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CONVERGENCE TO THE LIMIT SET OF LINEAR CELLULAR AUTOMATA

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Abstract. We investigate the topology of convergence to limit sets for one-dimensional linear cellular automata using $\mathbb{Z}_p$-valued upper semi-continuous functions, where p is a prime integer.

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§1. Introduction

Cellular automata are discrete dynamical systems with simple construction, and are used as models for physical or biological phenomena [9, 13]. In more detail, a cellular automaton consists of a finite-dimensional lattice of sites, each of which takes an element of a finite set $\mathbb{Z}_p = \{0, 1, \ldots, p-1\}$ of integers at each time step and the value of each site at any time step is determined as a function of the values of the neighbouring sites at the previous time step. Precisely speaking, the site values evolve by the rule

$$w(x, t+1) = f(w(x+k_1, t), \ldots, w(x+k_m, t)),$$

where $w(x,t)$ denotes the value of site $x \in \mathbb{Z}^d$ at time t and f is a $\mathbb{Z}_p$-valued function which determines the cellular automaton rule with $k_j \in \mathbb{Z}^d$ $(j = 1, \ldots, m)$, which are neighbouring sites of the origin. When f is a linear map, that is,

$$w(x, t+1) = \sum_{j=1}^{m} \alpha_j w(x+k_j, t) \quad (\text{mod } p), \tag{1.1}$$

with $\alpha_j \in \mathbb{Z}_p$ $(j = 1, \ldots, m)$, then the cellular automaton is said to be a linear cellular automaton (LCA). It is known that linear cellular automata may generate patterns which have connection with fractal sets, while non-linear cellular automata may sometimes cause chaotic phenomena [3, 4].

The rule defined by (1.1) determines the values of sites at time $t+1$ from the values of finite sites at time t. Moreover if we introduce the set $\mathcal{P}$ of all configurations $a\colon \mathbb{Z}^d \to \mathbb{Z}_p$ with compact support (i.e., $\#\{i \mid a(i) \neq 0\} < \infty$), then $w(\cdot\,, t) \in \mathcal{P}$ implies $w(\cdot\,, t+1) \in \mathcal{P}$. So we can consider an operator L in $\mathcal{P}$ derived from (1.1), which maps $a \in \mathcal{P}$ to

$$(La)(x) = \sum_{j=1}^{m} \alpha_j a(x + k_j) \qquad (\mathrm{mod}\ p). \tag{1.2}$$

Then L is linear (=a $\mathbb{Z}_p$-module endomorphism) and is translation invariant, i.e., $(La_\tau)(x) = (La)(x-\tau)$, where $a_\tau(x) := a(x-\tau)$. The configuration of cellular automata at time step t is represented by operating L on the initial configuration by t times.

In case of $p = 2$, S. J. Willson [10] investigated the so-called limit set of LCA. For $n \in \mathbb{N}$ and $a \in \mathcal{P}$, he considered the set

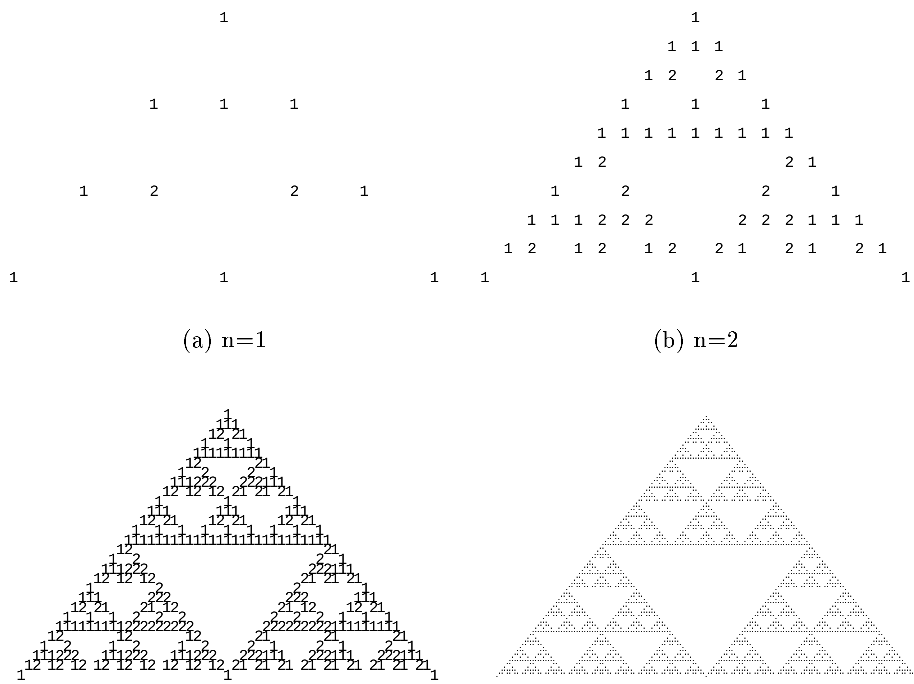
$$K(n, a) = \{(x, t) \in \mathbb{Z}^d \times \mathbb{Z}_+ \mid 0 \leq t \leq 2^n,\ (L^t a)(x) = 1\},$$

where L^t is the t-th power of L. The set $K(n, a)$ is determined by the values up to the 2^n-th time step of LCA with the initial configuration $a \in \mathcal{P}$. By contracting $K(n, a)$ with the rate $1/2^n$, he considered the set $\frac{K(n,a)}{2^n} = \{(\frac{x}{2^n}, \frac{t}{2^n}) \mid (x, t) \in K(n, a)\}$. Then the set $\frac{K(n,a)}{2^n}$ is a subset of $\mathbb{R}^d \times [0, 1]$. He showed that for any nonzero $a \in \mathcal{P}$ the following equality holds in the sense of Kuratowski limit [2, 7],

$$\liminf \frac{K(n, a)}{2^n} = \limsup \frac{K(n, a)}{2^n} = \liminf \frac{K(n, \delta)}{2^n} = \limsup \frac{K(n, \delta)}{2^n},$$

where $\delta \in \mathcal{P}$ satisfies $\delta(0) = 1$ and $\delta(x) = 0$ for $x \neq 0$. When $\liminf \frac{K(n,a)}{2^n}$ coincides with $\limsup \frac{K(n,a)}{2^n}$ in the sense of Kuratowski limit, both of them are called the limit set of $\frac{K(n,a)}{2^n}$. When the limit set does not depend on the initial configuration, the limit set is called the limit set of LCA. F.v. Haeseler, H.-O. Peitgen and G. Skordev [1] studied the existence of the limit set of LCA in Hausdorff metric by using matrix substitution systems and polynomials which induce LCA. As an extension of the result of Willson, S. Takahashi [8] investigated the case of an arbitrary prime number $p \geq 2$ and he considered the set

$$K(n, \delta) = \{(x, t) \in \mathbb{Z}^d \times \mathbb{Z}_+ \mid 0 \leq t \leq p^n - 1,\ (L^t \delta)(x) \neq 0\}$$



(a) n=1 (b) n=2

(c) n=3 (d) n=4

Figure 1: A sequence of $\psi_n(\delta)$ with $La(x) = a(x-2) + a(x-1) + a(x+1)$ (mod 3).

for $n \in \mathbb{N}$. By using the set $K(n,\delta)$, he also defined the limit set as a subset of $\mathbb{R}^d \times [0,1]$ in the same way as in the case of $p = 2$, and showed the existence of the limit set Y_δ of $\{\frac{K(n,\delta)}{p^n}\}$. Takahashi also investigated the limit set of "j-state" $K_j(n,\delta) = \{(x,t) \in \mathbb{Z}^d \times \mathbb{Z}_+ \mid 0 \le t \le p^n - 1,\ (L^t\delta)(x) = j\}$ for $j \in \{1,\ldots,p-1\}$.

We recall that the sets $\limsup \frac{K(n,\delta)}{p^n}$ and $\liminf \frac{K(n,\delta)}{p^n}$ in the sense of Kuratowski limit are the same with $\bigcap_{k=1}^\infty \overline{\bigcup_{n\ge k} \frac{K(n,\delta)}{p^n}}$ and $\overline{\bigcup_{k=1}^\infty \bigcap_{n\ge k} \frac{K(n,\delta)}{p^n}}$ respectively, since $\{\frac{K(n,\delta)}{p^n}\}$ is an increasing sequence. Hence the above results of [10] and [8] are concerned with the set $\bigcap_{k=1}^\infty \overline{\bigcup_{n\ge k} \frac{K(n,a)}{p^n}}$ and $\overline{\bigcup_{k=1}^\infty \bigcap_{n\ge k} \frac{K(n,a)}{p^n}}$. So it is natural to consider the limit set in the sense of set theory, which is defined if the set $\bigcap_{k=1}^\infty \bigcup_{n\ge k} \frac{K(n,a)}{p^n}$ coincides with the set $\bigcup_{k=1}^\infty \bigcap_{n\ge k} \frac{K(n,a)}{p^n}$ without taking its closure. Since the convergence of a sequence of sets $\{A_n\}$ in the sense of set theory corresponds to that of characteristic functions $\{1_{A_n}\}$ in the pointwise topology, we shall consider the convergence of the characteristic functions $\{\phi_n\}$ on $\mathbb{R}^d \times [0,1]$ corresponding to the set $\frac{K(n,\delta)}{p^n}$. Moreover, we shall consider a $\mathbb{Z}_p$-valued function $\psi_n(\delta)$ on $\mathbb{R}^d \times [0,1]$ which corresponds to the values of sites up to the p^n-th time step of LCA and expresses all the states simultaneously, though Takahashi considered the limit set of each state $K_j(n,\delta)$ separately. In Figure 1, we show an example of a sequence of $\psi_n(\delta)$ (defined in (2.3)), which suggests that the set $\{(x,t) \mid \psi_n(\delta)(x,t) \neq 0\}$ converges to a fractal set.

In this paper, we shall treat the linear cellular automata whose values are elements of $\mathbb{Z}_p$ for a prime number p, and investigate the topology of convergence of $\psi_n(\delta)$. In Section 2, we shall consider the convergence in the pointwise topology. We show that there exists $\lim \psi_n(\delta)$ in the pointwise topology (Theorems 2.2 and 2.5). We also show that there exists $\lim \phi_n$ in the pointwise topology which corresponds to the set $\bigcap_{k=1}^{\infty} \bigcup_{n \geq k} \frac{K(n,a)}{2^n} = \bigcup_{k=1}^{\infty} \bigcap_{n \geq k} \frac{K(n,a)}{p^n}$, where the closure is not taken (Theorem 2.6).

Though neither $\lim \phi_n$ nor $\lim \psi_n(\delta)$ corresponds to the characteristic function 1_{Y_δ}, the results of Willson [10] and Takahashi [8] indicate the existence of some topology in which $\psi_n(\delta)$ converges to a function f_δ corresponding to the limit set Y_δ of $\frac{K(n,\delta)}{p^n}$ in the sense of Kuratowski limit. Since f_δ and $\psi_n(\delta)$ are $\mathbb{Z}_p$-valued upper semi-continuous functions on $\mathbb{R}^d \times [0,1]$, we introduce the space USC of $\mathbb{Z}_p$-valued upper semi-continuous functions on $\mathbb{R}^d \times [0,1]$, where the order in $\mathbb{Z}_p$ is considered as a subset of $\mathbb{R}$ and investigate the topology in which $\psi_n(\delta)$ converges to f_δ. Section 3 is devoted to the study of two metrics d_f, D_f in USC. In Section 4, we investigate the convergence of $\{\psi_n(\delta)\}$ in these two metrics in the space USC of $\mathbb{Z}_p$-valued upper semi-continuous functions on $\mathbb{R} \times [0,1]$. We show that $\{\psi_n(\delta)\}$ is a Cauchy sequence in the metric d_f and $\psi_n(\delta)$ converges to the function f_δ in the metric D_f (Theorem 4.1). In Section 6, we show that the similar results hold for any initial configuration $a \in \mathcal{P}$ (Theorem 6.4) and that the limit function in the metric D_f is the same as f_δ if $a(0) \neq 0$ and also show that f_a is the upper envelope of g_a (Theorem 6.4).

When p is a prime number, we can show that the limit function in the metric D_f takes only two values by using Lemma 6.1 in [8]. However, when p is not a prime number, it occurs that the limit function in the metric D_f takes more than two values and the results in this case will be reported in [6].

§2. Convergence in the pointwise topology

Let p be a prime number and let $\mathcal{P}$ be the set of all configurations $a\colon \mathbb{Z}^d \to \mathbb{Z}_p$ with compact support. We define $\delta \in \mathcal{P}$ as

$$\delta(x) = \begin{cases} 1 & (x = 0), \\ 0 & (x \neq 0). \end{cases}$$

Let $L\colon \mathcal{P} \to \mathcal{P}$ be a linear operator derived from the cellular automata rule (1.1) as follows:

$$(La)(x) = \sum_{j \in G} \alpha_j a(x + k_j) \qquad \text{for } a \in \mathcal{P}, \tag{2.1}$$

where G is a finite subset of $\mathbb{Z}$ with $\sharp G \geq 2$, $k_j \in \mathbb{Z}^d$ $(j \in G)$ is a neighbouring site of origin, $\alpha_k \in \mathbb{Z}_p \backslash \{0\}$ and *the summation* $\sum$ *is taken as the summation with mod p throughout this paper.*

Let

$$X_n = \{(\frac{x}{p^n}, \frac{t}{p^n}) \in \mathbb{R}^d \times [0,1] \mid x \in \mathbb{Z}^d, t \in \mathbb{Z}_+, 0 \leq t \leq p^n\}$$

for $n \in \mathbb{N}$. Then $\{X_n\}$ is an increasing sequence, that is,

$$X_1 \subset X_2 \subset \cdots \subset X_n \subset X_{n+1} \subset \cdots .$$

For $j \in \mathbb{N}$, put

$$G_j = \{\ell \in \mathbb{Z}^d \mid (L^j \delta)(\ell) \neq 0\}. \tag{2.2}$$

Define a map ψ_n from $\mathcal{P}$ to the function space on $\mathbb{R}^d \times [0,1]$ for $a \in \mathcal{P}$ and $n \in \mathbb{N}$ by

$$(\psi_n(a))(\frac{x}{p^n}, \frac{t}{p^n}) = \begin{cases} (L^t a)(x) & \text{if } (\frac{x}{p^n}, \frac{t}{p^n}) \in X_n, \\ 0 & \text{if } (\frac{x}{p^n}, \frac{t}{p^n}) \in (\mathbb{R}^d \times [0,1]) \setminus X_n. \end{cases} \tag{2.3}$$

We shall quote the following useful lemma in order to consider the convergence of $\psi_n(\delta)$.

