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AN INTRINSIC VERSION OF HASHIGUCHI– ICHIJYŌ'S THEOREMS FOR WAGNER MANIFOLDS

Cs. Vincze

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Abstract. We present a new conceptual framework for Hashiguchi-Ichijyō's theorems concerning Wagner manifolds and prove them intrinsically.

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0. Introduction

In their joint work [4] M. Hashiguchi and Y. Ichijyō have explored the significance of Wagner manifolds relating them to the conformal changes of Riemann-Finsler metrics. One of the most important observations in [4] is that the class of Wagner manifolds is closed under the conformal change of the metric. In addition to this relevant (geometrical) property there are other aspects accounting for dealing with Wagner manifolds. In his paper [3], Hashiguchi suggested and (in some sense!) solved the problem: under what conditions does a Finsler manifold become conformal to a Berwald (or a locally Minkowski) manifold. "These conditions were, however, given in terms of very complicated systems of differential equations, for which appropriate geometrical meanings have been wanted", he wrote a year later in [4]. As it was shown these "appropriate geometrical meanings" were hidden in the notion of Wagner manifolds. Namely, in the classical terminology: "The condition that a Finsler space be conformal to a Berwald space is that the space becomes a Wagner space with respect to a gradient $\alpha_i(x)$." ([4], Theorem B.)

In the present paper, synthesizing our previous works [7] and [8], we insert these wonderful results in the conceptual and technical framework elaborated by J. Grifone [1], [2] (see also [5], [6] and [8]). We find it natural to consider this paper as a direct continuation of [8], so its terminology and notations will be retained. We shall refer to the results and formulas in [8] by their original numbering, adding a great Roman I/ ... if necessary.

1. Special Finsler connections and Finsler manifolds

Let (M, E) be a Finsler manifold. As it is well-known, the fundamental lemma of Finsler geometry (see [7]) guarantees the existence and uniqueness of a horizontal endomorphism h characterized by the following conditions:

(B1) $d_h E = 0$ (i.e., h is conservative),

(B2) the strong torsion of h vanishes.

Explicitly:

$$h = \frac{1}{2}(1 + [J, S]),$$

where S is the canonical spray and J is the vertical endomorphism, respectively (see e.g. [8]).

This h is called the *Barthel endomorphism* of the Finsler manifold (M, E) .

Theorem 1 ([5]). *Let (M, E) be a Finsler manifold and let h be a horizontal endomorphism on M. There is a unique Finsler connection $(\overset{\circ}{D}, h)$ on M such that*

(01) *the $(v)hv$-torsion $\overset{\circ}{\mathbb{P}}{}^1$ of $\overset{\circ}{D}$ vanishes,*

(02) *the $(h)hv$-torsion $\overset{\circ}{\mathbb{B}}$ of $\overset{\circ}{D}$ vanishes.*

Then $(\overset{\circ}{D}, h)$ is of Berwald-type, *i.e., the covariant derivatives with respect to $\overset{\circ}{D}$ can be calculated by the formulas*

(i) $\overset{\circ}{D}_{JX} JY = J[JX, Y],$

(ii) $\overset{\circ}{D}_{hX} JY = v[hX, JY],$

(iii) $\overset{\circ}{D}_{JX} hY = h[JX, Y],$

(iv) $\overset{\circ}{D}_{hX} hY = hF[hX, JY],$

where $v := 1 - h$, and F is the almost complex structure associated with h.

If $(\overset{\circ}{D}, h)$ satisfies the further conditions

(03) *h is conservative,*

(04) *the h-deflection $h^*(\overset{\circ}{D}C)$ vanishes,*

(05) *the $(h)h$-torsion $\overset{\circ}{\mathbb{A}}$ of $\overset{\circ}{D}$ vanishes,*

then h is just the Barthel endomorphism. In this special case $(\overset{\circ}{D}, h)$ is called the Berwald connection *of the Finsler manifold* (M, E) .

Definition ([6]). A Finsler manifold (M, E) is said to be a *Berwald manifold* if there is a linear connection ∇ on M such that

$$(\nabla_X Y)^v = \left[X^h, Y^v\right] \qquad \text{for } {}^\forall X, Y \in \mathfrak{X}(M)$$

where the horizontal lifting is taken with respect to the Barthel endomorphism.

