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SOME CLASS OF INFINITE DIMENSIONAL HYPERBOLIC SYSTEMS OF PDES

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Abstract. The initial value problem for a system of PDEs of the form

$$(*) \qquad \frac{\partial u_i(t,x)}{\partial t} + \frac{\partial u_i(t,x)}{\partial x}\lambda_i(t,x,u(t,x)) = f_i(t,x,u(t,x)), \quad 1 \le i \le N.$$

is solved. In $(*)$ t is a real variable. x is a variable moving in a real Banach space X. $u_1, \dots, u_N$ are unknown functions whose values are in a real Banach space G. u denotes the 'vector' $(u_1, \dots, u_N)$. Differentiation in x is taken in the sense of the Fréchet derivative. $f_1, \dots, f_N$ are G-valued given functions and $\lambda_1, \dots, \lambda_N$ are X-valued given functions.

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§1. Introduction

In the old textbook [6] on partial differential equations by Petrovsky there is a section with the title "*Cauchy problem for hyperbolic systems in two independent variables*". He considers there an initial value problem of the form

$$(1.1) \quad \frac{\partial u_i(t,x)}{\partial t} + \lambda_i(t,x)\frac{\partial u_i(t,x)}{\partial x} = \sum_{j=1}^{N} a_{ij}(t,x)u_j(t,x) + b_i(t,x), \ 1 \le i \le N,$$

$$(1.2) \quad u_i(0,x) = \varphi_i(x), \quad 1 \le i \le N.$$

In (1.1) t and x are variables moving in the real line $\mathbf{R}$. $u_1, \cdots, u_N$ are $\mathbf{R}$-valued unknown functions. λ_i, a_{ij} and b_i are given functions. According to the terminology of Petrovsky the system (1.1) is of the standard form of the first order linear hyperbolic system with two independent variables. It is shown in [6] that the problem (1.1)-(1.2) has a unique (local) C^1-solution under the assumption that the given functions λ_i, a_{ij}, b_i and φ_i are in the C^1 class. In a footnote of the book it is noted that the above result can be easily generalized to the case where the linear system (1.1) is replaced by the 'semi-linear' system

$$(1.3) \qquad \frac{\partial u_i(t,x)}{\partial t} + \lambda_i(t,x)\frac{\partial u_i(t,x)}{\partial x} = f_i(t,x,u(t,x)), \quad 1 \leq i \leq N,$$

where $u(t,x) = (u_1(t,x), \ldots, u_N(t,x))$.

In some other textbooks on partial differential equations such as Garabedian [3] and Williams [7] a more general problem is treated. In [3], [7] the semi-linear system (1.3) is replaced by the 'quasi-linear' system

$$(1.4) \quad \frac{\partial u_i(t,x)}{\partial t} + \lambda_i(t,x,u(t,x))\frac{\partial u_i(t,x)}{\partial x} = f_i(t,x,u(t,x)), \quad 1 \leq i \leq N.$$

The author of the present note believes that the initial value problem for a system of the form (1.4) was first solved by Friedrichs[2]. It was proved in [2] that the initial value problem (1.3)-(1.2) has a unique local C^1-solution, if the functions λ_i, f_i and φ_i are C^2-functions. See Theorem 7.2 of [2]. Lax [5] treated a similar problem mainly within the framework of analytic functions.

By the monogragh [4] of Jeffrey we know that the above mentioned initial value problem has good applications in some fields of physics such as gas dynamics.

The purpose of this note is to consider a problem that is more general than those treated in [2], [3] etc. Here we want to replace the real variable x in (1.4) by a variables moving in an arbitrary real Banach space X. We also want to consider the case where the values of the unknown functions $u_1, \cdots, u_N$ are in another arbitrary real Banach space G. We take the partial derivatives $\partial u_i(t,x)/\partial x$ in the sense of the theory of Fréchet derivatives. Note first, however, that it is difficult for us to consider the problem with no change in the form of the differential equations. This is because of the difference between the meaning of a partial derivative in the ordinary sense and that in the sense of the theory of Fréchet derivatives. In the theory of Fréchet derivatives the value of the partial derivative $\partial u_i(t,x)/\partial x$ is not a point of the space G, but a bounded linear operator from X into G. If we denote the set of all bounded linear operator from X into G by, according to Dieudonné[1], $\mathcal{L}(X;G)$, we can say that the partial derivative $\partial u_i(t,x)/\partial x$ is in the space $\mathcal{L}(X;G)$. In this interpretation of partial derivatives, therefore, the value of the second term $\lambda_i(t,x,u(t,x))\dfrac{\partial u_i(t,x)}{\partial x}$ in (1.4) is not in the space G, if we take the value of the 'coefficient' $\lambda_i(t,x,u(t,x))$ as a mere number. This difficulty can be overcome, if we consider $\lambda_i(t,x,u(t,x))$ to denote a point of the space

$\mathcal{L}(\mathcal{L}(X;G);G)$. However this interpretation of the coefficient $\lambda_i(\cdots)$ is a little cumbersome. So we take a little bit different way. We assume that the value of $\lambda_i(\cdots)$ is in the space X and consider, instead of (1.4), the system of the form

$$\frac{\partial u_i(t,x)}{\partial t} + \frac{\partial u_i(t,x)}{\partial x}\lambda_i(t,x,u(t,x)) = f_i(t,x,u(t,x)), \quad 1 \le i \le N. \tag{1.5}$$

If $\lambda_i(\cdots) \in X$, then the *product* $\dfrac{\partial u_i(t,x)}{\partial x}\lambda_i(\cdots)$ becomes an element of G and the left hand side of (1.5) as a whole becomes an element of G.

