-1})]
\end{aligned}$$

then the above inequality yields

$$\begin{aligned}
\Big(\int_{2^{-1}R_1}^{R_1}\int_{2^{-1}R_2}^{R_2} f(y_1,y_2)u(y_1,y_2)dy_1dy_2\Big)^{p'}\Big(\int_0^{2^{-1}R_1}\int_0^{2^{-1}R_2} v^{1-p'}(x_1,x_2) \\
\exp[-p'(R_1x_1^{-1} + R_2x_2^{-1})]dx_1dx_2\Big) \\
\leq C^{p'}\Big(\int_{2^{-1}R_1}^{R_1}\int_{2^{-1}R_2}^{R_2} f^{q'}(z_1,z_2)u(z_1,z_2)dz_1dz_2\Big)^{\frac{p'}{q'}}.
\end{aligned}$$

Taking $f(.,.) = 1$ on its support $]2^{-1}R_1, R_1[\times]2^{-1}R_2, R_2[$ then condition (3.9) appears immediately.

The Sufficient Part

The main point to get $\mathbf{L} : L^p_v \to L^q_u$ is to break the operator as follows

(5.8)

$$\begin{aligned}
\int_0^{\infty}\int_0^{\infty} & (\mathbf{L}f)^q(x_1,x_2)u(x_1,x_2)dx_1dx_2 \\
&\leq \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\Big[\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\lambda_{j_1}\lambda_{j_2}\Big(\int_{2^{-(2+j_1+k_1)}}^{2^{-(j_1+k_1)}}\int_{2^{-(2+j_2+k_2)}}^{2^{-(j_2+k_2)}} f^p(x_1,x_2) \\
&\qquad v(x_1,x_2)dx_1dx_2\Big)^{\frac{1}{p}} \times \Theta_{k_1,k_2,j_1,j_2}\Big]^q
\end{aligned}$$

for all functions $f(.,.) \geq 0$. Here

$$\lambda_j = \exp[-\varepsilon 2^j], \qquad \text{with} \qquad 0 < \varepsilon \leq 1$$

and

$$\Theta_{k_1,k_2,j_1,j_2} = \Theta_{k_1,k_2,j_1,j_2}(p,q,v,u) = \Big(\int_{2^{-(1+k_1)}}^{2^{-k_1}}\int_{2^{-(1+k_2)}}^{2^{-k_2}} u(y_1,y_2)dy_1dy_2\Big)^{\frac{1}{q}}$$

$$\times \Big(\int_{2^{-(j_1+2+k_1)}}^{2^{-(j_1+k_1)}} \int_{2^{-(j_2+2+k_2)}}^{2^{-(j_2+k_2)}} v^{1-p'}(z_1, z_2)$$
$$\times \exp[-4^{-1}(1-\varepsilon)p'(2^{-k_1}z_1^{-1} + 2^{-k_2}z_2^{-1})]dz_1 dz_2\Big)^{\frac{1}{p'}}.$$

The proof of (5.8) is postponed below.

It can be noted that $\sum_{j=0}^{\infty} \lambda_j = c_0 < \infty$. By condition (3.10), with $R_1 = 2^{-k_1}$ and $R_2 = 2^{-k_2}$, it is clear that

$$\Theta_{k_1,k_2,j_1,j_2} \leq A.$$

Using these last facts and the cutout (5.8), the boundedness $\mathbf{L} : L_v^p \to L_u^q$ arises as follows

$$\int_0^\infty \int_0^\infty (\mathbf{L}f)^q(x_1, x_2)u(x_1, x_2)dx_1 dx_2$$
$$\leq A^q \times$$
$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \Big[\sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \lambda_{j_1}\lambda_{j_2} \times$$
$$\Big(\int_{2^{-(2+j_1+k_1)}}^{2^{-(j_1+k_1)}} \int_{2^{-(2+j_2+k_2)}}^{2^{-(j_2+k_2)}} f^p(x_1, x_2)v(x_1, x_2)dx_1 dx_2\Big)^{\frac{1}{p}}\Big]^q$$
$$\leq A^q c_1 \times$$
$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \Big[\sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \lambda_{j_1}\lambda_{j_2} \times$$
$$\int_{2^{-(2+j_1+k_1)}}^{2^{-(j_1+k_1)}} \int_{2^{-(2+j_2+k_2)}}^{2^{-(j_2+k_2)}} f^p(x_1, x_2)v(x_1, x_2)dx_1 dx_2\Big]^{\frac{q}{p}}$$