Definition ([8]). Let (M, E) be a Finsler manifold. The triplet $(\overline{D}, \overline{h}, \alpha)$ is said to be a *Wagner connection* if it satisfies the following conditions:

(W0) $(\overline{D}, \overline{h})$ is a Finsler connection on M,

(W1) $\overline{D}$ is metrical with respect to the prolonged metric $g_{\overline{h}}$,

(W2) the $(v)v$-torsion $\overline{S}^1$ of $\overline{D}$ vanishes,

(W3) $\overline{D}$ is $(h)h$-semisymmetric, i.e., the $(h)h$-torsion $\overline{\mathbb{A}}$ of $\overline{D}$ has the following form:

$$\overline{\mathbb{A}} = d\alpha^v \otimes \overline{h} - \overline{h} \otimes d\alpha^v \qquad \text{for } {}^\forall \alpha \in C^\infty(M),$$

(W4) the h-deflection $\overline{h}^* \; (\overline{D}C)$ vanishes.

Then $\overline{h}$ is called a *Wagner endomorphism* on M.

Theorem 2 ([8]). *Retaining the hypothesis of the preceding definition, the Wagner endomorphism $\overline{h}$ and the Barthel endomorphism h of a Finsler manifold are related as follows:*

$$\overline{h} = h + \alpha^c J - E[J, \operatorname{grad} \alpha^v] - d_J E \otimes \operatorname{grad} \alpha^v. \tag{1}$$

Definition ([8]). Let (M, E) be a Finsler manifold endowed with the Wagner connection $(\overline{D}, \overline{h}, \alpha)$. (M, E) is said to be a *Wagner manifold* (with respect to $(\overline{D}, \overline{h}, \alpha)$) if there is a linear connection $\overline{\nabla}$ on M such that for any vector fields $X, Y \in \mathfrak{X}(M)$

$$\left(\overline{\nabla}_X Y\right)^v = \overline{D}_{X^{\overline{h}}} Y^v,$$

where the horizontal lifting is taken with respect to the Wagner endomorphism.

Theorem 3 ([8]). *Let $(\overline{D}, \overline{h}, \alpha)$ be a Wagner connection on the Finsler manifold (M, E) and let us consider the Finsler connection $(\overset{\circ}{\overline{D}}, \overline{h})$ described in Theorem 1. Then the following assertions are equivalent:*

(i) *(M, E) is a Wagner manifold with respect to $(\overline{D}, \overline{h}, \alpha)$,*

(ii) *the hv-curvature tensor $\overset{\circ}{\overline{\mathbb{P}}}$ of $\overset{\circ}{\overline{D}}$ vanishes.*

Proposition 1. *Let (M, E) be a Wagner manifold with respect to $(\overline{D}, \overline{h}, \alpha)$. Then the following assertions are equivalent:*

(i) $\overline{R} = 0$ *(i.e., $\overline{h}$ is integrable),*

(ii) *the h-curvature tensor $\overline{\mathbb{R}}$ of $\overline{D}$ vanishes,*

(iii) *the h-curvature tensor $\overset{\circ}{\overline{\mathbb{R}}}$ of $\overset{\circ}{\overline{D}}$ vanishes.*

Proof. As it was shown in Proposition 8 of [8], the second Cartan tensor $\overline{\mathcal{C}}'$ be-long–ing to $\overline{h}$ vanishes in any Wagner manifold. This means that the Bianchi identity IV (see [8], Corollary 5) reduces to the formula

$$\begin{aligned} (2) \qquad & \left(\overline{D}_{\overline{h}X}\overline{\mathbb{P}}\right)(Y, Z) - \left(\overline{D}_{\overline{h}Y}\overline{\mathbb{P}}\right)(X, Z) + \left(\overline{D}_{JZ}\overline{\mathbb{R}}\right)(X, Y) \\ &= -\overline{\mathbb{R}}(X, \overline{F}\mathcal{C}(Y, Z)) + \overline{\mathbb{R}}(Y, \overline{F}\mathcal{C}(X, Z)) \\ &- \overline{\mathbb{P}}(\overline{\mathbb{A}}(X, Y), Z) - \overline{\mathbb{Q}}(\overline{F}\,\overline{R}(X, Y), Z). \end{aligned}$$

Substituting a semispray S_0 into (2), by Corollary 4 of [8] we get the relation

$$\left(\overline{D}_{JZ}\overline{R}\right)(X, Y) - \overline{\mathbb{R}}(X, Y)Z = -\overline{R}(X, \overline{F}\mathcal{C}(Y, Z)) + \overline{R}(Y, \overline{F}\mathcal{C}(X, Z)).$$

Therefore the implication (i) $\Rightarrow$ (ii) holds. The converse is an immediate consequence of the relation

$$\overline{\mathbb{R}}(X, Y)S_0 = \overline{R}(X, Y)$$

(see [8], Corollary 4).