In what follows we denote by ∂_i the partial differentiation with respect to the ith variable for a function of several variables. By this notational convention the system (1.5) is rewritten as

$$\partial_1 u_i(t,x) + \partial_2 u_i(t,x)\lambda_i(t,x,u(t,x)) = f_i(t,x,u(t,x)), \quad 1 \le i \le N. \tag{1.6}$$

We want to solve the system (1.6) under the initial condition (1.2). But, as is usual, there is no loss of generality in considering the homogeneous initial condition

$$u_i(0,x) \equiv 0, \quad 1 \le i \le N, \tag{1.7}$$

only, instead of the general condition (1.2). In this note we consider, therefore, the condition of the form (1.7) only as the initial condition.

To sum up, we want to solve the initial value problem (1.6)-(1.7) with respect to the G-valued unknown functions $u_1, \ldots, u_N$ of the variable $(t,x) \in \mathbf{R} \times X$. In what follows we write IVP instead of 'initial value problem'. So our purpose is to solve the IVP (1.6)-(1.7).

In the next section, §2, we solve the IVP (1.6)-(1.7) in the 'semi-linear case', i.e., in the case where $\lambda_i(t,x,u)$'s do not depend on u. This problem is solved by a method similar to the one in Petrovsky[6], i.e., the method using 'the characteristic curves'. A result for the semi-linear IVP will be stated in §2 as Theorem 2.1. In §3 we consider a system of the form

$$\partial_1 u_i(t,x) + \partial_2 u_i(t,x)\mu_i(t,x,v(t,x)) = f_i(t,x,u(t,x)), \quad 1 \le i \le N, \tag{1.8}$$

where v is a G^N-valued given function. This system is still semi-linear, not quasi-linear. So the existence of the solution of the IVP (1.8)-(1.7) directly follows from the result of §2. Our purpose of §3 is to investigate how the solution u of this IVP depends on the given function v. Using the results in §3, the quasi-linear IVP (1.6)-(1.7) will be solved in §4. The final result of this note will be stated in §4 as Theorem 4.1. It is hoped that this result may have applications such as in Jeffrey's book in a wider context.

For the possibilities of further extending the result of this note see the remark at the end of §4.

§2. Solution of a Semi-linear IVP

In this section we want to solve the IVP (1.8)-(1.7) in the semi-linear case. In other words we want to solve here an IVP of the form

$$\partial_1 u_i(t,x) + \partial_2 u_i(t,x)\lambda_i(t,x) = f_i(t,x,u(t,x)), \quad 1 \le i \le N, \tag{2.1}$$

$$u(0,x) \equiv 0. \tag{2.2}$$

In (2.1) t is a real variable. x is a variable moving in a real Banach space X. $u_1, \dots, u_N$ are unknown functions whose values are in a real Banach space G.

Now, before starting detailed discussions on the system (2.1), we want to simplify the expression of the system. We write

$$\partial_i u(t,x) = (\partial_i u_1(t,x), \dots, \partial_i u_N(t,x)) \quad (i = 1,2), \tag{2.3}$$

$$\lambda(t,x) = (\lambda_1(t,x), \dots, \lambda_N(t,x)), \tag{2.4}$$

$$f(t,x,u) = (f_1(t,x,u), \dots, f_N(t,x,u)) \tag{2.5}$$

and

$$\partial_2 u(t,x) \odot \lambda(t,x) = (\partial_2 u_1(t,x)\lambda_1(t,x), \dots, \partial_2 u_N(t,x)\lambda_N(t,x)). \tag{2.6}$$

Then the system (2.1) is rewritten as the single equation

$$\partial_1 u(t,x) + \partial_2 u(t,x) \odot \lambda(t,x) = f(t,x,u(t,x)). \tag{2.7}$$

We seek a solution of the IVP (2.7)-(2.2). In order to state and prove a result on this IVP we need to make some more notational agreement. If A, R and L are positive constants, we write

$$\mathcal{D}(A,R) = \{(t,x) \in \mathbf{R} \times X \ ; \ 0 \le t \le A, \ \|x\| < R\},$$
$$\Omega(R,L,A) = \{(t,x) \in \mathbf{R} \times X \ ; \ 0 \le t \le A, \ \ \|x\| < R - Lt\},$$
$$\tilde{\Omega}(R,L,A) = \{(t,t_0,x_0) \in \mathbf{R} \times \mathbf{R} \times X \ ; \ (t_0,x_0) \in \Omega(R,L,A), \ 0 \le t \le t_0\}.$$

Further, if S is a positive constant or $S = \infty$, then we write

$$\mathcal{W}(A,R,S) = \{(t,x,u) \in \mathbf{R} \times X \times G \ ; \ 0 \le t \le A, \ \|x\| < R, \ \|u\| < S\}.$$

Next, if Y is a normed space and h is a Y-valued function defined in a set $\mathcal{S}$, then let us agree to write

$$\|h\|_{\mathcal{S}} = \sup_{z \in \mathcal{S}} \|h(z)\|.$$

Using the above notational agreement, the main result in this section is stated as the following theorem.

Theorem 2.1. *Let X, G be real Banach spaces. Let A, R, L_0, L_1, K_0 and K_1 be positive constants. Let S be a positive constant or $S = \infty$. Let N be a positive integer. Let $\lambda(t,x) = (\lambda_1(t,x), \dots, \lambda_N(t,x))$ be an X^N-valued C^1-function of $(t,x) \in \mathcal{D}(A,R)$ and $f(t,x,u)$ a G^N-valued C^1-function of $(t,x,u) \in \mathcal{W}(A,R,S)$. Assume that*

$$(2.8)\qquad \|\lambda\|_{\mathcal{D}(A,R)}{}^{1)} \le L_0, \quad \|\partial_2\lambda\|_{\mathcal{D}(A,R)} \le L_1,$$

$$(2.9)\qquad \|f\|_{\mathcal{W}(A,R,S)} \le K_0, \quad \|\partial_j f\|_{\mathcal{W}(A,R,S)} \le K_1 \ (j = 2,3).$$

Assume further that $\partial_2 f$ and $\partial_3 f$ are uniformly continuous. Let B be a positive constant satisfying

$$(2.10)\qquad B \le \min\left\{A, \frac{1}{2L_1}, \frac{S}{2K_0}, \frac{1}{4K_1}\right\}.$$

Then the IVP (2.7)-(2.2) has a unique C^1-solution $u : \Omega(R, L_0, B) \to G^N$.