Lemma 2.1 ([8],Theorem 3.1). *Let L be a linear operator derived from the cellular automata rule* (1.1) *with p prime. Then for $j, n \in \mathbb{N}$, we have*

$$(L^{jp^n} \delta)(x) = \begin{cases} (L^j \delta)(y) & \text{if there exists } y \text{ such that } p^n y = x, \\ 0 & \text{otherwise.} \end{cases}$$

By using this lemma, we have

Theorem 2.2. *$\psi_n(\delta)$ converges to a function on $\mathbb{R}^d \times [0,1]$ in the pointwise topology.*

Proof. By Lemma 2.1, we have $(\psi_n(\delta))(\frac{x}{p^n}, \frac{t}{p^n}) = (\psi_{n+1}(\delta))(\frac{xp}{p^{n+1}}, \frac{tp}{p^{n+1}})$ for $(\frac{x}{p^n}, \frac{t}{p^n}) \in X_n$. So $\{\psi_n\}$ is a bounded increasing sequence and there exists a limit function in the pointwise topology. □

The above theorem concerns $\delta \in \mathcal{P}$. Now for $a \in \mathcal{P}$ with $a(0) \neq 0$, we shall consider the convergence of $\psi_n(a)$.

For a configuration $a \in \mathcal{P}$, put

$$l(a) = \max\{|i-j|+1 \mid a(i) \neq 0 \text{ and } a(j) \neq 0\},$$
$$n_a = \min\{n \in \mathbb{N} \mid p^n > l(a)\}.$$

The following lemma shows the relation between $L^{jp^n+i}a$ and $L^j\delta$.

Lemma 2.3. *For $a \in \mathcal{P}$ and $j, n, i \in \mathbb{N}$, we have*

$$(L^{jp^n+i}a)(x) = \sum_{\ell \in G_j} (L^j\delta)(\ell)(L^i a)(x - \ell p^n). \tag{2.4}$$

Proof. By the linearity of L, we have

$$(L^{jp^n+i}a)(x) = \sum_{i' \in \mathbb{Z}^d} (L^i a)(i')(L^{jp^n}\delta)(x - i').$$

For $x \in \mathbb{Z}^d$, put $G' = \{\ell \in \mathbb{Z}^d \mid \ell = \frac{x-i'}{p^n}$ with $n \in \mathbb{N}$ and $i' \in \mathbb{Z}^d\}$. Then G_j defined in (2.2) is included in G' and $(L^j\delta)(\ell) = 0$ holds for $\ell \in G' \setminus G_j$ by the definition of G_j. Therefore by using Lemma 2.1, we have $(L^{jp^n+i}a)(x) = \sum_{\ell \in G'}(L^i a)(x - \ell p^n)(L^j\delta)(\ell) = \sum_{\ell \in G_j}(L^i a)(x - \ell p^n)(L^j\delta)(\ell)$. □

By referring to the above lemma, we shall define an operator T as follows.

Define a map $S_{\ell,j} \colon \mathbb{R}^d \times [0,1] \to \mathbb{R}^d \times [\frac{j}{p}, \frac{j+1}{p}]$ by

$$S_{\ell,j}(x,t) = (\frac{x}{p}, \frac{t}{p}) + (\frac{\ell}{p}, \frac{j}{p}) \tag{2.5}$$

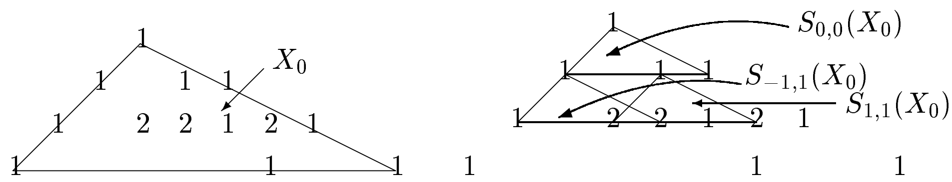
as shown in Figure 2, where the positive direction of t (virtical) is downward. For a function g on $\mathbb{R}^d \times [0,1]$, by using maps $S_{\ell,j}$ define a function Tg on $\mathbb{R}^d \times [0,1]$ by

$$Tg(y,q) = \sum_{\ell \in G_j} (L^j\delta)(\ell) g(S_{\ell,j}^{-1}(y,q)) \tag{2.6}$$

for $\frac{j}{p} < q \leq \frac{j+1}{p}$ with $0 \leq j \leq p-1$ and

$$Tg(y,0) = g(py,0).$$

Then we have



(a) n=1 (b) n=2

Figure 2: An exapmle of maps $S_{\ell,j}$ with $La(x) = a(x-2)+a(x-1)+a(x+1) \pmod 3$.

Proposition 2.4. *The following relation holds for $a \in \mathcal{P}$ and $n \in \mathbb{N}$:*

$$T^n(\psi_1(a)) = T(\psi_n(a)) = \psi_{n+1}(a).$$

Proof. For $(\frac{x}{p^{n+1}}, \frac{jp^n+i}{p^{n+1}}) \in X_{n+1}$ with $j \in \mathbb{N}$, $i \in \{0, \dots, p^n-1\}$, by using (2.3) and (2.4) we have

$$(\psi_{n+1}(a))(\frac{x}{p^{n+1}}, \frac{jp^n+i}{p^{n+1}}) = (L^{jp^n+i}a)(x) = \sum_{\ell \in G_j} (L^j\delta)(\ell)(L^i a)(x-\ell p^n).$$

By the relation $(\psi_n a)(S_{\ell,j}^{-1}(\frac{x}{p^{n+1}}, \frac{jp^n+i}{p^{n+1}})) = (\psi_n a)(\frac{x-\ell p^n}{p^n}, \frac{i}{p^n}) = (L^i a)(x-\ell p^n)$, we have

$$\begin{aligned}(T(\psi_n(a)))(\frac{x}{p^{n+1}}, \frac{jp^n+i}{p^{n+1}}) &= \sum_{\ell \in G_j} (L^j\delta)(\ell)(\psi_n(a))(S_{\ell,j}^{-1}(\frac{x}{p^{n+1}}, \frac{jp^n+i}{p^{n+1}})) \\ &= \sum_{\ell \in G_j} (L^j\delta)(\ell)(L^i a)(x-\ell p^n).\end{aligned}$$

Since $(T(\psi_n(a)))(y,q) = (\psi_{n+1}(a))(y,q) = 0$ for $(y,q) \in \mathbb{R}^d \times [0,1] \backslash X_{n+1}$, we have $T(\psi_n(a)) = \psi_{n+1}(a)$, from which we also have $T^n(\psi_1(a)) = \psi_{n+1}(a)$. $\square$

By using the above lemmas, we prove the following theorem.

Theorem 2.5. *For $a \in \mathcal{P}$ with $a(0) \neq 0$, we have the following assertions:*

(1) *$\psi_n(a)$ converges to a function on $\mathbb{R}^d \times [0,1]$ in the pointwise topology.*

(2) *The limit function g_a of the sequence $\{\psi_n(a)\}$ in the pointwise topology is T-invariant, that is, $Tg_a = g_a$.*

(3) *As for the limit functions g_δ and g_a of $\{\psi_n(\delta)\}$ and $\{\psi_n(a)\}$ respectively, we have $a(0)g_\delta = g_a$.*

Proof. (1) For $(\frac{xp^{n_a}}{p^n}, \frac{tp^{n_a}}{p^n}) \in X_{n-n_a}$, with $n > n_a$ and $m > n$, we have

$$(2.7)\qquad \begin{aligned} (\psi_m(a))(\frac{xp^{n_a}}{p^n}, \frac{tp^{n_a}}{p^n}) &= (\psi_m(a))(\frac{xp^{n_a+m-n}}{p^m}, \frac{tp^{n_a+m-n}}{p^m}) \\ &= (L^{tp^{n_a+m-n}}a)(p^{n_a+m-n}x) \\ &= \sum_{\ell \in G_t} (L^t\delta)(\ell)a(p^{n_a+m-n}(x-\ell)) \\ &= a(0)(L^t\delta)(x), \end{aligned}$$

by Lemma 2.3. So $\lim_{m\to\infty}(\psi_m(a))(\frac{xp^{n_a}}{p^n}, \frac{tp^{n_a}}{p^n}) = a(0)(L^t\delta)(x)$.

For $(y,q) \in (\mathbb{R}^d \times [0,1]) \setminus \bigcup_{k=n_a+1}^{\infty} X_k$, we have

$$(2.8)\qquad (\psi_n(a))(y,q) = 0 \quad \text{for any } n \in \mathbb{N}$$

by the definition of ψ_n and there exists $\lim_{n\to\infty}(\psi_n(a))(y,q)$. So the sequence $\{\psi_n(a)\}$ converges to a function on $\mathbb{R}^d \times [0,1]$ in the pointwise topology.

(2) By using the relations $\lim_{n\to\infty}(\psi_n(a))(y,q) = g_a(y,q)$ for each $(y,q) \in \mathbb{R}^d \times [0,1]$ and $(T\psi_n)(a)(y,q) = (\psi_{n+1}(a))(y,q)$, we have $Tg_a = g_a$.

(3) By equations (2.7), (2.8) and Theorem 2.2, we get the conclusion. □

Now we have investigated the convergence of $\psi_n(a)$ in the pointwise topology. Since Takahashi investigated the limit set of the set $K(n,\delta) = \{(x,t) \in \mathbb{Z}^d \times \mathbb{Z}_+ \mid 0 \le t \le 2^n,\ (\psi_n(\delta))(x,t) \neq 0\}$ in the sense of Kuratowski limit, we shall consider the characteristic function corresponding to the set $\frac{K(n,\delta)}{p^n} = \{(\frac{x}{p^n}, \frac{t}{p^n}) \mid (x,t) \in K(n,\delta)\}$ and the limit function in the pointwise topology. As shown in the following theorem, the limit function in the pointwise topology is the characteristic function corresponding to the limit set in the sense of set theory, where $\bigcap_{k=1}^{\infty} \bigcup_{n\ge k} \frac{K(n,a)}{p^n}$ and $\bigcup_{k=1}^{\infty} \bigcap_{n\ge k} \frac{K(n,a)}{p^n}$ are the same.

Theorem 2.6. *For $a \in \mathcal{P}$ with $a(0) \neq 0$ and $n \in \mathbb{N}$, let*

$$(\phi_n(a))(y,q) = \begin{cases} 1 & \text{if } (\psi_n(a))(y,q) \neq 0, \\ 0 & \text{if } (\psi_n(a))(y,q) = 0. \end{cases}$$

Then

(1) *$(\phi_n(a))(y,q)$ converges to a function on $\mathbb{R}^d \times [0,1]$ in the pointwise topology.*

(2) *The limit function ϕ_a of the sequence $\{\phi_n(a)\}$ in the pointwise topology is the characteristic function corresponding to the limit set in the sense of the set theory, that is, $\bigcap_{k=1}^{\infty} \bigcup_{n\ge k} \frac{K(n,a)}{p^n}$, which is the same as the set $\bigcup_{k=1}^{\infty} \bigcap_{n\ge k} \frac{K(n,a)}{p^n}$.*

(3) *As for the limit functions* ϕ_δ *and* ϕ_a *of* $\{\phi_n(\delta)\}$ *and* $\{\phi_n(a)\}$ *respectively, we have* $\phi_\delta = \phi_a$.

Proof. The proof is done in a similar way to that of Theorem 2.5. □

As shown in the above theorems, the limit function of $\{\psi_n(\delta)\}$ in the pointwise topology does not correspond to the characteristic function of the limit set in the sense of Kuratowski limit. However, the results of Willson [10] and Takahashi [8] indicate the existence of some topology in which $\psi_n(\delta)$ converges to a function f_δ corresponding to the limit set Y_δ of $\frac{K(n,\delta)}{p^n}$ in the sense of Kuratowski limit. So we shall investigate the topological space in which $\psi_n(\delta)$ converges to a function related to 1_{Y_δ}. Since $\psi_n(a)$ and 1_{Y_δ} are upper semi-continuous functions on $\mathbb{R}^d \times [0,1]$, we introduce the space *USC* of $\mathbb{Z}_p$-valued upper semi-continuous functions on $\mathbb{R}^d \times [0,1]$ and investigate the topology in which $\psi_n(\delta)$ converges to the function related to 1_{Y_δ}.

§3. Metrics in the space *USC*

In this section, we shall introduce two metrics d_f, D_f in the space of $\mathbb{Z}_p$-valued upper semi-continuous functions on a compact subset of $\mathbb{R}^d \times [0,1]$, which have some relation to the convergence of $\{\psi_n(\delta)\}$ to 1_{Y_δ}. Let *USC* be the space of $\mathbb{Z}_p$-valued upper semi-continuous functions on $\mathbb{R}^d \times [0,1]$, where $\mathbb{Z}_p$-valued upper semi-continuous functions mean upper semi-continuous functions embedded in $\mathbb{R}$-valued function spaces. For functions $f,\ g \in USC$, the order $f \geq g$ is defined by $f(y,q) \geq g(y,q)$ for any $(y,q) \in \mathbb{R}^d \times [0,1]$ by considering $\mathbb{Z}_p$ as a subset of $\mathbb{R}$. For functions $\{f_\lambda\}_{\lambda \in \Lambda} \subset USC$ having an upper bound, let

$$g_1(y,q) = \inf\{g(y,q) \mid g \in USC, g \geq f_\lambda \text{ for any } \lambda \in \Lambda\}$$

and

$$g_2(y,q) = \inf\{f_\lambda(y,q) \mid \text{ for any } \lambda \in \Lambda\}.$$

Then g_1 and g_2 belong to *USC* and g_1 is the least upper bound function $\bigvee f_\lambda$ and g_2 is the greatest lower bound function $\bigwedge f_\lambda$ in *USC*. So the space *USC* is an order complete lattice.