Finally, by Proposition 7 of [8],

$$\overline{\mathbb{R}}(X, Y)Z = \overset{\circ}{\overline{\mathbb{R}}}(X, Y)Z + \mathcal{C}(\overline{F}\,\overline{R}(X, Y), Z),$$

so (ii) and (iii) are also equivalent. □

Corollary 1. *If (M, E) is a Berwald manifold, then the following assertions are equivalent:*

(i) $R = 0$ *(i.e., h is integrable),*

(ii) *the h-curvature tensor $\mathbb{R}$ of the Cartan connection (D, h) vanishes,*

(iii) *the h-curvature tensor $\overset{\circ}{\mathbb{R}}$ of the Berwald connection $(\overset{\circ}{D}, h)$ vanishes.*

Definition ([6]). Let (M, E) be a Berwald manifold. If one, and therefore all, of the conditions

$$\text{(i)} \quad R = 0, \qquad \text{(ii)} \quad \mathbb{R} = 0, \qquad \text{(iii)} \quad \overset{\circ}{\mathbb{R}} = 0$$

are satisfied, then (M, E) is called a *locally Minkowski manifold.*

2. Conformal changes of Riemann-Finsler metrics

Definition [7]. Consider two Finsler manifolds (M, E) and $(M, \widetilde{E})$ and denote by g and $\widetilde{g}$ their Riemann-Finsler metrics. We say that g and $\widetilde{g}$ are *conformally equivalent* if there exists a function $\beta \in C^\infty(M)$ satisfying the condition

$$\text{(CE)} \qquad \widetilde{g} = \varphi g \qquad (\varphi := \exp \circ \beta^v = \exp \circ \beta \circ \pi).$$

The function φ is called the *scale function* or the *proportionality function* and $\widetilde{g}$ is called a *conformal change* of the metric g.

Lemma 1. *If a Finsler manifold (M, E) with the Riemann-Finsler metric g, and a function $\beta \in C^\infty(M)$, are given, then $\widetilde{g} = \varphi g$ $(\varphi := \exp \circ \beta^v)$ is the Riemann-Finsler metric of the Finsler manifold $(M, \widetilde{E})$, where $\widetilde{E} := \varphi E$.*

Proof. It is enough to show that the form $\widetilde{\omega} := dd_J \widetilde{E}$ is nondegenerate. Since $\widetilde{E} := \varphi E$ we get immediately the relation

$$\widetilde{\omega} = d\varphi \wedge d_J E + \varphi \omega. \tag{3}$$

Then the following assertions are equivalent:

$$0 = i_X \widetilde{\omega}, \tag{4}$$

$$0 = (X\varphi) d_J E - JX(E) d\varphi + \varphi i_X \omega. \tag{5}$$

Applying both sides of (5) to a vertical vector field JY $(Y \in \mathfrak{X}(TM))$ we have

$$0 = \varphi i_X \omega(JY) := \varphi \omega(X, JY) \overset{\mathrm{I}/(18)}{=} -\varphi g(JX, JY).$$

Therefore $JX = 0$ and thus $X \in \mathfrak{X}^v(TM)$. Hence

$$0 = i_X \widetilde{\omega} \Leftrightarrow 0 = \varphi i_X \omega \Leftrightarrow X = 0,$$

which means that $\widetilde{\omega}$ is nondegenerate.

Finally, for any vector fields $X, Y \in \mathfrak{X}(TM)$,

$$\widetilde{g}(JX, JY) := \widetilde{\omega}(JX, Y) \overset{(3)}{=} \varphi \omega(JX, Y) = \varphi g(JX, JY),$$

i.e., the Riemann-Finsler metrics g and $\widetilde{g}$ are conformally equivalent. □

Proposition 2 ([7]). *Under the conformal change $\widetilde{g} = \varphi g$ $(\varphi = \exp \circ \beta^v)$ of the Riemann-Finsler metric g, the Barthel endomorphisms are related as follows:*

$$\widetilde{h} = h - \frac{1}{2}(\beta^c J + d\beta^v \otimes C) + \frac{1}{2} E[J, \operatorname{grad} \beta^v] + \frac{1}{2} d_J E \otimes \operatorname{grad} \beta^v. \tag{6}$$

Proposition 3. *Let (M, E) be a Finsler manifold and $\alpha \in C^\infty(M)$. Then the tensors*

$$E[J, \operatorname{grad} \alpha^v], \qquad d_J E \otimes \operatorname{grad} \alpha^v$$

are invariant under any conformal changes of the metric g.