Proof. In order to solve the IVP (2.7)-(2.2) we first consider the following N ordinary differential equations:

$$(2.11)\qquad \frac{d}{dt}\xi(t) = \lambda_i(t, \xi(t)), \quad 1 \le i \le N.$$

$\xi(t)$ in (2.11) is an X-valued unknown function of $t \in \mathbf{R}$. It is necessary for us to solve the above differential equations under the initial condition

$$(2.12)\qquad \xi(t_0) = x_0,$$

where (t_0, x_0) is a given point in the set $\mathcal{D}(A,R)$. Since λ_i is a C^1-function, the IVP (2.11)-(2.12) has a unique C^1-solution in a neighborhood of t_0. We denote it by $\xi_i(t, t_0, x_0)$. For any given point (t_0, x_0) in $\mathcal{D}(A,R)$ the map $t \mapsto \xi_i(t, t_0, x_0)$ is defined (at least) for t such that

$$(2.13)\qquad 0 \le t \le t_0, \quad L_0|t - t_0| \le (R - \|x_0\|).$$

In particular, if $(t_0, x_0) \in \Omega(R, L_0, A)$, then $\xi_i(t, t_0, x_0)$ is defined for all $t \in [0, t_0]$. Therefore $\xi_i(t, t_0, x_0)$ is defined for all $(t, t_0, x_0) \in \tilde{\Omega}(R, L_0, A)$ and satisfies

$$(2.14)\ \|\xi_i(t, t_0, x_0)\| \le \|x_0\| + L_0(t_0 - t) \le R - L_0 t_0 + L_0(t_0 - t) = R - L_0 t.$$

This means that $(t, \xi_i(t, t_0, x_0)) \in \Omega(R, L_0, A)$ for all $(t, t_0, x_0) \in \tilde{\Omega}(R, L_0, A)$. Further, by a well-known theorem on the smooth dependence on the initial

1) $\|\lambda\|_{\mathcal{D}(A,R)} = \max_{1 \le i \le N} \|\lambda_i\|_{\mathcal{D}(A,R)}$. Other similar symbols such as $\|f\|_{\mathcal{W}(A,R,S)}$, $\|\partial_2\lambda\|_{\mathcal{D}(A,R)}$ etc. are interpreted similarly.

condition of the solution of an ordinary differential equation, we know that $\xi_i(t, t_0, x_0)$ is a C^1-function of $(t, t_0, x_0) \in \tilde{\Omega}(R, L_0, A)$. For our later purpose we need to estimate the magnitude of $\|\partial_3 \xi_i(t, t_0, x_0)\|$. Since $\xi_i(t, t_0, x_0)$ is the solution of the IVP (2.11)-(2.12), we have, for $(t, t_0, x_0) \in \tilde{\Omega}(R, L_0, B)$,

$$\xi_i(t, t_0, x_0) = x_0 + \int_{t_0}^{t} \lambda_i(\sigma, \xi_i(\sigma, t_0, x_0))\, d\sigma \tag{2.15}$$

and

$$\partial_3 \xi_i(t, t_0, x_0) = 1_X + \int_{t_0}^{t} \partial_2 \lambda_i(\sigma, \xi_i(\sigma, t_0, x_0)) \partial_3 \xi_i(\sigma, t_0, x_0)\, d\sigma, \tag{2.16}$$

where 1_X is the identity map in the space X. From (2.16) and (2.10) we obtain

$$\begin{aligned} \|\partial_3 \xi_i(t, t_0, x_0)\| &\le 1 + \int_{t}^{t_0} \|\partial_2 \lambda_i(\sigma, \xi_i(\sigma, t_0, x_0))\| \cdot \|\partial_3 \xi_i(\sigma, t_0, x_0)\|\, d\sigma \\ &\le 1 + L_1 B \|\partial_3 \xi_i\|_{\tilde{\Omega}(R, L_0, B)} \le 1 + 2^{-1} \|\partial_3 \xi_i\|_{\tilde{\Omega}(R, L_0, B)} \end{aligned}$$

and

$$\|\partial_3 \xi_i\|_{\tilde{\Omega}(R, L_0, B)} \le 2. \tag{2.17}$$

Now assume that $u(t, x) = (u_1(t, x), \ldots, u_N(t, x))$ is a C^1-function of $(t, x) \in \Omega(R, L_0, B)$ and satisfies the partial differential equation (2.7). Then we have

$$\begin{aligned} \frac{\partial}{\partial t}\{u_i(t, \xi_i(t, t_0, x))\}^{2)} &= \partial_1 u_i(t, \xi_i(\cdots)) + \partial_2 u_i(t, \xi_i(\cdots)) \partial_1 \xi_i(\cdots) \\ &= \partial_1 u_i(\cdots) + \partial_2 u_i(\cdots) \lambda_i(\cdots) = f_i(t, \xi_i(t, t_0, x), u(t, \xi_i(\cdots))). \end{aligned} \tag{2.18}$$

If, further, $u(t, x)$ satisfies the initial condition (1.7), then we have

$$u_i(0, \xi_i(0, t_0, x)) = 0. \tag{2.19}$$

From (2.18) and (2.19) we obtain

$$u_i(t, \xi_i(t, t_0, x)) = \int_0^t f_i(\sigma, \xi_i(\sigma, t_0, x), u(\sigma, \xi_i(\sigma, t_0, x)))\, d\sigma. \tag{2.20}$$

It is easy to show, conversely, that a C^1-function $u : \Omega(R, L_0, B) \to G^N$ satisfying (2.20) becomes a solution of the IVP (2.7)-(2.2). Therefore, seeking a C^1-solution in $\Omega(R, L_0, B)$ of the IVP (2.7)-(2.2) is the same as seeking C^1-functions $u_i : \Omega(R, L_0, B) \to G, 1 \le i \le N$, that satisfy the system (2.20) of integral equations.