Let K be a compact subset of $\mathbb{R}^d \times [0,1]$ and (y_0, q_0) be a point of $(\mathbb{R}^d \times [0,1]) \setminus K$. Let

$$USC|_K = \{g \in USC \mid \text{ support of } g \subset K\}.$$

By using the Hausdorff distance $D(A,B)$ of non-empty compact sets A and B in $\mathbb{R}^d \times [0,1]$, we shall define the pseudodistance $D_0(A,B)$ of A and B in $\mathbb{R}^d \times [0,1]$ by

$$D_0(A,B) = D(A \cup \{(y_0,q_0)\}, B \cup \{(y_0,q_0)\})$$

and metrics d_f, D_f in $USC|_K$ as follows:

$$d_f(g_1, g_2) = \max_{1 \le j \le p-1} D_0(\overline{g_1^{-1}(j)}, \overline{g_2^{-1}(j)}),$$

$$D_f(g_1, g_2) = \max_{1 \le s \le p-1} D_0(g_1^{-1}[s+], g_2^{-1}[s+]),$$

for $g_1, g_2 \in USC|_K$, where $g^{-1}[s+] = \{(x,t) \mid g(x,t) \ge s\}$ and $\overline{g_1^{-1}(j)}$ is the closure of the set $g_1^{-1}(j) = \{(x,t) \mid g(x,t) = j\}$. It is easy to see that d_f and D_f satisfy the axioms of metric in $USC|_K$.

As for the relation between distance D and pseudodistance D_0, we can easily show the following

Proposition 3.1. *For compact sets A and B in $\mathbb{R}^d \times [0,1]$, we have the following*:

(1) *If both A and B are non-empty subsets of K, then*

$$D_0(A, B) = D(A, B).$$

(2) *If both A and B are empty sets, then*

$$D_0(A, B) = 0.$$

Concerning the Hausdorff distance, we shall recall the following lemmas.

Lemma 3.2. *Let $A_1, A_2, \dots$ and $B_1, B_2, \dots$ be non-empty compact subsets of $\mathbb{R}^d \times [0,1]$ included in a fixed compact set. Then the following relation holds*:

$$D(\overline{\cup_j A_j}, \overline{\cup_j B_j}) \le \max_j D(A_j, B_j)$$

Lemma 3.3. *Let $\{A_n\}_n$ be a monotone decreasing sequence of non-empty compact subsets of $\mathbb{R}^d \times [0,1]$, that is, $A_n \supset A_{n+1}$ and satisfy*

$$D(A_n, A_m) \to 0 \text{ as } n, m \to \infty.$$

Then the following relation holds:

$$D(A_n, \textstyle\bigcap_{m \ge 1} A_m) \to 0 \text{ as } n \to \infty.$$

By using above lemmas, we have

Proposition 3.4. *For $f_n \in USC|_K$, suppose $d_f(f_n, f_m) \to 0$ as $n, m \to \infty$. Then for any $n \in \mathbb{N}$, there exists the least upper bound $g_n = \bigvee_{k \ge n} f_k$ in $USC|_K$ and we have the following assertions*:

(1) *$D_f(g_n, g_m) \to 0$ as $n, m \to \infty$,*

(2) $D_f(f_n, g_n) \to 0$ *as* $n \to \infty$.

Proof. Since the support f_n is contained in a compact set K and f_n is bounded, there exists the least upper bound $\bigvee_{k\geq n} f_k$. By assumption,

$$D_0(\overline{f_n^{-1}(j)}, \overline{f_m^{-1}(j)}) \to 0 \quad \text{for all } j \in \{1, 2, \cdots, p-1\}$$

as $n \to \infty$. So for any $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for $n, m > n_0$,

$$D_0(\overline{f_n^{-1}(j)}, \overline{f_m^{-1}(j)}) < \varepsilon \qquad \text{for all } j \in \{1, 2, \cdots, p-1\}.$$

Therefore by Lemma 3.2 and $f_n^{-1}[s+] = \bigcup_{s\leq j\leq p-1} \overline{f_n^{-1}(j)}$,

$$D_0(f_n^{-1}[s+], f_m^{-1}[s+]) \leq \max_{s\leq j\leq p-1} D_0(\overline{f_n^{-1}(j)}, \overline{f_m^{-1}(j)}) < \varepsilon$$

holds for $n, m > n_0$. By using the relation $g_n^{-1}[s+] = \overline{\bigcup_{k\geq n} f_k^{-1}[s+]}$ and Lemma 3.2, we get

$$D_0(g_n^{-1}[s+], g_m^{-1}[s+]) < \varepsilon \qquad \text{for } n, m > n_0,$$

which implies (1). Put

$$\begin{aligned} D_{0,r}(A, B) &= \sup\{d(A \cup \{(y_0, q_0)\}, y) \mid y \in B \cup \{(y_0, q_0)\}\}, \\ D_{0,\ell}(A, B) &= \sup\{d(x, B \cup \{(y_0, q_0)\}) \mid x \in A \cup \{(y_0, q_0)\}\}. \end{aligned}$$

Then $D_0(A, B) = \max\{D_{0,\ell}(A, B), D_{0,r}(A, B)\}$ and

$$D_{0,\ell}(f_n^{-1}[s+], g_n^{-1}[s+]) = 0 \ \text{ for all } n \in \mathbb{N}.$$

By $D_{0,r}(f_n^{-1}[s+], g_n^{-1}[s+]) \leq \sup_{m\geq n} D_0(f_n^{-1}[s+], f_m^{-1}[s+])$,

$$D_{0,r}(f_n^{-1}[s+], g_n^{-1}[s+]) < \varepsilon \qquad \text{for } n > n_0$$

holds. So $D_0(f_n^{-1}[s+], g_n^{-1}[s+]) < \varepsilon$, which implies (2). □

Theorem 3.5. *For* $f_n \in USC|_K$, *suppose* $d_f(f_n, f_m) \to 0$ *as* $n, m \to \infty$. *Let* $g = \bigwedge_{k=1}^{\infty} \bigvee_{n\geq k} f_n$. *Then we have*

$$D_f(f_n, g) \to 0 \text{ as } n \to \infty.$$

Proof. Put $g_n = \bigvee_{k\geq n} f_k$. Then $g_n^{-1}[s+] \supset g_{n+1}^{-1}[s+]$ and

$$g^{-1}[s+] = \bigcap_{m\geq 1} g_m^{-1}[s+]$$

hold. By using the relation

$$D_0(f_n^{-1}[s+], g^{-1}[s+]) = D_0(f_n^{-1}[s+], \textstyle\bigcap_{m\geq 1} g_m^{-1}[s+])$$

$$\leq D_0(f_n^{-1}[s+], g_n^{-1}[s+]) + D_0(g_n^{-1}[s+], \textstyle\bigcap_{m\geq 1} g_m^{-1}[s+]),$$

we get the conclusion by Proposition 3.4 and Lemma 3.3. □

Remark 1. We don't know whether the metric space $(USC|_K, d_f)$, or $(USC|_K, D_f)$ is complete or not. However, Theorem 3.5 shows that any Cauchy sequence in $(USC|_K, d_f)$ converges to an element in $USC|_K$ in the metric D_f.

§4. Convergence of $\psi_n(\delta)$ in case of $\mathbb{R} \times [0,1]$

In this section, we will consider the convergence of $\{\psi_n(\delta)\}_n$ in the space of $\mathbb{Z}_p$-valued upper semi-continuous functions on a compact subset of $\mathbb{R} \times [0,1]$. At first we give some notations.

Let α_k be as defined in (2.1). Suppose that at least one of α_k is nonzero. Then put

$$k_- = \min\{k \in G \mid \alpha_k \neq 0\},$$
$$k_+ = \max\{k \in G \mid \alpha_k \neq 0\}$$

and

$$k_0 = k_+ - k_-.$$

For $j \in \{0, 1, \dots, p\}$ and $i \in \{1, \dots, jk_0 + 1\}$, put

$$r_j = j + \frac{j(j-1)}{2} k_0 \tag{4.1}$$

and $\ell = r_j + i$.

Then the map $S_{\ell,j}$ defined by (2.5) with double suffix can be reindexed with single suffix for those satisfying $S_{\ell,j}(X_0) \subset X_0$ (X_0 is defined in (4.4)) as follows:

$$S_\ell(y,q) = (\frac{y}{p}, \frac{q}{p}) + (\ \frac{-jk_+ + i - 1}{p},\ \frac{j}{p}\ ). \tag{4.2}$$

Put

$$c_\ell = L^j \delta(-jk_+ + i - 1)$$

with $\ell = r_j + i$ and

$$\Lambda = \{\ell \in \{1, \dots, r_p\} \mid c_\ell \neq 0\}.$$

Then for $(y,q) \in \mathbb{R} \times [0,1]$ satisfying $\frac{j}{p} \leq q \leq \frac{j+1}{p}$ with $0 \leq j \leq p-1$, we have

$$(\psi_{n+1}(\delta))(y,q) = \sum_{\ell=r_j+1}^{r_{j+1}} c_\ell (\psi_n(\delta))(S_\ell^{-1}(y,q)) \tag{4.3}$$

by using (2.3), (2.6) and Proposition 2.4.

Let X_0 be the smallest convex subset of $\mathbb{R} \times [0,1]$ containing the support of $\psi_1(\delta)$, that is,

$$X_0 = \{(y,q) \in \mathbb{R} \times [0,1] \mid 0 \le q \le 1, -qk_+ \le y \le -qk_-\}. \tag{4.4}$$

Then for any $n \in \mathbb{N}$, the support of $\psi_n(\delta)$ is contained in X_0 and for $\ell \in \Lambda$, $S_\ell(X_0)$ is also contained in X_0. So we consider the space $USC|_{X_0}$ and the metrics d_f, D_f in $USC|_{X_0}$ as in Section 3 and prove the following theorem.

Theorem 4.1. *For $\psi_n(\delta) \in USC|_{X_0}$, we have the following:*

(1) $d_f(\psi_n(\delta), \psi_m(\delta)) \to 0$ *as* $n, m \to \infty$.

(2) *Put* $f_\delta = \bigwedge_{k \ge 1} \bigvee_{n \ge k} \psi_n(\delta)$, *where* $\bigwedge$ *and* $\bigvee$ *are lattice operations in* USC. *Then we have*

$$D_f(\psi_n(\delta), f_\delta) \to 0 \qquad as \qquad n \to \infty.$$

Since the proof of the theorem is pretty complicated, we shall describe the strategy of the proof.

4.1. Strategy of proof

Roughly speaking, our fundamental idea is to show the estimate

$$d_f(\psi_{n+1}, \psi_{m+1}) \le \frac{1}{p} d_f(\psi_n, \psi_m). \tag{4.5}$$

Indeed, if $\{(S_\ell(X_0))^\circ\}_{\ell \in \Lambda}$ is mutually disjoint, the inequality (4.5) can be verified without too much work, where $(S_\ell(X_0))^\circ$ is the interior of $S_\ell(X_0)$. However, we cannot prove (4.5) if $\{(S_\ell(X_0))^\circ\}_{\ell \in \Lambda}$ are mutually overlapped. So we introduce an auxiliary quantity $M_0^{n,n'}$ and show the following estimates:

M-1) $d_f(\psi_{n+1}(\delta), \psi_{n'+1}(\delta)) \le \frac{1}{p} M_0^{n,n'}$ (Proposition 4.5);

M-2) $M_0^{n+1,n'+1} \le \frac{1}{p} M_0^{n,n'}$ (Proposition 4.6).

$M_0^{n,n'}$ is devised to compensate the fact that $\{(S_\ell(X_0))^\circ\}_{\ell \in \Lambda}$ may overlap mutually. To define $M_0^{n,n'}$, we use two divisions $\{E_\gamma\}$ and $\{A_{b,j,s}\}$ of X_0 and functions $\{h_v^n\}$.

4.2. The definition of $\{E_\gamma\}$ and $\{A_{b,j,s}\}$

We shall divide X_0 into subsets $\{E_\gamma\}$ and $\{A_{b,j,s}\}$ as follows (see Figure 3). Let

$$\Gamma = \{(1,j,s) \mid 1 \le s \le pk_0, 1 \le j \le s\} \bigcup \{(2,j,s) \mid 2 \le s \le pk_0, 1 \le j \le s-1\}.$$

We define $\{E_\gamma\}(\gamma \in \Gamma)$ as follows:

In case of $\gamma = (1,j,s) \in \Gamma$, let

$$E_\gamma = \{(y,q) \mid \frac{s-1}{pk_0} \le q \le \frac{s}{pk_0}, \ \ -k_+ q + \frac{j-1}{p} \le y \le -k_- q - \frac{s-j}{p}\};$$

in case of $\gamma = (2,j,s) \in \Gamma$

$$E_\gamma = \{(y,q) \mid \frac{s-1}{pk_0} \le q \le \frac{s}{pk_0}, \ \ -k_- q - \frac{s-j}{p} \le y \le -k_+ q + \frac{j}{p}\}.$$

Let for $1 \le s \le k_0$ and $1 \le j \le s$,

$$A_{1,j,s} = \{(y,q) \mid \frac{s-1}{k_0} \le q \le \frac{s}{k_0}, \ \ -k_+ q + \frac{j-1}{p} \le y \le -k_- q - \frac{s-j}{p}\}$$

and for $2 \le s \le k_0$ and $1 \le j \le s-1$,

$$A_{2,j,s} = \{(y,q) \mid \frac{s-1}{k_0} \le q \le \frac{s}{k_0}, \ \ -k_- q - \frac{s-j}{p} \le y \le -k_+ q + \frac{j}{p}\}.$$

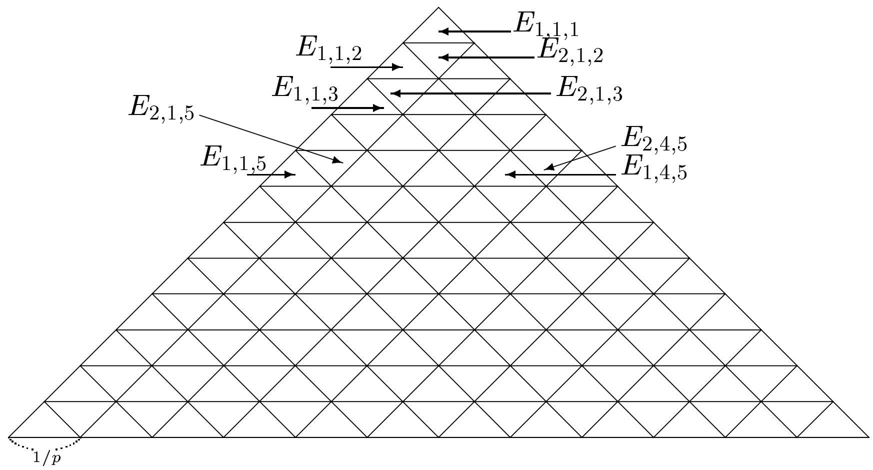
Then we have the following properties.