by using the Hölder inequality for the inside term and $\sum_{l=0}^{\infty} \lambda_l = c_0$

$$\leq A^q c_1 \times$$
$$\Big(\sum_{j_1=0}^{\infty} \lambda_{j_1} \sum_{j_2=0}^{\infty} \lambda_{j_2} \times$$
$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \int_{2^{-(2+j_1+k_1)}}^{2^{-(j_1+k_1)}} \int_{2^{-(2+j_2+k_2)}}^{2^{-(j_2+k_2)}} f^p(x_1, x_2)v(x_1, x_2)dx_1 dx_2\Big)^{\frac{q}{p}}$$

since $\dfrac{q}{p} \geq 1$

$$\leq A^q c_2 \times \left(\sum_{j_1=0}^{\infty} \lambda_{j_1} \sum_{j_2=0}^{\infty} \lambda_{j_2} \int_0^\infty \int_0^\infty f^p(x_1,x_2) v(x_1,x_2) dx_1 dx_2 \right)^{\frac{q}{p}}$$
$$= A^q c_3 \times \left(\int_0^\infty \int_0^\infty f^p(x_1,x_2) v(x_1,x_2) dx_1 dx_2 \right)^{\frac{q}{p}}.$$

Now to prove inequality (5.8), let us introduce

$$C = C(\varepsilon, p) = 4^{-1}(1-\varepsilon)p'.$$

So this inequality appears as follows

$$\int_0^\infty \int_0^\infty (\mathbf{L}f)^q(x_1,x_2) u(x_1,x_2) dx_1 dx_2$$
$$= \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \int_{2^{-(k_1+1)}}^{2^{-k_1}} \int_{2^{-(k_2+1)}}^{2^{-k_2}} \Bigg[\sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \int_{2^{-(j_1+1)}x_1}^{2^{-j_1}x_1} \int_{2^{-(j_2+1)}x_2}^{2^{-j_2}x_2} f(y_1,y_2)$$
$$\times \exp[-(x_1 y_1^{-1} + x_2 y_2^{-1})] dy_1 dy_2 \Bigg]^q u(x_1,x_2) dx_1 dx_2$$
$$\leq \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \int_{2^{-(k_1+1)}}^{2^{-k_1}} \int_{2^{-(k_2+1)}}^{2^{-k_2}} \Bigg[\sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \int_{2^{-(j_1+1)}x_1}^{2^{-j_1}x_1} \int_{2^{-(j_2+1)}x_2}^{2^{-j_2}x_2} f(y_1,y_2)$$
$$\times \exp[-(2^{j_1} + 2^{j_2})] dy_1 dy_2 \Bigg]^q u(x_1,x_2) dx_1 dx_2$$
$$\leq \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \Bigg[\sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \Big(\int_{2^{-(j_1+2+k_1)}}^{2^{-(j_1+k_1)}} \int_{2^{-(j_2+2+k_2)}}^{2^{-(j_2+k_2)}} f(y_1,y_2) dy_1 dy_2 \Big)$$
$$\times \Big(\int_{2^{-(k_1+1)}}^{2^{-k_1}} \int_{2^{-(k_2+1)}}^{2^{-k_2}} u(x_1,x_2) dx_1 dx_2 \Big)^{\frac{1}{q}} \exp[-(2^{j_1} + 2^{j_2})] \Bigg]^q$$
$$\leq \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty}$$
$$\Bigg[\sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \Big(\int_{2^{-(j_1+2+k_1)}}^{2^{-(j_1+k_1)}} \int_{2^{-(j_2+2+k_2)}}^{2^{-(j_2+k_2)}} f^p(y_1,y_2) v(y_1,y_2) dy_1 dy_2 \Big)^{\frac{1}{p}}$$
$$\times \Big(\int_{2^{-(j_1+2+k_1)}}^{2^{-(j_1+k_1)}} \int_{2^{-(j_2+2+k_2)}}^{2^{-(j_2+k_2)}} v^{1-p'}(z_1,z_2) dz_1 dz_2 \Big)^{\frac{1}{p'}}$$
$$\times \Big(\int_{2^{-(k_1+1)}}^{2^{-k_1}} \int_{2^{-(k_2+1)}}^{2^{-k_2}} u(x_1,x_2) dx_1 dx_2 \Big)^{\frac{1}{q}} \times \exp[-(2^{j_1} + 2^{j_2})] \Bigg]^q$$
$$\leq \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty}$$