Letting $t_0 = t$ in (2.20), we obtain

$$u_i(t, x) = \int_0^t f_i(\sigma, \xi_i(\sigma, t, x), u(\sigma, \xi_i(\sigma, t, x)))\, d\sigma. \tag{2.21}$$

[2)] Note that $(t, \xi_i(t, t_0, x)) \in \Omega(R, L_0, B)$ if $(t, t_0, x) \in \tilde{\Omega}(R, L_0, B)$. This is seen by (2.14).

Conversely, suppose that (2.21) is satisfied for all $(t,x)\in\Omega(R,L_0,B)$. Then we have

(2.22)
$$u_i(t,\xi_i(t,t_0,x))=\int_0^t f_i(\sigma,\xi_i(\sigma,t,\xi_i(t,t_0,x)),u(\sigma,\xi_i(\sigma,t,\xi_i(t,t_0,x))))\,d\sigma$$

for $(t,t_0,x)\in\tilde{\Omega}(R,L_0,B)$. From (2.22) and the relation

$$\xi_i(\sigma,t,\xi_i(t,t_0,x))=\xi_i(\sigma,t_0,x) \tag{2.23}$$

we obtain (2.20). We have thus shown that seeking a C^1-solution in $\Omega(R,L_0,B)$ of the IVP (2.7)-(2.2) is the same as seeking C^1-functions $u_i:\Omega(R,L_0,B)\to G, 1\le i\le N$, that satisfy the system (2.21) of integral equations.

The system (2.21) of integaral equations is solved as follows. If a function $u:\Omega(R,L_0,B)\to G^N$ is continuous and has the continuous partial derivative $\partial_2 u:\Omega(R,L_0,B)\to\mathcal{L}(X;G^N)$, then we say that u is a G^N-valued $C^{(0,1)}$-function in $\Omega(R,L_0,B)$ and denote by $\mathcal{F}_B$ the set of all G^N-valued $C^{(0,1)}$-functions in $\Omega(R,L_0,B)$. Further we write

$$\Gamma_B=\{u\in\mathcal{F}_B\ ;\ \|u\|_{\Omega(R,L_0,B)}\le S/2,\ \ \|\partial_2 u\|_{\Omega(R,L_0,B)}\le 1\}.$$

For each element $u\in\Gamma_B$ we can define an element $w=(w_1,\ldots,w_N)\in\mathcal{F}_B$ by

$$w_i(t,x)=\int_0^t f_i(\sigma,\xi_i(\sigma,t,x),u(\sigma,\xi_i(\sigma,t,x)))\,d\sigma. \tag{2.24}$$

We denote the map $u\mapsto w$ by Φ. If $w=\Phi u$, then we have

$$\|w_i(t,x)\|\le\int_0^t\|f_i(\sigma,\xi_i(\sigma,t,x),u(\sigma,\xi_i(\sigma,t,x)))\|\,d\sigma\le K_0t\le K_0B$$

for $(t,x)\in\Omega(R,L_0,B)$. Therefore we have

$$\|\Phi u\|_{\Omega(R,L_0,B)}\le K_0B\le S/2. \tag{2.25}$$

In order to estimate $\|\partial_2(\Phi u)\|_{\Omega(R,L_0,B)}$ we differentiate (2.24) in x. Then we obtain

$$\begin{aligned}\partial_2 w_i(t,x)&=\int_0^t\partial_2 f_i(\sigma,\xi_i(\sigma,t,x),u(\sigma,\xi_i(\sigma,t,x)))\partial_3\xi_i(\sigma,t,x)\,d\sigma\\&\quad+\int_0^t\partial_3 f_i(\sigma,\xi_i(\cdots),u(\sigma,\xi_i(\cdots)))\partial_2 u(\sigma,\xi_i(\cdots))\partial_3\xi_i(\cdots)\,d\sigma.\end{aligned} \tag{2.26}$$

From (2.26), (2.17) and (2.9) we obtain

$$\|\partial_2 w_i(t,x)\|\le\{K_1\cdot 2+K_1\cdot 1\cdot 2\}t\le 4K_1B\le 1$$

and

$$\|\partial_2(\Phi u)\|_{\Omega(R,L_0,B)} \leq 1. \tag{2.27}$$

(2.25) and (2.27) show that Φ maps the set Γ_B into itself.

Therefore we can construct a sequence $\{u^k\}_{k=0,1,\ldots}$ of elements of Γ_B by setting $u^0(t,x) \equiv 0$ and $u^k = \Phi u^{k-1}$ for $k = 1, 2, \ldots$. If we write $u^k = (u_1^k, \ldots, u_N^k)$, we have

$$u_i^{k+1}(t,x) = \int_0^t f_i(\sigma, \xi_i(\sigma,t,x), u^k(\sigma, \xi_i(\sigma,t,x)))d\sigma. \tag{2.28}$$

From (2.28) we obtain

$$\begin{aligned} &u_i^{k+1}(t,x) - u_i^k(t,x) \\ &= \int_0^t \{f_i(\sigma, \xi_i(\cdots), u^k(\sigma, \xi_i(\cdots))) - f_i(\sigma, \xi_i(\cdots), u^{k-1}(\sigma, \xi_i(\cdots)))\}d\sigma. \end{aligned} \tag{2.29}$$