Proposition 4.2. (1) *The sets $\{E_\gamma\}$ have the following properties.*

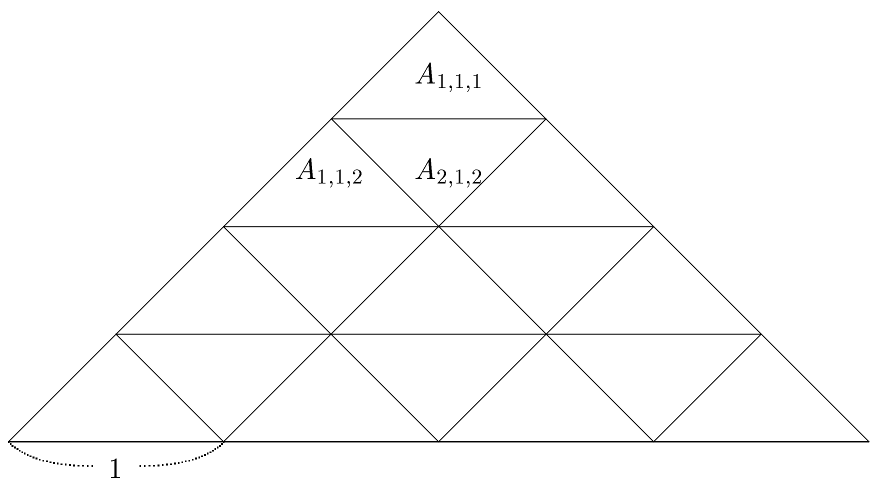
- *E-1) For $\gamma = (b,j,s), \gamma' = (b,j',s) \in \Gamma$, E_γ is the shift of $E_{\gamma'}$ in the the first coordinate direction for any s and $b \in \{1,2\}$.*
- *E-2) $E_\gamma^\circ \cap E_{\gamma'}^\circ = \emptyset$ if $\gamma \ne \gamma'$.*
- *E-3) If $(S_\ell(X_0))^\circ \cap (S_{\ell'}(X_0))^\circ \ne \emptyset$, then $S_\ell(X_0) \cap S_{\ell'}(X_0)$ is the union of some E_γ's.*
- *E-4) $X_0 = \bigcup_{\gamma \in \Gamma} E_\gamma$.*

(2) *The sets $\{A_{b,j,s}\}_{b,j,s}$ have the following properties.*

- *A-1) For any $A_{b,j,s}$, there exist $\gamma \in \Gamma$ and $\ell \in \{1, \cdots, r_p\}$ such that $A_{b,j,s} = S_\ell^{-1}(E_\gamma)$.*
- *A-2) $X_0 = \bigcup_{b=1}^{2} \bigcup_{s=1}^{k_0} \bigcup_{j=1}^{s} A_{b,j,s}$.*
- *A-3) $A_{b,j,s}^\circ \cap A_{b',j',s'}^\circ = \emptyset \quad$ if $\quad (b,j,s) \ne (b',j',s')$.*



(a) The locations of $E_{1,1,1}$, $E_{1,1,2}$, $E_{2,1,2}$, $E_{1,1,3}$, $E_{2,1,3}$, $E_{1,4,5}$, $E_{2,4,5}$, $E_{1,1,5}$ and $E_{2,1,5}$.



(b) The locations of $A_{1,1,1}$, $A_{1,1,2}$ and $A_{2,1,2}$.

Figure 3: $\{E_\gamma\}_\gamma$ and $\{A_{2,j,s}\}_{2,j,s}$ for $La(x) = a(x-2) + a(x-1) + a(x+1) + a(x+2) \pmod 3$.

Proof. By the definition, we can easily get the result. □

Put

$$\Lambda_\gamma = \{\ell \in \Lambda \mid E_\gamma \subset S_\ell(X_0)\}$$

for $\gamma \in \Gamma$ and let $\ell_\gamma \in \Lambda_\gamma$ be the minimum number in Λ_γ. Then equation (4.3) can be rewritten as follow:

$$(\psi_{n+1}(\delta)1_{E_\gamma})(y,q) = \sum_{\ell \in \Lambda_\gamma} c_\ell(\psi_n(\delta))(S_\ell^{-1}(y,q)), \tag{4.6}$$

where $\sum_{\ell\in\Lambda_\gamma} b_\ell$ means 0 if $\Lambda_\gamma = \emptyset$. Then (4.3) and (4.6) is the same, since the left side of (4.6) is zero if $\Lambda_\gamma = \emptyset$.
If $(\psi_{n+1}(\delta))^{-1}(j)$ and $(\psi_{m+1}(\delta))^{-1}(j)$ are nonempty and $\#\Lambda_\gamma = 1$, then by using (4.6) for $j = c_\ell \times j'$ (mod p)

$$\begin{aligned} &D((\psi_{n+1}(\delta))^{-1}(j) \cap E_\gamma, (\psi_{m+1}(\delta))^{-1}(j) \cap E_\gamma) \\ &\quad = \frac{1}{p} D((\psi_n(\delta))^{-1}(j') \cap S_\ell^{-1}(E_\gamma), (\psi_m(\delta))^{-1}(j') \cap S_\ell^{-1}(E_\gamma)). \end{aligned}$$

Though equation (4.2) is useful to prove the convergence of $\{\psi_n\}$, (4.2) does not hold if $\#\Lambda_\gamma > 1$. So we introduce the function h_v^n.

4.3. The definition of $\{h_v^n\}$ and their fundamental properties

We shall define the function h_v^n as follows. Put

$$\begin{aligned} V =&\{v = (\gamma_1, \dots, \gamma_m) \mid m \in \mathbb{N}, \gamma_1 \in \Gamma \text{ with } \#\Lambda_{\gamma_1} \geq 1, \gamma_k \in \Gamma \\ &\text{with } \#\Lambda_{\gamma_k} \geq 1 \ \text{ and } \ S_{\ell_{\gamma_{k-1}}}(E_{\gamma_k}) \subset E_{\gamma_{k-1}} \ \text{ for any } k \in \{2, \dots, m\}\}. \end{aligned}$$

For $v = (\gamma_1, \dots, \gamma_m) \in V$ and $n \in \mathbb{N}$, define

$$\begin{aligned} h_v^n(y,q) = &\sum_{\ell_1 \in \Lambda_{\gamma_1}} \cdots \sum_{\ell_m \in \Lambda_{\gamma m}} c_{\ell_1} \dots c_{\ell_m} \\ &\times (\psi_n(\delta))(S_{\ell_m}^{-1} \dots S_{\ell_1}^{-1} S_{\ell_{\gamma_1}} \dots S_{\ell_{\gamma m}}(y,q)) 1_{S_{\ell_{\gamma m}}^{-1}(E_{\gamma m})}(y,q) \end{aligned} \tag{4.7}$$

for $(y,q) \in \mathbb{R} \times [0,1]$.

When $v = (\gamma)$, h_v^n satisfies

$$h_v^n(y,q) = \sum_{\ell \in \Lambda_\gamma} c_\ell(\psi_n(\delta))(S_\ell^{-1} S_{\ell_\gamma}(y,q)) 1_{S_{\ell_\gamma}^{-1}(E_\gamma)}(y,q),$$

and

$$h_v^n(S_{\ell_\gamma}^{-1}(y,q)) = (\psi_{n+1}(\delta))(y,q)1_{E_\gamma}(y,q)$$

for $n \in \mathbb{N}$. Since the length of v is one, h_v^n has the relation with ψ_{n+1}.

If the length of v is m, then h_v^n has the relation with ψ_{n+m} and this is useful in estimating the metric $d_f(h_v^n, h_v^{n'})$ as shown in the following Lemma.

Lemma 4.3. *For $v = (\gamma_1, \gamma_2, \ldots, \gamma_m) \in V$, $k \in \{1, \ldots, m\}$ and $(y,q) \in \mathbb{R} \times [0,1]$, put*

$$F_k(y,q) = S_{\ell_{\gamma_1}}(S_{\ell_{\gamma_2}}(\ldots(S_{\ell_{\gamma_k}}(y,q))\ldots)).$$

Then we have

(1) *for $(y,q) \in F_{m-1}(E_{\gamma_m})$,*

$$h_v^n(F_m^{-1}(y,q)) = (\psi_{n+m}(\delta))(y,q)1_{F_{m-1}(E_{\gamma_m})}(y,q)$$

and

(2) *if the sets $\{j \in \{1, \ldots, p-1\} \mid (h_v^n)^{-1}(j) = \emptyset\}$ and $\{j \in \{1, \ldots, p-1\} \mid (h_v^{n'})^{-1}(j) = \emptyset\}$ are the same, then*

$$d_f(h_v^n, h_v^{n'}) = p^m d_f(\psi_{n+m}(\delta)1_{F_{m-1}(E_{\gamma_m})}, \psi_{n'+m}(\delta)1_{F_{m-1}(E_{\gamma_m})})$$

for any n, $n' \in \mathbb{N}$.

Proof. (1) By using (4.6) and (4.7), we have

$$\begin{aligned}
&h_v^n(F_m^{-1}(y,q)) \\
&= \sum_{\ell_1 \in \Lambda_{\gamma_1}} \cdots \sum_{\ell_m \in \Lambda_{\gamma_m}} c_{\ell_1} \ldots c_{\ell_m} (\psi_n(\delta))(S_{\ell_m}^{-1} \ldots S_{\ell_1}^{-1}(y,q))1_{S_{\ell_{\gamma_m}}^{-1}(E_{\gamma_m})}(F_m^{-1}(y,q)) \\
&= \sum_{\ell_1 \in \Lambda_{\gamma_1}} \cdots \sum_{\ell_{m-1} \in \Lambda_{\gamma_{m-1}}} c_{\ell_1} \ldots c_{\ell_{m-1}} \\
&\qquad \times (\psi_{n+1}(\delta))(S_{\ell_{m-1}}^{-1} \ldots S_{\ell_1}^{-1}(y,q))1_{F_{m-1}(E_{\gamma_m})}(y,q) \\
&\ \ \vdots \\
&= (\psi_{n+m}(\delta))(y,q)1_{F_{m-1}(E_{\gamma_m})}(y,q).
\end{aligned}$$

(2) By the assumption, if $(h_v^n)^{-1}(j) \neq \emptyset$, then

$$D_0((h_v^n)^{-1}(j), (h_v^{n'})^{-1}(j)) = D((h_v^n)^{-1}(j), (h_v^{n'})^{-1}(j))$$

and

$$
\begin{aligned}
(h_v^n)^{-1}(j) &= \{(y,q) \mid h_v^n(y,q) = j\} \\
&= \{(y,q) \mid (\psi_{n+m}(\delta))(F_m(y,q))1_{F_{m-1}(E_{\gamma m})}(F_m(y,q)) = j\} \\
&= \{F_m^{-1}(y',q') \mid (\psi_{n+m}(\delta))(y',q')1_{F_{m-1}(E_{\gamma m})}(y',q') = j\} \\
&= F_m^{-1}(\{(y',q') \mid (\psi_{n+m}(\delta))(y',q')1_{F_{m-1}(E_{\gamma m})}(y',q') = j\}) \qquad (4.8)
\end{aligned}
$$

hold. If both $(h_v^n)^{-1}(j)$ and $(h_v^{n'})^{-1}(j)$ are empty, then

$$D_0((h_v^n)^{-1}(j), (h_v^{n'})^{-1}(j)) = 0 \qquad (4.9)$$

holds. So the relations (4.8) and (4.9) imply

$$d_f(h_v^n, h_v^{n'}) = p^m d_f(\psi_{n+m}(\delta)1_{F_{m-1}(E_{\gamma m})}, \psi_{n'+m}(\delta)1_{F_{m-1}(E_{\gamma m})}).$$

□

In Lemma 4.3 (2), we put assumption that the sets $\{j \in \{1, \ldots, p-1\} \mid (h_v^n)^{-1}(j) = \emptyset\}$ and $\{j \in \{1, \ldots, p-1\} \mid (h_v^{n'})^{-1}(j) = \emptyset\}$ are the same. However, this is true for sufficiently large n, n', which is proved by using the following proposition and Lemma 4.3 (1).

Proposition 4.4. *For sufficiently large $n \in \mathbb{N}$, the following assertions are equivalent for any $v = (\gamma_1, \ldots, \gamma_{m_v}) \in V$, $\ell \in \mathbb{Z}_p \backslash \{0\}$.*

(1) $(\psi_{n+m_v}(\delta))^{-1}(\ell) \cap (F_{m_v-1}(E_{\gamma_{m_v}}))^\circ \neq \emptyset$.

(2) $(\psi_{n+m_v+1}(\delta))^{-1}(\ell) \cap (F_{m_v-1}(E_{\gamma_{m_v}}))^\circ \neq \emptyset$.

The proof is given in Section 5.

4.4. The definition of $\{M_0^{n,n'}\}$ and their properties

By using h_v^n, we shall define $M_0^{n,n'}$ by

$$M_0^{n,n'} = \sup\{d_f(h_v^n, h_v^{n'}) \mid v \in V\}.$$

Then we have the following crucial estimates in Propositions 4.5 and 4.6.

Proposition 4.5. (1) $\sup\{M_0^{n,n'} \mid n, n' \in \mathbb{N}\} < \infty$.