Proof. Let us consider the conformal change $\widetilde{g} = \varphi g$ ($\varphi = \exp \circ \beta^v$). Since $\operatorname{grad} \alpha^v \in \mathfrak{X}^v(TM)$ (see [7]), we get from (3) the relation

$$i_{\operatorname{grad} \alpha^v} \widetilde{\omega} = \varphi i_{\operatorname{grad} \alpha^v} \omega = \varphi d\alpha^v \Rightarrow \frac{1}{\varphi} \operatorname{grad} \alpha^v = \widetilde{\operatorname{grad}} \alpha^v.$$

Therefore

$$\widetilde{E}[J, \widetilde{\operatorname{grad}} \alpha^v] = \varphi E[J, \frac{1}{\varphi} \operatorname{grad} \alpha^v] = E[J, \operatorname{grad} \alpha^v]$$

since φ is a vertical lift.

In the same way

$$d_J \widetilde{E} \otimes \widetilde{\operatorname{grad}} \alpha^v = d_J(\varphi E) \otimes \frac{1}{\varphi} \operatorname{grad} \alpha^v = \varphi d_J E \otimes \frac{1}{\varphi} \operatorname{grad} \alpha^v = d_J E \otimes \operatorname{grad} \alpha^v. \quad \square$$

3. Hashiguchi-Ichijyō's theorems for Wagner manifolds

Theorem 4. *Let (M, E) be a Wagner manifold with respect to $(\overline{D}, \overline{h}, \alpha)$ and let us consider the conformal change $\widetilde{g} = \varphi g$ ($\varphi = \exp \circ \beta^v$) of the metric g. Then the Finsler manifold $(M, \widetilde{E})$ is also a Wagner manifold with respect to the Wagner connection induced by $\frac{1}{2}\beta + \alpha \in C^\infty(M)$.*

Proof. Let us consider the Wagner endomorphism $\overline{\overline{h}}$ induced by the function $\frac{1}{2}\beta + \alpha$. We get from the relation (1)

$$\overline{\overline{h}} = \widetilde{h} + \left(\frac{1}{2}\beta + \alpha\right)^c J - \widetilde{E}[J, \widetilde{\operatorname{grad}} \left(\frac{1}{2}\beta + \alpha\right)^v] - d_J \widetilde{E} \otimes \widetilde{\operatorname{grad}} \left(\frac{1}{2}\beta + \alpha\right)^v$$

$$\overset{\text{Prop. 3.}}{=} \widetilde{h} + \left(\frac{1}{2}\beta + \alpha\right)^c J - E[J, \operatorname{grad} \left(\frac{1}{2}\beta + \alpha\right)^v] - d_J E \otimes \operatorname{grad} \left(\frac{1}{2}\beta + \alpha\right)^v$$

$$= \widetilde{h} + \frac{1}{2}\beta^c J - \frac{1}{2} E[J, \operatorname{grad} \beta^v] - \frac{1}{2} d_J E \otimes \operatorname{grad} \beta^v$$

$$+ \alpha^c J - E[J, \operatorname{grad} \alpha^v] - d_J E \otimes \operatorname{grad} \alpha^v \overset{(6)}{=} h - \frac{1}{2} d\beta^v \otimes C + \alpha^c J$$

$$- E[J, \operatorname{grad} \alpha^v] - d_J E \otimes \operatorname{grad} \alpha^v \overset{(1)}{=} \overline{h} - \frac{1}{2} d\beta^v \otimes C.$$

Using this form of $\overline{\overline{h}}$, we easily obtain that the hv-curvature tensor of the Berwald-type connection associated with $\overline{\overline{h}}$ vanishes (see Theorem 1). This means that $(M, \widetilde{E})$ is a Wagner manifold. $\square$

Definition. A Finsler manifold (M, E) is said to be *conformal* to a Berwald (or a locally Minkowski) manifold if there is a conformal change $\widetilde{g} = \varphi g$ ($\varphi = \exp \circ \beta^v$) such that $(M, \widetilde{E})$ is a Berwald (or a locally Minkowski) manifold.