From (2.29) and the inequality

$$\begin{aligned} &\|f_i(\sigma, \xi_i(\cdots), u^k(\sigma, \xi_i(\cdots))) - f_i(\sigma, \xi_i(\cdots), u^{k-1}(\sigma, \xi_i(\cdots)))\| \\ &\leq \|\partial_3 f\|_{\mathcal{W}(A,R,S)} \cdot \|u^k - u^{k-1}\|_{\Omega(R,L_0,B)} \leq K_1\|u^k - u^{k-1}\|_{\Omega(R,L_0,B)} \end{aligned}$$

we obtain

$$\|u_i^{k+1}(t,x) - u_i^k(t,x)\| \leq K_1 t\|u^k - u^{k-1}\|_{\Omega(R,L_0,B)} \leq K_1 B\|u^k - u^{k-1}\|_{\Omega(\cdots)}$$

and, using (2.10),

$$\|u^{k+1} - u^k\|_{\Omega(R,L_0,B)} \leq K_1 B\|u^k - u^{k-1}\|_{\Omega(\cdots)} \leq 4^{-1}\|u^k - u^{k-1}\|_{\Omega(\cdots)}. \tag{2.30}$$

It follows that

$$\lim_{j\to\infty} \sup_{k,l\geq j} \|u^k - u^l\|_{\Omega(R,L_0,B)} = 0 \tag{2.31}$$

and that the sequence $\{u^k\}$ of functions $u^k : \Omega(R, L_0, B) \to G^N$ converges to a continuous function $u^\infty : \Omega(R, L_0, B) \to G^N$ uniformly in $\Omega(R, L_0, B)$. Letting $k \to \infty$ in (2.28), we see that $u^\infty = (u_1^\infty, \ldots, u_N^\infty)$ satisfies

$$u_i^\infty(t,x) = \int_0^t f_i(\sigma, \xi_i(\sigma,t,x), u^\infty(\sigma, \xi_i(\sigma,t,x)))d\sigma. \tag{2.32}$$

Next let us see that the sequence $\{\partial_2 u^k\}$ converges. For this purpose we differentiate (2.28) in x. Then we obtain, just like (2.26),

$$\begin{aligned} \partial_2 u_i^{k+1}(t,x) &= \int_0^t \partial_2 f_i(\sigma, \xi_i(\sigma,t,x), u^k(\sigma, \xi_i(\sigma,t,x)))\partial_3\xi_i(\sigma,t,x)\, d\sigma \\ &+ \int_0^t \partial_3 f_i(\sigma, \xi_i(\cdots), u^k(\sigma, \xi_i(\cdots)))\partial_2 u^k(\sigma, \xi_i(\cdots))\partial_3\xi_i(\cdots)\, d\sigma. \end{aligned} \tag{2.33}$$

From (2.33) we obtain

$$
\begin{aligned}
(2.34)\ &\|\partial_2 u_i^{k+1}(t,x) - \partial_2 u_i^{l+1}(t,x)\| \\
&\leq \int_0^t \|\partial_2 f_i(\cdots, u^k(\cdots)) - \partial_2 f_i(\cdots, u^l(\cdots))\| \cdot \|\partial_3 \xi_i\|_{\tilde{\Omega}(R,L_0,B)}\, d\sigma \\
&\quad + \int_0^t \|\partial_3 f_i(\cdots, u^k(\cdots)) - \partial_3 f_i(\cdots, u^l(\cdots))\| \cdot \|\partial_2 u^k\|_{\Omega(\cdots)} \cdot \|\partial_3 \xi_i\|_{\tilde{\Omega}(\cdots)}\, d\sigma \\
&\quad + \int_0^t \|\partial_3 f_i(\cdots, u^l(\cdots))\| \cdot \|\partial_2 u^k - \partial_2 u^l\|_{\Omega(\cdots)} \cdot \|\partial_3 \xi_i\|_{\tilde{\Omega}(\cdots)}\, d\sigma \\
&\leq 2\int_0^t \|\partial_2 f_i(\cdots, u^k(\cdots)) - \partial_2 f_i(\cdots, u^l(\cdots))\|\, d\sigma \\
&\quad +2\int_0^t \|\partial_3 f_i(\cdots, u^k(\cdots)) - \partial_3 f_i(\cdots, u^l(\cdots))\|\, d\sigma \\
&\quad +2K_1\|\partial_2 u^k - \partial_2 u^l\|_{\Omega(R,L_0,B)} \cdot t
\end{aligned}
$$

Now suppose an $\varepsilon > 0$ is given. Then, by the assumption that the functions $\partial_2 f$ and $\partial_3 f$ are uniformly continuous, there is a $\delta > 0$ such that, u and v are in Γ_B and $\|u - v\|_{\Omega(R,L_0,B)} < \delta$, then the inequality

$$
(2.35)\quad \|\partial_j f_i(\sigma, \xi_i(\cdots), u(\sigma, \xi_i(\cdots))) - \partial_j f_i(\sigma, \xi_i(\cdots), v(\sigma, \xi_i(\cdots)))\| < \varepsilon
$$

holds for all (σ, t, x). Therefore, if $\|u^k - u^l\|_{\Omega(R,L_0,B)} < \delta$, then we have

$$
\begin{aligned}
(2.36)\ \|\partial_2 u^{k+1} - \partial_2 u^{l+1}\|_{\Omega(R,L_0,B)} &\leq 4K_1 t \cdot \varepsilon + 2K_1 t\|\partial_2 u^k - \partial_2 u^l\|_{\Omega(R,L_0,B)} \\
&\leq 4K_1 B\varepsilon + 2K_1 B\|\partial_2 u^k - \partial_2 u^l\|_{\Omega(R,L_0,B)} \\
&\leq \varepsilon + 2^{-1}\|\partial_2 u^k - \partial_2 u^l\|_{\Omega(R,L_0,B)}.
\end{aligned}
$$