(2) $d_f(\psi_{n+1}(\delta), \psi_{n'+1}(\delta)) \leq \frac{1}{p} M_0^{n,n'}$ *holds for sufficiently large $n, n' \in \mathbb{N}$.*

Proof. (1) Put $c = \sup\{d((y_0, q_0), (y, q)) \mid (y, q) \in X_0\}$, where $(y_0, q_0) \notin X_0$ is the point defining the pseudodistnace D_0 as shown in Section 3 and $d(\cdot,\cdot)$ is the Euclidean distance in $\mathbb{R} \times [0, 1]$. Then c is finite, since X_0 is a bounded set. For any $g, f \in USC|_{X_0}$ and $j \in \{1, \cdots, p-1\}$,

$$D_0(g^{-1}(j), f^{-1}(j)) \leq \max\{c, |X_0|\},$$

where $|X_0|$ is a diameter of X_0. So we have

$$\sup\{M_0^{n,n'} \mid n, n' \in \mathbb{N}\} < \max\{c, |X_0|\}.$$

(2) By the definition,

$$\begin{aligned} &d_f(\psi_{n+1}(\delta), \psi_{n'+1}(\delta)) \\ &\qquad = \max_{1\leq \ell \leq p-1} D_0(\overline{(\psi_{n+1}(\delta))^{-1}(\ell)}, \overline{(\psi_{n'+1}(\delta))^{-1}(\ell)}). \end{aligned} \tag{4.10}$$

By using Proposition 3.1 and Lemma 3.2,

$$\begin{aligned} &D_0((\psi_{n+1}(\delta))^{-1}(\ell), (\psi_{n'+1}(\delta))^{-1}(\ell)) \\ &\qquad \leq \max_{\gamma\in\Gamma} D_0((\psi_{n+1}(\delta))^{-1}(\ell) \cap E_\gamma, (\psi_{n'+1}(\delta))^{-1}(\ell) \cap E_\gamma). \end{aligned} \tag{4.11}$$

For sufficiently large n, n', we have by Proposition 4.4 and by Lemma 4.3 (2)

$$\begin{aligned} D_0(\psi_{n+1}(\delta)^{-1}(\ell) \cap E_\gamma, \psi_{n'+1}(\delta)^{-1}(\ell) \cap E_\gamma) &\leq d_f(\psi_{n+1}(\delta)1_{E_\gamma}, \psi_{n'+1}(\delta)1_{E_\gamma}) \\ &\leq \frac{1}{p} d_f(h_v^n, h_v^{n'}). \end{aligned} \tag{4.12}$$

By using the inequalities (4.10), (4.11) and (4.12), we get the conclusion. ☐

Proposition 4.6. *For sufficiently large n, n', we have*

$$M_0^{n+1,n'+1} \leq \frac{1}{p} M_0^{n,n'}.$$

Proof. For $v = (\gamma_1, \ldots, \gamma_m) \in V$, let F_m be as defined in Lemma 4.3. Then

$$\begin{aligned} h_v^{n+1}(y, q) = &\sum_{\ell_1\in\Lambda_{\gamma_1}} \cdots \sum_{\ell_m\in\Lambda_{\gamma_m}} c_{\ell_1} \ldots c_{\ell_m} \\ &\times (\psi_{n+1}(\delta))(S_{\ell_m}^{-1} \ldots S_{\ell_1}^{-1} F_m(y, q)) 1_{S_{\ell_{\gamma_m}}^{-1}(E_{\gamma_m})}(y, q). \end{aligned}$$

For $E_{\gamma_{m+1}} \subset S^{-1}_{\ell_{\gamma_m}}(E_{\gamma_m})$, let $v_{\gamma_{m+1}} = (\gamma_1, \dots, \gamma_{m+1})$. Then we have

$$
\begin{aligned}
&h_v^{n+1}(y,q)1_{E_{\gamma_{m+1}}}(y,q)\\
&= \sum_{\ell_1 \in \Lambda_{\gamma_1}} \dots \sum_{\ell_m \in \Lambda_{\gamma_m}} c_{\ell_1} \dots c_{\ell_m}(\psi_{n+1}(\delta))(S^{-1}_{\ell_m} \dots S^{-1}_{\ell_1} F_m(y,q))1_{E_{\gamma_{m+1}}}(y,q)\\
&= \sum_{\ell_1 \in \Lambda_{\gamma_1}} \dots \sum_{\ell_{m+1} \in \Lambda_{\gamma_{m+1}}} c_{\ell_1} \dots c_{\ell_{m+1}}(\psi_n(\delta))(S^{-1}_{\ell_{m+1}} \dots S^{-1}_{\ell_1} F_m(y,q))1_{E_{\gamma_{m+1}}}(y,q)\\
&= \sum_{\ell_1 \in \Lambda_{\gamma_1}} \dots \sum_{\ell_{m+1} \in \Lambda_{\gamma_{m+1}}} c_{\ell_1} \dots c_{\ell_{m+1}}\\
&\quad \times (\psi_n(\delta))(S^{-1}_{\ell_{m+1}} \dots S^{-1}_{\ell_1} F_m S_{\ell_{\gamma_{m+1}}} S^{-1}_{\ell_{\gamma_{m+1}}}(y,q))1_{S^{-1}_{\ell_{\gamma_{m+1}}}(E_{\gamma_{m+1}})}(S^{-1}_{\ell_{\gamma_{m+1}}}(y,q))\\
&= h^n_{v_{\gamma_{m+1}}}(S^{-1}_{\ell_{\gamma_{m+1}}}(y,q))1_{S^{-1}_{\ell_{\gamma_{m+1}}}(E_{\gamma_{m+1}})}(S^{-1}_{\ell_{\gamma_{m+1}}}(y,q)).
\end{aligned}
$$

By using above equations, we have

$$
\begin{aligned}
(h_v^{n+1})^{-1}(j) &= \bigcup_{E_{\gamma_{m+1}} \subset S^{-1}_{\ell_{\gamma_m}}(E_{\gamma_m})} \{(y,q) \mid h^n_{v_{\gamma_{m+1}}}(S^{-1}_{\ell_{\gamma_{m+1}}}(y,q)) = j\}\\
&= \bigcup_{E_{\gamma_{m+1}} \subset S^{-1}_{\ell_{\gamma_m}}(E_{\gamma_m})} S_{\ell_{\gamma_{m+1}}}(\{(y',q') \mid h^n_{v_{\gamma_{m+1}}}(y',q') = j\}),
\end{aligned}
$$

and

$$
d_f(h_v^{n+1}, h_v^{n'+1}) \le \frac{1}{p} \max\{d_f(h^n_{v_{\gamma_{m+1}}}, h^{n'}_{v_{\gamma_{m+1}}}) \mid \ \gamma_{m+1} \subset S^{-1}_{\ell_{\gamma_m}}(E_{\gamma_m})\},
$$

for sufficiently large n, n' satisfying the condition in Proposition 4.4. So we get the conclusion. □

4.5. Proof of Theorem 4.1

By using above propositions, we shall prove Theorem 4.1.

(1) By Propositions 4.5 (2) and 4.6, we have

$$
\lim_{n,m \to \infty} M_0^{n,m} = 0.
$$

By Proposition 4.5 (1) , we have

$$
d_f(\psi_{n+1}(\delta), \psi_{m+1}(\delta)) \le \frac{1}{p} M_0^{n,m}.
$$

So we obtain the conclusion.

(2) We get the result from (1) and Theorem 3.5. □

§5. Proof of Proposition 4.4

In this section, we shall prove Proposition 4.4. We will prove it in the case of $m_v = 1$, since in the other case the proof is similar to the case of $m_v = 1$.

At first, we shall consider the property of $\psi_n(\delta)1_{E_\gamma^\circ}$.

Lemma 5.1. *Let $\gamma \in \Gamma$ and suppose that $\psi_n(\delta)1_{E_\gamma^\circ} = 0$ for some $n \geq 2$. Then $\psi_{n-1}(\delta)1_{E_\gamma^\circ} = 0$.*

Proof. It is clear by the relation

$$\psi_{n-1}(\delta)1_{E_\gamma^\circ}(\frac{x}{p^{n-1}}, \frac{t}{p^{n-1}}) = \psi_n(\delta)1_{E_\gamma^\circ}(\frac{px}{p^n}, \frac{pt}{p^n}).$$

□

In order to prove Lemma 5.4, we prepare the following Lemmas 5.2 and 5.3.

Lemma 5.2. *For $s \in \{2, \dots, k_0\}$, $b \in \{1, 2\}$, $m, n' \geq 1$, $\beta_\ell \in \mathbb{Z}_p (\ell = 1, \dots, m)$ and $n \in \mathbb{N}$ satisfying $p^{n-3} > 2k_0$, suppose that*

$$\sum_{\ell=1}^{m} \beta_\ell (\psi_n(\delta)1_{A_{b,\ell,s}^\circ})(y + \ell - 1, q) = 0 \tag{5.1}$$

and

$$\sum_{\ell=1}^{m} \beta_\ell (\psi_{n+n'}(\delta)1_{E_{b,p(\ell-1)+1,p(s-1)+1}^\circ})(y + \ell - 1, q) = 0 \tag{5.2}$$

hold for any $(y, q) \in \mathbb{R} \times [0, 1]$. Then

$$\sum_{\ell=1}^{m} \beta_\ell (\psi_{n+n'}(\delta)1_{A_{b,\ell,s}^\circ})(y + \ell - 1, q) = 0 \tag{5.3}$$

holds for any $(y, q) \in \mathbb{R} \times [0, 1]$.

Proof. At first consider the case of $b = 2$. Put $t_0 = \min\{t \in \mathbb{Z}_+ \mid \exists x \in \mathbb{Z} \ s.t. \ (x/p^n, t/p^n) \in A_{2,\ell,s}^\circ\}$ and set

$$B_\ell = \{(y, q) \in A_{2,\ell,s}^\circ \mid q \geq t_0/p^n\} \quad \text{(see Figure 4).}$$

Then by (5.1) and Lemma 2.1, we have

$$\sum_{\ell=1}^{m} \beta_\ell (\psi_{n+n'}(\delta)1_{A_{2,\ell,s}^\circ})(y + \ell - 1, \frac{t_0 p^{n'}}{p^{n+n'}}) = 0$$

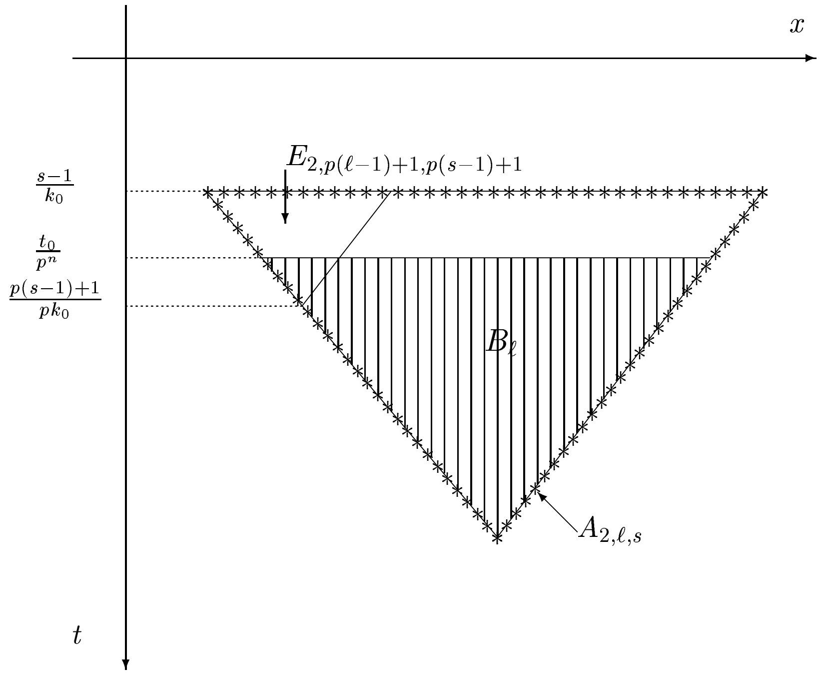


Figure 4: The positions of B_ℓ, $E_{2,p(\ell-1)+1,p(s-1)+1}$ and $A_{2,\ell,s}$.

for any $y \in \mathbb{R}$. Hence $\sum_{\ell=1}^{m} \beta_\ell(\psi_{n+n'}(\delta)1_{B_\ell})(y+\ell-1,q) \equiv 0$ by (2.1) and (2.3). So we shall only show

$$\sum_{\ell=1}^{m} \beta_\ell(\psi_{n+n'}(\delta)1_{(A^\circ_{2,\ell,s}\setminus(B_\ell\cup E^\circ_{2,p(\ell-1)+1,p(s-1)+1}))})(y+\ell-1,q) \equiv 0.$$

To do so, note that the equations $(\psi_{n+n'}(\delta))((x-i)/p^{n+n'}, t/p^{n+n'}) = 0$ for all $i \in \{1,\dots,k_0\}$ and $(\psi_{n+n'}(\delta))((x-k_+)/p^{n+n'}, (t+1)/p^{n+n'}) = 0$ imply $(\psi_{n+n'}(\delta))(x/p^{n+n'}, t/p^{n+n'}) = 0$. By virtue of this property, beginning from the lower left corner to the upper right corner, we can successively verify that

$$\sum_{\ell=1}^{m} \beta_\ell(\psi_{n+n'}(\delta)1_{A^\circ_{2,\ell,s}})(y+\ell-1,q) = 0$$

at every lattice point (y,q) in $(A^\circ_{2,\ell,s}\setminus(B_\ell\cup E^\circ_{2,p(\ell-1)+1,p(s-1)+1}))\cap X_{n+n'}$. For the case $b=1$, we can prove in a similar way that (5.3) holds. □

Let

$$n_0 = \min\{n \in \mathbb{N} \mid k_0+1 < p^{n-3}\}. \tag{5.4}$$

Then it is easy to show that $E_\gamma \cap X_n \neq \emptyset$ holds for any $\gamma \in \Gamma$ and any $n \geq n_0$.

Lemma 5.3. *Let* $\gamma = (b, j, s) \in \Gamma$ *satisfy* $E_\gamma \subset \bigcup_{\ell=1}^{r_p} S_\ell(X_0)$. *Then there exist an infinite sequence* $\{s_k\}_k$ *and a finite or infinite sequence* $\{\{\beta_\ell^k\}_{\ell=1}^{s_k-b+1}\}_k$ *satisfying the following* (1) $\sim$ (3).