Theorem 5. *A Finsler manifold is conformal to a Berwald manifold if and only if it is a Wagner manifold with respect to a gradient vector field α.*

Proof. Let us suppose that the Finsler manifold (M, E) is conformal to a Berwald manifold, i.e., there is a conformal change $\widetilde{g} = \varphi g$ ($\varphi = \exp \circ \beta^v$) such that $(M, \widetilde{E})$ is a Berwald manifold. Since the Berwald manifolds are, in particular, Wagner manifolds (c.f. [8], Prop. 6), in view of Theorem 4, the conformal change $g = \frac{1}{\varphi}\widetilde{g}$ yields a Wagner manifold with respect to the Wagner connection induced by $-\frac{1}{2}\beta \in C^\infty(M)$. Explicitly, the Wagner endomorphism $\overline{h}$ and the Barthel endomorphism $\widetilde{h}$ of the Berwald manifold $(M, \widetilde{E})$ are related as follows:

$$\overline{h} = \widetilde{h} + \frac{1}{2} d\beta^v \otimes C.$$

Conversely, let us suppose that (M, E) is a Wagner manifold with respect to $(\overline{D}, \overline{h}, \alpha)$ where α is a gradient. Then, in view of Theorem 4, the conformal change $\widetilde{g} = \varphi g$ ($\varphi := \exp \circ \beta^v$, $\beta := -2\alpha$) yields a Wagner manifold whose Wagner connection is induced by the function $\frac{1}{2}\beta + \alpha = -\alpha + \alpha = 0$. Therefore (c.f. [8], Prop. 6) $(M, \widetilde{E})$ is a Berwald manifold. The Barthel endomorphism $\widetilde{h}$ and the Wagner endomorphism $\overline{h}$ of the Wagner manifold (M, E) are related as follows:

$$\widetilde{h} = \overline{h} + d\alpha^v \otimes C. \qquad \square$$

Theorem 6. *A Finsler manifold is conformal to a locally Minkowski manifold if and only if it is a Wagner manifold and one (therefore all) of the conditions*

$$\text{(i)} \quad \overline{R} = 0, \qquad \text{(ii)} \quad \overline{\mathbb{R}} = 0, \qquad \text{(iii)} \quad \overset{\circ}{\overline{\mathbb{R}}} = 0$$

are satisfied.

Proof. Let us suppose that the Finsler manifold (M, E) is conformal to a locally Minkowski manifold, i.e., there is a conformal change $\widetilde{g} = \varphi g$ ($\varphi = \exp \circ \beta^v$) such that $(M, \widetilde{E})$ is a locally Minkowski manifold. Then, in view of Theorem 5, (M, E) is a Wagner manifold with respect to the Wagner connection induced by the function $-\frac{1}{2}\beta \in C^\infty(M)$ and

$$\overline{h} = \widetilde{h} + \frac{1}{2} d\beta^v \otimes C.$$

Since $(M, \widetilde{E})$ is a locally Minkowski manifold, $\widetilde{R} := -\frac{1}{2}[\widetilde{h}, \widetilde{h}] = 0$. This implies by an easy (but little lengthy) calculation, that $\overline{R} := -\frac{1}{2}[\overline{h}, \overline{h}]$ also vanishes.

Conversely, if (M, E) is a Wagner manifold with respect to $(\overline{D}, \overline{h}, \alpha)$ then, in view of Theorem 5, the conformal change $\widetilde{g} = \varphi g (\varphi := \exp \circ \beta^v,\ \beta := -2\alpha)$ yields a Berwald manifold with the Barthel endomorphism $\widetilde{h}$ such that

$$\widetilde{h} = \overline{h} + d\alpha^v \otimes C.$$

Now applying the further condition $\overline{R} = 0$, we get:

$$\begin{aligned}
\widetilde{R} &:= -\frac{1}{2}[\widetilde{h}, \widetilde{h}] = -\frac{1}{2}\big([\overline{h}, \overline{h}] + 2[\overline{h}, d\alpha^v \otimes C] \\
&\quad + [d\alpha^v \otimes C, d\alpha^v \otimes C]\big) = -[\overline{h}, d\alpha^v \otimes C] - \frac{1}{2}[d\alpha^v \otimes C, d\alpha^v \otimes C] \\
&\overset{\mathrm{I}/(6)}{=} -d_{\overline{h}} d\alpha^v \otimes C + d\alpha^v \otimes [\overline{h}, C] - \frac{1}{2} d_{d\alpha^v \otimes C} d\alpha^v \otimes C \\
&\quad + \frac{1}{2} d\alpha^v \wedge [d\alpha^v \otimes C, C] \overset{\mathrm{I/Cor.\ 1.}}{=} \frac{1}{2} d\alpha^v \wedge [d\alpha^v \otimes C, C] \overset{\mathrm{I}/(5)}{=} 0. \qquad \square
\end{aligned}$$

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Cs. Vincze
Institute of Mathematics and Informatics
Lajos Kossuth University
H-4010 Debrecen, P.O. Box 12
Hungary