By (2.36) and (2.31) we know that the inequality

$$
(2.37)\quad \|\partial_2 u^{k+1} - \partial_2 u^{l+1}\|_{\Omega(R,L_0,B)} \leq \varepsilon + 2^{-1}\|\partial_2 u^k - \partial_2 u^l\|_{\Omega(R,L_0,B)}
$$

holds for all sufficiently large k and l. It follows that the sequence $\{\partial_2 u^k\}$ converges uniformly in $\Omega(R, L_0, B)$. From this fact and the fact that the sequence $\{u^k\}$ converges to u^∞ it follows that $u^\infty(t,x)$ is continuously differentiable in x. From this fact and the relation (2.32) it follows that $u^\infty(t,x)$ is continuously differentiable in t, too.

Thus the proof of the fact that, if the positive number B satisfies (2.10), then the system (2.21) of integral equations has a C^1-solution in $\Omega(R, L_0, B)$ is now complete. The uniqueness of the solution of the system (2.21) of integral equations is proved as follows. First note that, if $u : \Omega(R, L_0, B) \to G^N$ is a C^1-solution of the equation (2.21), then $\|u\|_{\Omega(R,L_0,B)} \leq K_0 B < \infty$. Next suppose that there are two C^1-solutions $u = (u_1, \ldots, u_N)$ and $v = (v_1, \ldots, v_N)$ in $\Omega(R, L_0, B)$ of the system (2.21). Then we have

$$\begin{aligned}
&\|u_i(t,x) - v_i(t,x)\| \\
&\le \int_0^t \|f_i(\sigma, \xi_i(\sigma,t,x), u(\sigma, \xi_i(\sigma,t,x))) - f_i(\sigma, \xi_i(\sigma,t,x), v(\sigma, \xi_i(\sigma,t,x)))\| d\sigma \\
&\le \|\partial_3 f\|_{\mathcal{W}(\cdots)} \cdot \|u - v\|_{\Omega(\cdots)} \cdot B \le K_1 B \|u - v\|_{\Omega(\cdots)} \le \frac{1}{2}\|u - v\|_{\Omega(\cdots)} < \infty,
\end{aligned}$$

from which we can conclude that $\|u - v\|_{\Omega(R,L_0,B)} = 0$ and $u = v$. **QED**

In Theorem 2.1 it is assumed that $\lambda(t,x)$ is defined in the set $\mathcal{D}(A,R)$. However, as is easily seen by the above proof of the theorem given above, this assumption can be weakened as is stated in the following proposition.

Proposition 2.1. *Let X, G, A, R, L_0, L_1, K_0, K_1, S and N be the same as in Theorem 2.1. Let $\lambda(t,x) = (\lambda_1(t,x), \ldots, \lambda_N(t,x))$ be an X^N-valued C^1-function of $(t,x) \in \Omega(R, L_0, A)$ and $f(t,x,u)$ a G^N-valued C^1-function of (t,x,u) in the set $\mathcal{W}(A,R,S)$. Assume that* (2.9) *and, instead of* (2.8),

$$\|\lambda\|_{\Omega(R,L_0,A)} \le L_0, \quad \|\partial_2 \lambda\|_{\Omega(R,L_0,A)} \le L_1. \tag{2.38}$$

Assume further that the partial derivatives $\partial_2 f$ and $\partial_3 f$ are uniformly continuous. Let B be a positive constant satisfying (2.10)*. Then the IVP* (2.7)-(2.2) *has a unique C^1-solution $u(t,x)$ in the range $\Omega(R, L_0, B)$. Moreover, it satisfies the inequalities*

$$\|u\|_{\Omega(R,L_0,B)} \le \frac{S}{2}, \quad \|\partial_2 u\|_{\Omega(R,L_0,B)} \le 1. \tag{2.39}$$

For our later purpose, however, we have to modify the above proposition further. Let us prove the following proposition.

Proposition 2.2. *Let X, G, A, R, L_0, L_1, K_0, K_1, S and N be the same as in Theorem 2.1. Let $\lambda(t,x) = (\lambda_1(t,x), \ldots, \lambda_N(t,x))$ be an X^N-valued C^1-function of $(t,x) \in \Omega(R, L_0, A)$ satisfying* (2.38) *and $f(t,x,u)$ a G^N-valued C^1-function of (t,x,u) in the set $\mathcal{W}(A,R,S)$ satisfying* (2.9)*. Assume further that the following Lipschitz type inequalities hold:*

$$\|\partial_2 \lambda(t,x_1) - \partial_2 \lambda(t,x_2)\| \le L_1 \|x_1 - x_2\|, \tag{2.40}$$

$$\|\partial_i f(t,x_1,u_1) - \partial_i f(t,x_2,u_2)\| \le K_1(\|x_1 - x_2\| + \|u_1 - u_2\|) \quad (i = 2,3). \tag{2.41}$$

Let B be a positive constant satisfying

$$B \leq \min\left\{A, \frac{1}{2L_1}, \frac{S}{2K_0}, \frac{1}{28K_1}\right\}. \tag{2.42}$$

Then the IVP (2.7)-(2.2) *has a unique C^1-solution $u : \Omega(R, L_0, B) \to G^N$ that satisfies* (2.39) *and the Lipschitz condition*

$$\|\partial_2 u(t, x_1) - \partial_2 u(t, x_2)\| \leq \|x_1 - x_2\|. \tag{2.43}$$