(If $b=1$, $\{\{\beta_\ell^k\}_{\ell=1}^{s_k-b+1}\}_k$ *is an infinite sequence. If* $b=2$, *then* $\{\beta_\ell^k\}_{\ell=1}^{s_k-b+1}$ *is defined as long as* $s_k \geq 2$ ($\Leftrightarrow s_k - b + 1 \geq 1$). *)*

(1) $s_1 = s \pmod{k_0}$, $s_{k+1} = p(s_k - 1) + 1 \pmod{k_0}$, $s_k \in \{1, \ldots, k_0\}$ *and* $\beta_\ell^k \in \mathbb{Z}_p$ *for* $k \in \mathbb{N}$ *and* $\ell \in \{1, \ldots, s_k - b + 1\}$.

(2) *For any* $(y, q) \in E_{b,j,s}$, *there exists* $(y_0, q_0) \in A^\circ_{b,1,s_1}$ *such that*

$$(\psi_m(\delta) 1_{E^\circ_{b,j,s}})(y, q) = \sum_{\ell=1}^{s_1-b+1} \beta_\ell^1 (\psi_{m-1}(\delta) 1_{A^\circ_{b,\ell,s_1}})(y_0 + \ell - 1, q_0) \tag{5.5}$$

for any $m > n_0 + 1$, *where the mapping* $(y, q) \mapsto (y_0, q_0)$ *is an affine isomorphism from* $E_{b,j,2}$ *to* $A_{b,1,s_1}$.

(3) *For any* $m > n_0$ *and* $k = 2, \ldots, m - n_0$, *where* n_0 *is defined in* (5.4),

$$\begin{aligned}&\sum_{\ell=1}^{s_{k-1}-b+1} \beta_\ell^{k-1} (\psi_{m-k+1}(\delta) 1_{E^\circ_{b,p(\ell-1)+1,p(s_{k-1}-1)+1}})(y + \ell - 1, q) \\ &\quad = \sum_{\ell=1}^{s_k-b+1} \beta_\ell^k (\psi_{m-k}(\delta) 1_{A^\circ_{b,\ell,s_k}})(S^{-1}_{r_{i_k}+1}(y, q) + (\ell - 1, 0))\end{aligned} \tag{5.6}$$

holds for any $(y, q) \in \mathbb{R} \times [0, 1]$ *as long as* $s_k - b + 1 \geq 1$, *where* r_k *is defined in* (4.1) *and* $i_k \in \mathbb{Z}_+$ *is determined by the following inequality*:

$$i_k \leq p(s_{k-1} - 1)/k_0 < i_k + 1.$$

note. *The relation between* $E_{2,p(\ell-1)+1,p(s-1)+1}$ *and* $A_{2,\ell,s}$ *is shown in Figure* 4.

Proof. At first consider the case of $(2, j, s) \in \Gamma$. Let $s_1 \in \{1, \ldots, k_0\}$ satisfy $s_1 = s \pmod{k_0}$ and $i_1 \in \{0, \ldots, p-1\}$ satisfy $i_1 \leq \frac{s-1}{k_0} < \frac{s}{k_0} \leq i_1 + 1$. Then for any $(y, q) \in E^\circ_{2,j,s}$, $\frac{i_1}{p} < q < \frac{i_1+1}{p}$ holds. So there exists an $\ell_0 \in \{1, \ldots, s_1 - 1\}$ such that $S_{r_{i_1}+j-\ell_0+1}(X_0) \supset E^\circ_{2,j,s}$. Then $S^{-1}_{r_{i_1}+j-\ell_0+1}(E^\circ_{2,j,s}) = A^\circ_{2,\ell_0,s_1}$. Let $(y_0, q_0) = S^{-1}_{r_{i_1}+j-\ell_0+1}(y, q) - (\ell_0 - 1, 0)$. Then $(y_0, q_0) \in A^\circ_{2,1,s_1}$ and the mapping $(y, q) \mapsto (y_0, q_0)$ is an affine isomorphism from $E_{b,j,2}$ to $A_{b,1,s_1}$. As for $S^{-1}_{r_i+k}(y, q)$, we have

$$S^{-1}_{r_i+k}(y, q) = S^{-1}_{r_i+1}(y, q) - (k - 1, 0) \qquad (1 \leq k \leq s_i - 1) \tag{5.7}$$

and

$$S^{-1}_{r_i+1}(y+\ell,q)=S^{-1}_{r_i+1}(y,q)+(p\ell,0). \tag{5.8}$$

Since $S_{r_{i_1}+j-\ell+1}(X_0)\cap E^\circ_{2,j,s}=\emptyset$ for $\ell\geq s_1$, by using (4.3) and (5.7), we obtain

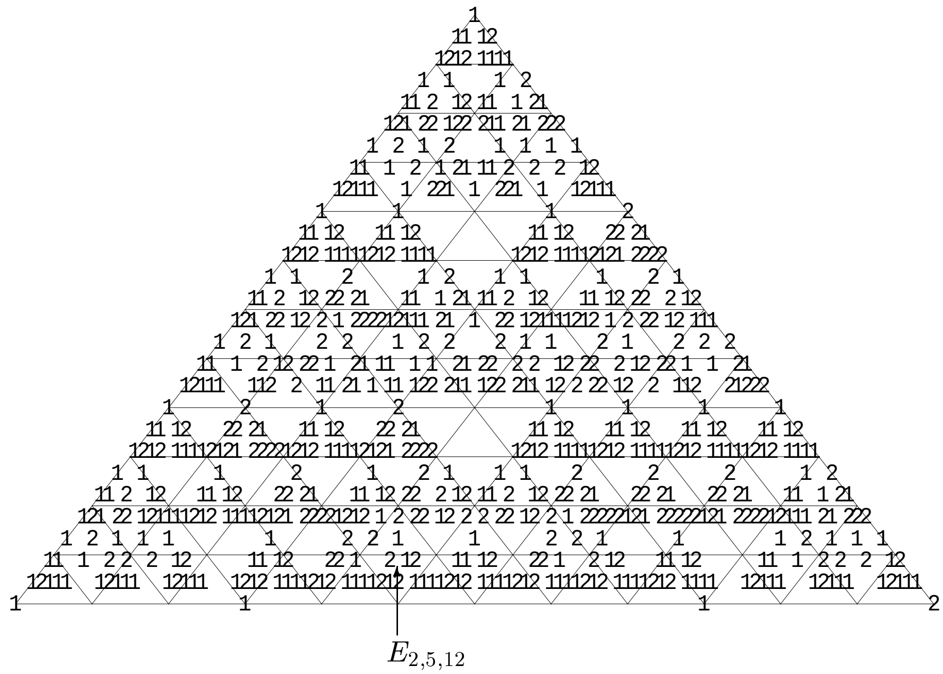
$$\begin{aligned}
&(\psi_m(\delta)1_{E^\circ_{2,j,s}})(y,q)\\
&=\sum_{\ell=1\vee(j+1-(r_{i_1+1}-r_{i_1}))}^{j\wedge(s_1-1)} c_{r_{i_1}+j-\ell+1}\\
&\times(\psi_{m-1}(\delta)1_{S^{-1}_{r_{i_1}+j-\ell+1}(E^\circ_{2,j,s})})(S^{-1}_{r_{i_1}+j-\ell+1}(y,q))\\
&=\sum_{\ell=1\vee(j+1-(r_{i_1+1}-r_{i_1}))}^{j\wedge(s_1-1)} c_{r_{i_1}+j-\ell+1}(\psi_{m-1}(\delta)1_{A^\circ_{2,l,s_1}})(y_0+\ell-1,q_0)\\
&=\sum_{\ell=1}^{s_1-1}\beta^1_\ell(\psi_{m-1}(\delta)1_{A^\circ_{2,\ell,s_1}})(y_0+\ell-1,q_0),
\end{aligned}$$

for any $(y,q)\in E_{2,j,s}$, where $\beta^1_\ell=c_{r_{i_1}+j-\ell+1}$ (mod p) if $(1\vee(j+1-(r_{i_1+1}-r_{i_1})))\leq l\leq(j\wedge(s_1-1))$ and $\beta^1_\ell=0$ if $1\leq\ell<(1\vee(j+1-(r_{i_1+1}-r_{i_1})))$ or $(j\wedge(s_1-1))<\ell\leq s_1-1$ (see Figure 5). Thus we have shown equation (5.5) in the case of $k=1$.

In the case of $k=2$, let $s_2\in\{1,\dots,k_0\}$ satisfy $s_2=p(s_1-1)+1$ (mod k_0) and $i_2\in\{0,\dots,p-1\}$ satisfy $i_2\leq p(s_1-1)/k_0<i_2+1$. If $s_2=1$, then the sequence $\{\{\beta^k_\ell\}^{s_k-b+1}_{\ell=1}\}_k$ consists of only one element $\{\{\beta^1_\ell\}^{s_1-b+1}_{\ell=1}\}$. If $s_2>1$, then $E_{2,p(\ell-1)+1,p(s_1-1)+1}\subset\bigcup^{r_p}_{\ell=1}S_\ell(X_0)$ and we have for each $\ell\in\{1,\dots,s_1-1\}$, $m>n_0+2$ and $(y,q)\in\mathbb{R}\times[0,1]$, by using (4.3)

$$\begin{aligned}
&(\psi_{m-1}(\delta)1_{E^\circ_{2,p(\ell-1)+1,p(s_1-1)+1}})(y+\ell-1,q)\\
&=\sum_{\ell'=1\vee(p(\ell-1)+2-(r_{i_2+1}-r_{i_2}))}^{(p(\ell-1)+1)\wedge(s_2-1)} c_{r_{i_2}+p(\ell-1)-\ell'+2}\\
&\qquad\times(\psi_{m-2}(\delta)1_{A^\circ_{2,\ell',s_2}})(S^{-1}_{r_{i_2}+p(\ell-1)-\ell'+2}(y+\ell-1,q)).
\end{aligned}\tag{5.9}$$

So by using (5.7), (5.8) and (5.9), we obtain for any $m>n_0+2$ and $(y,q)\in\mathbb{R}\times[0,1]$,



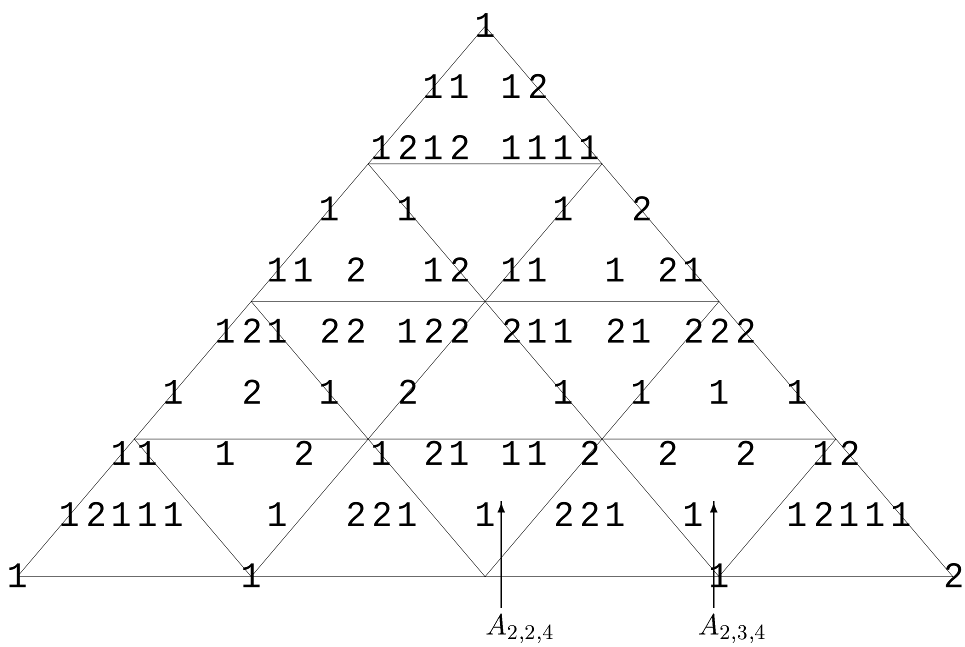
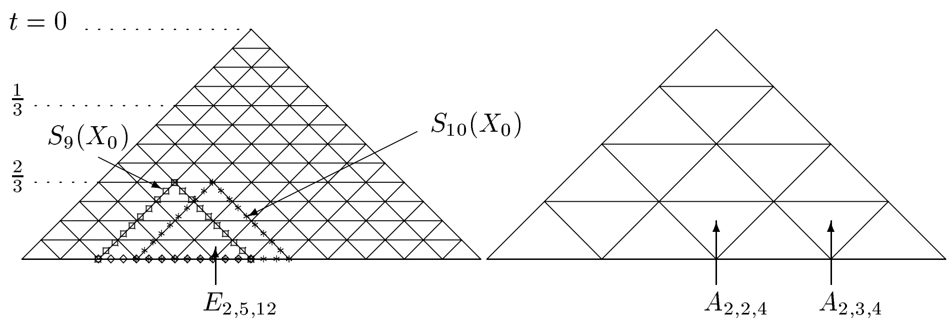
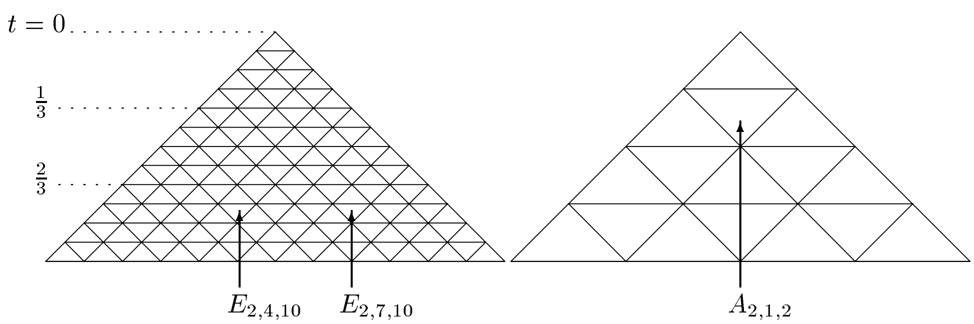


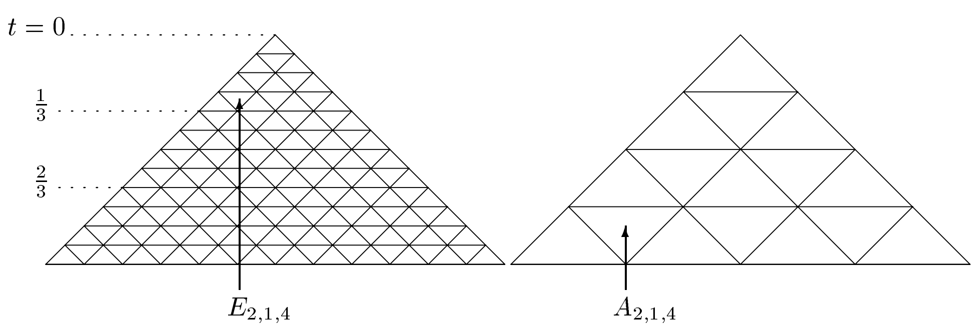
Figure 5: $(\psi_3(\delta)1_{E^\circ_{2,5,12}})(y,q) = 2(\psi_2(\delta)1_{A^\circ_{2,2,4}})(S_{10}^{-1}(y,q)) + (\psi_2(\delta)1_{A^\circ_{2,3,4}})(S_9^{-1}(y,q))$



(a) $s_1 = 4$ and $i_1 = 2$. $(\psi_n(\delta)1_{E_{2,5,12}})(y,q) = 2(\psi_{n-1}(\delta)1_{A_{2,2,4}})(S_{10}^{-1}(y,q)) + (\psi_{n-1}(\delta)1_{A_{2,3,4}})(S_9^{-1}(y,q))$.