$$\Big[\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\lambda_{j_1}\lambda_{j_2}\Big(\int_{2^{-(j_1+2+k_1)}}^{2^{-(j_1+k_1)}}\int_{2^{-(j_2+2+k_2)}}^{2^{-(j_2+k_2)}} f^p(y_1,y_2)v(y_1,y_2)dy_1dy_2\Big)^{\frac{1}{p}}$$

$$\times\Big(\int_{2^{-(j_1+2+k_1)}}^{2^{-(j_1+k_1)}}\int_{2^{-(j_2+2+k_2)}}^{2^{-(j_2+k_2)}} v^{1-p'}(z_1,z_2)\exp[-C(2^{-k_1}z_1^{-1}+2^{-k_2}z_2^{-1})]dz_1dz_2\Big)^{\frac{1}{p'}}$$

$$\times\Big(\int_{2^{-(k_1+1)}}^{2^{-k_1}}\int_{2^{-(k_2+1)}}^{2^{-k_2}} u(x_1,x_2)dx_1dx_2\Big)^{\frac{1}{q}}\Big]^q$$

$$=\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}$$

$$\Big[\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\lambda_{j_1}\lambda_{j_2}\Big(\int_{2^{-(j_1+2+k_1)}}^{2^{-(j_1+k_1)}}\int_{2^{-(j_2+2+k_2)}}^{2^{-(j_2+k_2)}} f^p(y_1,y_2)v(y_1,y_2)dy_1dy_2\Big)^{\frac{1}{p}}$$

$$\times\Theta_{k_1,k_2,j_1,j_2}\Big]^q.$$

Proof of Theorem 3.5

The Necessary Part

The proof is similar to the Necessary Part of Theorem 3.4, since the term $\exp[-p'(R_1x_1^{-1}+R_2x_2^{-1})]$ is just replaced by $\exp[-p'(R_1x_1^{-1})]$.

The Sufficient Part

As in the proof of Theorem 3.4, the main point to get $\mathbf{L}_1 : L_v^p \to L_u^q$ is the inequality

(5.9)
$$\int_0^{\infty}\int_0^{\infty}(\mathbf{L}_1 f)^q(x_1,x_2)u(x_1,x_2)dx_1dx_2$$
$$\leq\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\Big[\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\lambda_{j_1}\tau_{j_2}\Big(\int_{2^{-(2+j_1+k_1)}}^{2^{-(j_1+k_1)}}\int_{2^{-(2+j_2+k_2)}}^{2^{-(j_2+k_2)}} f^p(x_1,x_2)$$
$$v(x_1,x_2)dx_1dx_2\Big)^{\frac{1}{p}}\times\widetilde{\Theta}_{k_1,k_2,j_1,j_2}\Big]^q$$

for all functions $f(.,.)\geq 0$. Here

$$\lambda_j=\exp[-\varepsilon 2^j]\qquad\text{with}\qquad 0<\varepsilon\leq 1,$$

the sequence τ_j is defined as in condition (3.12)

and

$$\widetilde{\Theta}_{k_1,k_2,j_1,j_2}=\widetilde{\Theta}_{k_1,k_2,j_1,j_2}(p,q,v,u)=\Big(\int_{2^{-(1+k_1)}}^{2^{-k_1}}\int_{2^{-(1+k_2)}}^{2^{-k_2}} u(y_1,y_2)dy_1dy_2\Big)^{\frac{1}{q}}$$

$$\times\, \tau_{j_2}^{-1} \times \Big(\int_{2^{-(j_1+2+k_1)}}^{2^{-(j_1+k_1)}} \int_{2^{-(j_2+2+k_2)}}^{2^{-(j_2+k_2)}} v^{1-p'}(z_1, z_2)$$
$$\times \exp[-4^{-1}(1-\varepsilon)p'(2^{-k_1}z_1^{-1})]dx_1dx_2 \Big)^{\frac{1}{p'}}.$$

The proof of (5.9) can be easily seen by adapting the one of (5.8).

Note also that $\sum_{j=0}^{\infty} \lambda_j = c_0 < \infty$ and $\sum_{j=0}^{\infty} \tau_j = c_1 < \infty$. And by condition (3.12), with $R_1 = 2^{-k_1}$ and $R_2 = 2^{-k_2}$, then

$$\widetilde{\Theta}_{k_1,k_2,j_1,j_2} \leq A.$$

Using these last facts and the cutout (5.9), the boundedness $\mathbf{L}_1 : L^p_v \to L^q_u$ can be shown as it is done for $\mathbf{L} : L^p_v \to L^q_u$. The details are left to the readers.

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