Proof. The condition imposed on λ, f and B in the present proposition is stronger than that in Proposition 2.1 [3]. Therefore we can use the result of Proposition 2.1 for the present situation. So we see that there is a unique C^1-solution $u : \Omega(R, L_0, B) \to G^N$ that satisfies (2.39). The only thing we have to do here is, therefore, to prove that $\partial_2 u$ satisfies the Lipschitz condition (2.43). To do so, however, we have to prove first that $\partial_3 \xi(t, t_0, x_0)$ satisfies the Lipschitz condition with respect to x_0. From (2.16), (2.40) and (2.17) we obtain

$$\begin{aligned}
&\|\partial_3\xi_i(t, t_0, x_0) - \partial_3\xi_i(t, t_0, x_1)\| \\
&\leq \int_{t_0}^{t} \|\partial_2\lambda_i(\sigma, \xi_i(\sigma, t_0, x_0)) - \partial_2\lambda_i(\sigma, \xi_i(\sigma, t_0, x_1))\| \cdot \|\partial_3\xi_i(\sigma, t_0, x_0)\|\, d\sigma \\
&\quad + \int_{t_0}^{t} \|\partial_2\lambda_i(\sigma, \xi_i(\sigma, t_0, x_1))\| \cdot \|\partial_3\xi_i(\sigma, t_0, x_0) - \partial_3\xi_i(\sigma, t_0, x_1)\|\, d\sigma \\
&\leq \int_{t_0}^{t} L_1\|\xi_i(\sigma, t_0, x_0) - \xi_i(\sigma, t_0, x_1)\| \cdot \|\partial_3\xi_i\|_{\tilde{\Omega}(R, L_0, B)}\, d\sigma \\
&\quad + \int_{t_0}^{t} \|\partial_2\lambda\|_{\Omega(R, L_0, A)} \cdot \|\partial_3\xi_i(\sigma, t_0, x_0) - \partial_3\xi_i(\sigma, t_0, x_1)\|\, d\sigma \\
&\leq L_1 B \|\partial_3\xi_i\|^2_{\tilde{\Omega}(\cdots)} \|x_0 - x_1\| + L_1 \int_{t_0}^{t} \|\partial_3\xi_i(\sigma, t_0, x_0) - \partial_3\xi_i(\sigma, t_0, x_1)\|\, d\sigma \\
&\leq 4L_1 B\|x_0 - x_1\| + L_1 B \sup_{0 \leq \sigma \leq t_0} \|\partial_3\xi_i(\sigma, t_0, x_0) - \partial_3\xi_i(\sigma, t_0, x_1)\|.
\end{aligned}$$

Therefore we see, since $2L_1 B \leq 1$, that the inequality

$$\|\partial_3\xi_i(t, t_0, x_0) - \partial_3\xi_i(t, t_0, x_1)\| \leq 4\|x_0 - x_1\|. \tag{2.44}$$

holds.

[3] We cannot say in the complete sense that the Lipschitz condition (2.41) is stronger than the condition in Theorem 2.1 or Proposition 2.1 that $\partial_2 f$ and $\partial_3 f$ are uniformly continuous. But the Lipschitz condition (2.41) is stronger than the condition actually used in the proof of Theorem 2.1.

Next we prove by mathematical induction that each member of the sequence $\{u^k\}$ of successive approximations of the solution of the IVP (2.7)-(2.2) satisfies the Lipschitz inequality

$$\|\partial_2 u^k(t,x_1) - \partial_2 u^k(t,x_2)\| \le \|x_1 - x_2\|. \tag{2.45}$$

The inequality (2.45) holds for $k=0$, because $u^0(t,x)\equiv 0$. Assume now that (2.45) holds for some $k\ge 0$. In order to prove that (2.45) holds for $k+1$, too, we use (2.33). From (2.33) we obtain

$$\begin{aligned}
&\partial_2 u_i^{k+1}(t,x_1) - \partial_2 u_i^{k+1}(t,x_2)\\
&= \int_0^t \{\partial_2 f_i(\sigma,\xi_i(\sigma,t,x_1),u^k(\sigma,\xi_i(\sigma,t,x_1)))\\
&\quad -\partial_2 f_i(\sigma,\xi_i(\cdots,x_2),u^k(\sigma,\xi_i(\cdots,x_2)))\}\partial_3\xi_i(\sigma,t,x_1)\,d\sigma\\
&\quad + \int_0^t \partial_2 f_i(\sigma,\xi_i(\cdots,x_2),u^k(\sigma,\xi_i(\cdots,x_2)))\{\partial_3\xi_i(\cdots,x_1)-\partial_3\xi_i(\cdots,x_2)\}\,d\sigma\\
&\quad + \int_0^t \{\partial_3 f_i(\sigma,\xi_i(\cdots,x_1),u^k(\sigma,\xi_i(\cdots,x_1)))\\
&\quad -\partial_3 f_i(\sigma,\xi_i(\cdots,x_2),u^k(\sigma,\xi_i(\cdots,x_2)))\}\partial_2 u^k(\sigma,\xi_i(\cdots,x_1))\partial_3\xi_i(\cdots,x_1)\,d\sigma\\
&\quad + \int_0^t \partial_3 f_i(\sigma,\xi_i(\sigma,t,x_2),u^k(\sigma,\xi_i(\sigma,t,x_2)))\\
&\qquad \{\partial_2 u^k(\sigma,\xi_i(\sigma,t,x_1)) - \partial_2 u^k(\sigma,\xi_i(\sigma,t,x_2))\}\partial_3\xi_i(\sigma,t,x_1)\,d\sigma\\
&\quad + \int_0^t \partial_3 f_i(\sigma,\xi_i(\sigma,t,x_2),u^k(\sigma,\xi_i(\sigma,t,x_2)))\\
&\qquad \partial_2 u^k(\sigma,\xi_i(\sigma,t,x_2))\{\partial_3\xi_i(\sigma,t,x_1)-\partial_3\xi_i(\sigma,t,x_2)\}\,d\sigma.
\end{aligned}$$