(b) $s_2 = 2$ and $i_2 = 2$. $(\psi_{n-1}(\delta)1_{E_{2,4,10}})(y,q) = 2(\psi_{n-2}(\delta)1_{A_{2,1,2}})(S_{10}^{-1}(y,q))$ and $(\psi_{n-1}(\delta)1_{E_{2,7,10}})(y,q) = (\psi_{n-2}(\delta)1_{A_{2,1,2}})(S_{12}^{-1}(y,q))$.



(c) $s_3 = 4$ and $i_3 = 0$. $(\psi_{n-2}(\delta)1_{E_{2,1,4}})(y,q) = (\psi_{n-3}(\delta)1_{A_{2,1,4}})(S_1^{-1}(y,q))$.

Figure 6: An example of s_k and i_k for $La(x) = a(x-2) + a(x-1) + a(x+1) + a(x+2) \pmod 3 (k_0 = 4)$ and $E_{2,5,12}$. See also Figure 5.

$$\begin{aligned}
&\sum_{\ell=1}^{s_1-1} \beta_\ell^1 (\psi_{m-1}(\delta) 1_{E^\circ_{2,p(\ell-1)+1,p(s_1-1)+1}})(y+\ell-1,q) \\
&\quad = \sum_{\ell=1}^{s_1-1} \beta_\ell^1 \sum_{\ell'=1\vee(p(\ell-1)+2-(r_{i_2+1}-r_{i_2}))}^{(p(\ell-1)+1)\wedge(s_2-1)} c_{r_{i_2}+p(\ell-1)-\ell'+2} \\
&\qquad\qquad \times (\psi_{m-2}(\delta) 1_{A^\circ_{2,\ell',s_2}})(S^{-1}_{r_{i_2}+1}(y,q)+(\ell'-1,0)) \\
&\quad = \sum_{\ell'=1}^{s_2-1} \beta_{\ell'}^2 (\psi_{m-2}(\delta) 1_{A^\circ_{2,\ell',s_2}})(S^{-1}_{r_{i_2}+1}(y,q)+(\ell'-1,0)),
\end{aligned}$$

where $\beta_{\ell'}^2 = \sum_{\ell=1}^{s_1-1} \beta_\ell^1 c_{r_{i_2}+p(\ell-1)-\ell'+2} \pmod p$ if $(1 \vee (p(\ell-1)+2-(r_{i_2+1}-r_{i_2}))) \leq \ell' \leq ((p(\ell-1)+1) \wedge (s_2-1))$ and $\beta_{\ell'}^2 = 0$ if $1 \leq \ell' < (1 \vee (p(\ell-1)+2-(r_{i_2+1}-r_{i_2})))$ or $((p(\ell-1)+1) \wedge (s_2-1)) < \ell' \leq s_2-1$.

Thus we have established (5.6) for $k=2$. In the case of $k>2$, we obtain equation (5.6) by repeating the procedure above (see Figure 6).

For the case of $(1,\ell,s) \in \Gamma$, we can prove similarly that there exist infinite sequences $\{s_k\}_k$ and $\{\{\beta_\ell^k\}_{\ell=1}^{s_k-b+1}\}_k$ satisfying (1) $\sim$ (3). □

Lemma 5.4. *For each sufficiently large n, $\psi_n(\delta)1_{E^\circ_\gamma} = 0$ implies*

$$\psi_{n+1}(\delta)1_{E^\circ_\gamma} = 0$$

for any $\gamma \in \Gamma$.

Proof. Consider the case of $(2,j,s) \in \Gamma$. Suppose

$$\psi_n(\delta)1_{E^\circ_{2,j,s}} = 0 \tag{5.10}$$

holds for a sufficiently large $n \in \mathbb{N}$ such that the number $i_0 + j_2 k_1$ given below is less than $n - n_0$. If $E_{2,j,s} \subset \bigcup_{\ell=1}^{r_p} S_\ell(X_0)$ does not hold, then it is clear that $\psi_{n+1}(\delta)1_{E^\circ_{2,j,s}} = 0$. If $E_{2,j,s} \subset \bigcup_{\ell=1}^{r_p} S_\ell(X_0)$ holds, then by Lemma 5.3, there exists $\{s_k, \{\beta_\ell^k\}_{\ell=1}^{s_k-1}\}_k$ ($s_k \in \{1,\ldots,k_0\}$ and $\beta_\ell^k \in \mathbb{Z}_p$) satisfying the relations of Lemma 5.3 (1) $\sim$ (3).

At first we shall consider the case that $s_k \geq 2$ holds for any $k \in \mathbb{N}$. By using the relation $E^\circ_{2,p(\ell-1)+1,p(s-1)+1} \subset A^\circ_{2,\ell,s}$, (5.10), (5.5) and (5.6) repeatedly, we have

$$\sum_{\ell=1}^{s_k-1} \beta_\ell^k (\psi_{n-k}(\delta)1_{A^\circ_{2,\ell,s_k}})(y+\ell-1,q) \equiv 0 \tag{5.11}$$

for any $k \in \mathbb{N}$ with $1 \leq k < n - n_0$.

Since $s_k \in \{1, \dots, k_0\}$, there exist $k_1 \in \mathbb{N}(1 \le k_1 \le k_0)$ and $i_0 \in \mathbb{N}$ such that $s_{i_0} = s_{i_0+k_1}$. So $s_{i_0} = s_{i_0+jk_1}$ for $j = 0, 1, \dots$ by Lemma 5.3 (1). Since $\beta_\ell^{i_0+jk_1} \in \{0, 1, \dots, p-1\}$, there exist $j_1, j_2 \in \mathbb{N}$ $(j_1 < j_2)$ such that $\beta_\ell^{i_0+j_1k_1} = \beta_\ell^{i_0+j_2k_1}$ for $\ell = 1, 2, \dots, s_{i_0} - 1$. Since $i_0 + j_2k_1$ does not depend on n and n is sufficiently large, we can assume that $i_0 + j_2k_1 < n - n_0$ holds.

So by applying (5.11) with $k = i_0 + j_1k_1$ and by using $\beta_\ell^{i_0+j_1k_1} = \beta_\ell^{i_0+j_2k_1}$ and $s_{i_0+jk_1} = s_{i_0+jk_2}$, we have

$$\sum_{\ell=1}^{s_{i_0+j_2k_1}-1} \beta_\ell^{i_0+j_2k_1}(\psi_{n-(i_0+j_1k_1)}(\delta)1_{A^\circ_{2,\ell,s_{i_0+j_2k_1}}})(y+\ell-1, q) \equiv 0. \tag{5.12}$$

Hence by applying (5.6) with $m = n+(j_2-j_1)k_1$, $k = i_0+j_2k_1-1$ and (5.12), we have

$$\sum_{\ell=1}^{s_{i_0+j_2k_1-1}-1} \beta_\ell^{i_0+j_2k_1-1} \times (\psi_{n-(i_0+j_1k_1)+1}(\delta)1_{E^\circ_{2,p(\ell-1)+1,p(s_{i_0+j_2k_1-1}-1)+1}})(y+\ell-1, q) \equiv 0. \tag{5.13}$$

We use (5.11) with $k = i_0 + j_2k_1 - 1$ and get

$$\sum_{\ell=1}^{s_{i_0+j_2k_1-1}-1} \beta_\ell^{i_0+j_2k_1-1}(\psi_{n-(i_0+j_2k_1)+1}(\delta)1_{A^\circ_{2,\ell,s_{i_0+j_2k_1-1}}})(y+\ell-1, q) \equiv 0, \tag{5.14}$$

which yields

$$\sum_{\ell=1}^{s_{i_0+j_2k_1-1}-1} \beta_\ell^{i_0+j_2k_1-1}(\psi_{n-(i_0+j_1k_1)+1}(\delta)1_{A^\circ_{2,\ell,s_{i_0+j_2k_1-1}}})(y+\ell-1, q) \equiv 0 \tag{5.15}$$

together with (5.13) by Lemma 5.2. The step from (5.12) to (5.15) increases the suffix of ψ by one and decreases the suffixes of s and A by one. So by repeating this step, we have

$$\sum_{\ell=1}^{s_1-1} \beta_\ell^1(\psi_{n+(j_2-j_1)k_1-1}(\delta)1_{A^\circ_{2,\ell,s_1}})(y+\ell-1, q) \equiv 0.$$

Hence by (5.5)

$$\psi_{n+(j_2-j_1)k_1}(\delta)1_{E^\circ_{2,j,s}} = 0.$$

By Lemma 5.1 and $(j_2 - j_1)k_1 \geq 1$, we have

$$\psi_{n+1}(\delta)1_{E^\circ_{2,j,s}} = 0,$$

which ends the proof of the first case.

For the other case where there exists a $k' \in \mathbb{N}$ such that $s_{k'+1} = 1$ and $s_{k'} \geq 2$, the equation $\sum_{\ell=1}^{s_{k'}-1} \beta_\ell^{k'}(\psi_{m'-k'}(\delta)1_{E^\circ_{2,p(\ell-1)+1,p(s_{k'}-1)+1}})(y+\ell-1,q) \equiv 0$ holds for any m' with $m' > n_0 + k'$. So by putting $m' = n+1$, we have

$$\sum_{\ell=1}^{s_{k'}-1} \beta_\ell^{k'}(\psi_{n+1-k'}(\delta)1_{E^\circ_{2,p(\ell-1)+1,p(s_{k'}-1)+1}})(y+\ell-1,q) \equiv 0. \tag{5.16}$$

On the other hand, for $k \in \{1, \ldots, k'\}$, equation (5.11) holds. By using Lemma 5.2 with $n = n - k'$ and $n' = 1$, we have

$$\sum_{\ell=1}^{s_{k'}-1} \beta_\ell^{k'}(\psi_{n+1-k'}(\delta)1_{A^\circ_{2,\ell,s_{k'}}})(y+\ell-1,q) \equiv 0.$$

Repeating the process from (5.12) to (5.15), we get

$$\sum_{\ell=1}^{s_1-1} \beta_\ell^{1}(\psi_{n}(\delta)1_{A^\circ_{2,\ell,s_1}})(y+\ell-1,q) \equiv 0$$

and

$$\psi_{n+1}(\delta)1_{E^\circ_{2,j,s}} = 0.$$

So we have proved the lemma for the case of $(2, j, s) \in \Gamma$.

For the case of $(1, j, s) \in \Gamma$, the lemma can be proved in a similar way. □

We use the following lemma to prove Lemma 5.6.

Lemma 5.5 ([8], Lemma 6.1). *For all $\ell \in \{0, \ldots, p-1\}$ there exist $x \in \mathbb{Z}$ and $t \in \mathbb{Z}_+$ such that $L^t\delta(x) = \ell$ and $0 \leq t \leq p^2$.*

Lemma 5.6. *For sufficiently large n, if $(\psi_n(\delta))^{-1}(\ell) \bigcap E^\circ_\gamma \neq \emptyset$ holds for some $\gamma \in \Gamma$, $\ell \in \{1, \ldots, p-1\}$, then*

$$(\psi_n(\delta))^{-1}(\ell') \bigcap E^\circ_\gamma \neq \emptyset \quad \text{for any } \ell' \in \{1, \ldots, p-1\}.$$

Proof. By Lemma 5.4, there exists an $n_1 \in \mathbb{N}$ such that $\psi_n(\delta)1_{E^\circ_\gamma} = 0$ implies $\psi_{n+1}(\delta)1_{E^\circ_\gamma} = 0$ for any $n \geq n_1$. Put

$$B_1 = \{(x/p^2, t/p^2) \mid 0 \leq t \leq p^2, -k_+t \leq x \leq -k_-t\}.$$

Let $n_2 \in \mathbb{N}$ satisfy $p^{n_2} > (k_0+1)p^2$ and

$$\{(x_1/p^{n_1}, t_1/p^{n_1}) + (y/p^{n_1+n_2-2}, q/p^{n_1+n_2-2}) \mid (y,q) \in B_1\} \subset E_\gamma^\circ \tag{5.17}$$

for any $(x_1/p^{n_1}, t_1/p^{n_1}) \in E_\gamma^\circ \cap X_{n_1}$. For an $n \geq n_1 + n_2$, suppose

$$(\psi_n(\delta))^{-1}(\ell) \bigcap E_\gamma^\circ \neq \emptyset$$

for some $\ell \in \{1, \ldots, p-1\}$. Then $\psi_{n_1}(\delta) 1_{E_\gamma^\circ} \neq 0$ holds by Lemma 5.4. So there exists $(x/p^{n_1}, t/p^{n_1}) \in E_\gamma^\circ$ such that $(\psi_{n_1}(\delta))(x/p^{n_1}, t/p^{n_1}) = \ell'$ for some $\ell' \in \{1, \ldots, p-1\}$. Lemma 2.1 implies

$$(\psi_{n_1+n_2}(\delta))((p^{n_2}x - i)/p^{n_1+n_2}, p^{n_2}t/p^{n_1+n_2}) = 0 \tag{5.18}$$

for $(k_0+1)p^2 \leq i \leq (k_0+1)p^2 (i \neq 0)$ and

$$(\psi_{n_1+n_2}(\delta))((p^{n_2}x)/p^{n_1+n_2}, p^{n_2}t/p^{n_1+n_2}) = \ell'. \tag{5.19}$$

So we have by using Lemma 5.5 with (5.17), (5.18) and (5.19)

$$(\psi_{n_1+n_2}(\delta))^{-1}(\ell'') \bigcap E_\gamma^\circ \neq \emptyset$$

for all $\ell'' \in \{1, \ldots, p-1\}$. So in the same way as the proof of Lemma 5.1, we have

$$(\psi_n(\delta))^{-1}(\ell'') \bigcap E_\gamma^\circ \neq \emptyset$$

for all $\ell'' \in \{1, \ldots, p-1\}$. □

By using these lemmas, we shall prove Proposition 4.4.

Proof of Proposition 4.4.

At first we prove this proposition in case of $m_v = 1$, that is, for $\gamma = (v) \in \Gamma$, $\ell \in \mathbb{Z}_p \backslash \{0\}$ and sufficiently large n, the following are equivalent.

(1-1) $(\psi_{n+1}(\delta))^{-1}(\ell) \cap E_\gamma^\circ \neq \emptyset$.

(2-1) $(\psi_{n+2}(\delta))^{-1}(\ell) \cap E_\gamma^\circ \neq \emptyset$.

(1-1) $\Longrightarrow$ (2-1) is shown in the same way as the proof of Lemma 5.1.

(2-1) $\Longrightarrow$ (1-1) : By Lemma 5.4, for $\gamma \in \Gamma$ and sufficiently large n,

$$\psi_{n+1}(\delta) 1_{E_\gamma^\circ} = 0 \quad \text{implies } \psi_{n+2}(\delta) 1_{E_\gamma^\circ} = 0. \tag{5.20}$$

Suppose $(\psi_{n+2}(\delta))^{-1}(\ell) \cap E_\gamma^\circ \neq \emptyset$. Then $(\psi_{n+2}(\delta)) 1_{E_\gamma^\circ} \neq 0$. So by (5.20), $\psi_{n+1}(\delta) 1_{E_\gamma^\circ} \neq 0$. By Lemma 5.6, we have $(\psi_{n+1}(\delta))^{-1}(\ell) \cap E_\gamma^\circ \neq \emptyset$.

When $m_v \geq 2$, we can prove in a similar way that (2-1) implies (1-1). □

§6. Convergence of $\psi_n(a)$ in case of $\mathbb{R}\times[0,1]$

In Section 4, we have investigated the convergence of $\{\psi_n(\delta)\}$ in the space of $\mathbb{Z}_p$-valued upper semi-continuous functions on a compact subset X_0 of $\mathbb{R}\times[0,1]$. In this section we shall show a similar result of the convergence of $\{\psi_n(a)\}$ for $a\in\mathcal{P}$ with $a(0)\neq 0$ and the relation between the limit function in the pointwise topology and that in the metric D_f. We first define $\{E^n_\gamma(a)\}_\gamma$ corresponding to $\{E_\gamma\}_\gamma$ in Section 4. Put

$$\begin{aligned}\max(a)&=\max\{x\in\mathbb{Z}\mid a(x)\neq 0\},\\ \min(a)&=\min\{x\in\mathbb{Z}\mid a(x)\neq 0\},\\ l(a)&=\max\{i\mid a(i)\neq 0\}-\min\{i\mid a(i)\neq 0\}+1,\\ n_1&=\min\{n\mid k_0+l(a)<p^n\}\end{aligned}$$

and

$$X_a=\{(y,q)\in\mathbb{R}\times[0,1]\mid 0\le q\le 1,-qk_++\min(a)\le y\le -qk_-+\max(a)\}.$$

Then the support of $\psi_n(a)$ is contained in X_a for any $n\in\mathbb{N}$.

Let Γ be the same as that in Section 4, that is,

$$\Gamma=\{(1,j,s)\mid 1\le s\le pk_0,1\le j\le s\}\bigcup\{(2,j,s)\mid 2\le s\le pk_0,1\le j\le s-1\}.$$

For $n\ge n_1+1$ and $\gamma=(1,j,s)\in\Gamma$, put

$$\begin{aligned}E^n_\gamma(a)=\{(y,q)\mid \frac{s-1}{pk_0}\le q\le\frac{s}{pk_0},&\\ -k_+q+\frac{\min(a)}{p^n}+\frac{j-1}{p}\le y\le -k_-q+\frac{\max(a)}{p^n}-\frac{s-j}{p}\}&\end{aligned}$$

and for $n\ge n_1+1$ and $\gamma=(2,j,s)\in\Gamma$, put

$$\begin{aligned}E^n_\gamma(a)=\{(y,q)\mid \frac{s-1}{pk_0}\le q\le\frac{s}{pk_0},&\\ -k_-q+\frac{\min(a)}{p^n}+\frac{s-j}{p}\le y\le -k_-q+\frac{\max(a)}{p^n}+\frac{j}{p}\}.&\end{aligned}$$

Then $X_a\supset\bigcup_{\gamma\in\Gamma}E^n_\gamma(a)$ holds and $\{(E^n_\gamma(a))^\circ\}_\gamma$ are not necessarily mutually disjoint. So for $\gamma\in\Gamma$, put

$$\tilde{E}^n_\gamma(a)=E^n_\gamma(a)\ \setminus\ \bigcup_{\gamma'\neq\gamma}E^n_{\gamma'}(a).$$

Let Γ_γ and V be the same as in Section 4.

For $v=(\gamma_1,\dots,\gamma_m)\in V$, $n\in\mathbb{N}$ and $a\in\mathcal{P}$, define

$$h^n_{a,v}(y,q)=\sum_{\ell_1\in\Lambda_{\gamma_1}}\cdots\sum_{\ell_m\in\Lambda_{\gamma_m}}c_{\ell_1}\dots c_{\ell_m}$$
$$\times(\psi_n(a))(S^{-1}_{\ell_m}\dots S^{-1}_{\ell_1}S_{\ell_{\gamma_1}}\dots S_{\ell_{\gamma_m}}(y,q))1_{S^{-1}_{\ell_{\gamma_m}}(E^n_{\gamma_m}(a))}(y,q),$$

and

$$\tilde{h}^n_{a,v}(y,q)=h^n_{a,v}(y,q)1_{S^{-1}_{\ell_{\gamma_m}}(\tilde{E}^n_{\gamma_m}(a))}(y,q).$$

Since $h^n_{a,v}$ and $\tilde{h}^n_{a,v}$ belong to the space $USC|_{X_a}$, we shall consider metrics d_f, D_f in $USC|_{X_a}$ as in Section 3.

$$M^{n,n'}_a=\sup\{d_f(h^n_{a,v},h^{n'}_{a,v})\mid v\in V\}$$

and

$$\tilde{M}^{n,n'}_a=\sup\{d_f(\tilde{h}^n_{a,v},\tilde{h}^{n'}_{a,v})\mid v\in V\}.$$

Then in a similar way to that in Section 4 we have the following.

Lemma 6.1. *For $n,n'\ge n_1+1$ and $m\ge 1$, we have the following assertions*:

(1) $\tilde{h}^n_{a,v}(F^{-1}_m(y,q))=(\psi_{n+m}(a))(y,q)1_{F_{m-1}(\tilde{E}^n_{\gamma_m}(a))}(y,q)$.

(2) $d_f(\tilde{h}^n_{a,v},\tilde{h}^{n'}_{a,v})=p^m d_f(\psi_{n+m}(a)1_{F_{m-1}(\tilde{E}^n_{\gamma_m}(a))},\psi_{n'+m}(a)1_{F_{m-1}(\tilde{E}^n_{\gamma_m}(a))})$.

(3) $\tilde{M}^{n+1,n'+1}_a\le\frac{1}{p}\tilde{M}^{n,n'}_a$.

(4) $\sup\{\tilde{M}^{n,n'}_a\mid n,n'\in\mathbb{N}\}<\infty$.

The above lemma concerns $\tilde{h}^n_{a,v}$ and $\tilde{M}^{n,n'}_a$, but the convergence of $\{\psi_n(a)\}$ concerns $M^{n,n'}_a$ and $h^n_{a,v}$ as follows.

Lemma 6.2.

$$d_f(\psi_{n+1}(a),\psi_{n'+1}(a))\le\frac{1}{p}M^{n,n'}_a.$$

Proof. We can prove this lemma by using the following relations in a similar way to that in Proposition 4.5.

$$\begin{aligned}
&D(\overline{(\psi_{n+1}(a))^{-1}(\ell)},\overline{(\psi_{n'+1}(a))^{-1}(\ell)})\\
\le\ &\max_{\gamma\in\Gamma}D(\overline{(\psi_{n+1}(a))^{-1}(\ell)\cap E^{n+1}_\gamma(a)},\overline{(\psi_{n'+1}(a))^{-1}(\ell)\cap E^{n+1}_\gamma(a)})\\
\le\ &\max_{\gamma\in\Gamma}d_f(\psi_{n+1}(a)1_{E^{n+1}_\gamma(a)},\psi_{n'+1}(a)1_{E^{n+1}_\gamma(a)})\\
\le\ &\max_{v\in V}\frac{1}{p}d_f(h^n_{a,v},h^{n'}_{a,v}).
\end{aligned}$$

□

As for the relation between $h^n_{a,v}$ and $\tilde{h}^n_{a,v}$, we have

Lemma 6.3. (1) $D(E^n_\gamma(a), \tilde{E}^n_\gamma(a)) \to 0$ *as* $n \to \infty$.

(2) *If* $E^n_\gamma(a) \setminus \tilde{E}^n_\gamma(a) \neq \emptyset$, *then there exists* γ' *in* Γ *such that* $\tilde{E}^n_{\gamma'}(a) \neq \emptyset$ *and* $E^n_\gamma(a) \cap E^n_{\gamma'}(a) \neq \emptyset$.

Using above lemmas, we will get the following theorem in a similar way to that in Theorem 4.1.

Theorem 6.4. *For* $a \in \mathcal{P}$ *with* $a(0) \neq 0$, *the following assertions hold*:

(1) $d_f(\psi_n(a), \psi_m(a)) \to 0$ *as* $n, m \to \infty$.

(2) *Put* $f_a = \bigwedge_{k\geq 1} \bigvee_{n \geq k} \psi_n(a) \in USC$. *Then we have*

$$D_f(\psi_n(a), f_a) \to 0 \qquad as \qquad n \to \infty.$$

We will consider the relation among f_δ in Theorem 4.1, f_a in Theorem 6.4 and the limit set $Y_a = \bigcap_{k=1}^\infty \overline{\bigcup_{n\geq k} \frac{K(n,a)}{p^n}}$.

Let $\hat{g}$ be the upper envelope of g, that is,

$$\hat{g}(y,q) = \inf\{\phi(y,q) \mid \phi \in USC, \phi(y,q) \geq g(y,q)\}.$$

Then the upper envelope of $\hat{g}_a$ of the limit function g_a in the pointwise topology has the relation with the limit set in the sense of Kuratowski limit.

Theorem 6.5. *For* $a \in \mathcal{P}$ *with* $a(0) \neq 0$, *let* $Y_a = \bigcap_{k=1}^\infty \overline{\bigcup_{n\geq k} \frac{K(n,a)}{p^n}}$ *and* g_a *be defined by* $g_a(y,q) = \lim_{n\to\infty} (\psi_n(a))(y,q)$ *in the pointwise topology. Then the following assertions hold*:

(1) *The characteristic function* 1_{Y_a} *of the set* Y_a *satisfies*

$$\hat{g}_a = (p-1)1_{Y_a}$$

and

$$\hat{g}_a = \bigwedge_{k\geq 1} \bigvee_{n\geq k} \psi_n(a).$$

(2) *Though* g_a *is not necessarily the same as* g_δ *for any* $a \in \mathcal{P}$ *as shown in Theorem* 2.5, *the upper envelope* $\hat{g}_a$ *of* g_a *is the same, that is,*

$$\hat{g}_a = \hat{g}_\delta = f_a = f_\delta,$$

where f_a *is defined in Theorem* 6.4.

Proof. (1) Let $(y, q) \in Y_a$. Then there exists a sequence $\{(y_{n_j}, q_{n_j}) \in \frac{K(n_j,\delta)}{p^{n_j}}\}$ such that $\lim_{j\to\infty}(y_{n_j}, q_{n_j}) = (y, q)$. So $g_\delta(y_{n_j}, q_{n_j}) \neq 0$. By Lemma 5.6, there exists a sequence $\{(z_{n_j}, w_{n_j}) \in \frac{K(n_j,a)}{p^{n_j}}\}$ such that $\lim_{j\to\infty}(z_{n_j}, w_{n_j}) = (y, q)$ and $g_a(z_{n_j}, w_{n_j}) = p - 1$. So $\hat{g}_\delta(y, q) = p - 1$. If $(y, q) \notin Y_a$, then there exists a neighborhood U of (y, q) and k such that $U \cap \frac{K(n,a)}{p^n} = \emptyset$ for any $n \geq k$. So $\hat{g}_\delta(y, q) = 0$. Therefore we obtain $(p - 1)1_{Y_a} = \hat{g}_a$.

By Lemma 5.6, we have

$$\bigvee_{n\geq k} \psi_n(a) = (p - 1)1_{Y_{a,k}},$$

where $Y_{a,k} = \overline{\bigcup_{n\geq k} \frac{K(n,a)}{p^n}}$. So $\hat{g}_a = \bigwedge_{k\geq 1} \bigvee_{n\geq k} \psi_n(a)$ holds.

(2) By Theorem 2.5 (3), $Y_\delta = Y_a$. By using (1), we have $f_\delta = \hat{g}_\delta = \hat{g}_a = f_a$. □

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