We denote the five integrals in the right member of the last equality by $I_1,\ldots,I_5$. I_1 is estimated as follows. In virtue (2.41), (2.17) we have

$$\begin{aligned}
&\|\partial_2 f_i(\sigma,\xi_i(\sigma,t,x_1),u^k(\sigma,\xi_i(\sigma,t,x_1))) - \partial_2 f_i(\sigma,\xi_i(\sigma,t,x_2),u^k(\sigma,\xi_i(\sigma,t,x_2)))\|\\
&\le K_1(\|\xi_i(\sigma,t,x_1)-\xi_i(\sigma,t,x_2)\| + \|u^k(\sigma,\xi_i(\sigma,t,x_1)) - u^k(\sigma,\xi_i(\sigma,t,x_2))\|)\\
&\le K_1(1+\|\partial_2 u^k\|_{\Omega(R,L_0,B)})\|\xi_i(\sigma,t,x_1)-\xi_i(\sigma,t,x_2)\|\\
&\le 2K_1\|\partial_3\xi_i\|_{\tilde\Omega(R,L_0,B)}\|x_1-x_2\|.
\end{aligned} \tag{2.46}$$

Therefore, using (2.17), we see that

$$\|I_1\| \le 2K_1 B(\|\partial_3\xi_i\|_{\tilde\Omega(R,L_0,B)})^2\|x_1-x_2\| \le 8K_1B\|x_1-x_2\|.$$

Similarly we have

$$\|I_3\| \le 2K_1B(\|\partial_3\xi_i\|_{\tilde\Omega(\cdots)})^2\cdot\|\partial_2 u^k\|_{\Omega(\cdots)}\cdot\|x_1-x_2\| \le 8K_1B\|x_1-x_2\|.$$

Using (2.45), (2.17) and (2.9), we obtain $\|I_4\| \leq 4K_1B\|x_1 - x_2\|$. Using (2.44), we obtain $\|I_2\| \leq 4K_1B\|x_1 - x_2\|$ and $\|I_5\| \leq 4K_1B\|x_1 - x_2\|$. Using the above estimation of $I_1, \ldots, I_5$, we obtain

$$\text{(2.47)} \quad \|\partial_2 u^{k+1}(t, x_1) - \partial_2 u^{k+1}(t, x_2)\| \leq 28K_1B\|x_1 - x_2\| \leq \|x_1 - x_2\|,$$

which completes the proof of the fact that (2.45) holds for all k.

Letting $k \to \infty$ in (2.45), we obtain (2.43). **QED**

§3. Preparation for Solving the Quasi-linear IVP

Our final purpose is to solve the quasi-linear IVP (1.6)-(1.7). As a preparation for this purpose we consider in this section the semi-linear equation obtained by substituting a known function $v(t, x)$ for $u(t, x)$ in $\mu(t, x, u(t, x))$. So we consider here the IVP (1.8)-(1.7). The system (1.8) is rewritten as

$$\text{(3.1)} \qquad \partial_1 u(t, x) + \partial_2 u(t, x) \odot \mu(t, x, v(t, x)) = f(t, x, u(t, x)).$$

The relation between (1.8) and (3.1) is just like the relation between (2.1) and (2.7). Throughout this section we denote by X and G two fixed real Banach spaces and by N a fixed natural number. Using Proposition 2.2 in §2 we can easily prove the following proposition on the IVP for the equation (3.1).

Proposition 3.1. *Let A, R, M_0, M_1, K_0 and K_1 be positive constants. Let S be a positive constant or $S = \infty$. Let $\mu(t, x, u)$ be an X^N-valued C^1-function of $(t, x, u) \in \mathcal{W}(A, R, S)$ satisfying*

$$\text{(3.2)} \qquad \|\mu\|_{\mathcal{W}(A,R,S)} \leq M_0, \quad \|\partial_j \mu\|_{\mathcal{W}(A,R,S)} \leq M_1 \quad (j = 2, 3)$$

and $f(t, x, u)$ a G^N-valued C^1-function of $(t, x, u) \in \mathcal{W}(A, R, S)$ satisfying (2.9). Assume further that Lipschitz type inequalities (2.41) and

$$\text{(3.3)} \quad \|\partial_i \mu(t, x_1, u_1) - \partial_i \mu(t, x_2, u_2)\| \leq M_1(\|x_1 - x_2\| + \|u_1 - u_2\|) \quad (i = 2, 3)$$

hold, where (t, x_1, u_1), $(t, x_2, u_2) \in \mathcal{W}(A, R, S)$. Let B be a positive constant satisfying

$$\text{(3.4)} \qquad B \leq \min\left\{A, \frac{1}{10M_1}, \frac{S}{2K_0}, \frac{1}{28K_1}\right\}$$

and $v : \Omega(R, M_0, B) \to G^N$ a C^1-function such that

$$\text{(3.5)} \qquad \|v(t, x)\| < S, \quad \|\partial_2 v(t, x)\| \leq 1$$

and

$$\text{(3.6)} \qquad \|\partial_2 v(t, x_1) - \partial_2 v(t, x_2)\| \leq \|x_1 - x_2\|.$$

Then the IVP (2.7)-(2.2) *has a unique* C^1*-solution* $u : \Omega(R, M_0, B) \to G^N$ *that satisfies the inequalities* (2.39) *and the Lipschitz condition* (2.43).

Proof. Write

$$\lambda(t, x) = \mu(t, x, v(t, x)). \tag{3.7}$$

If $(t, x) \in \Omega(R, M_0, B)$, then $v(t, x) \in G^N$, $\|v(t, x)\| < S$ and $(t, x, v(t, x)) \in \mathcal{W}(B, R, S) \subset \mathcal{W}(A, R, S)$. Therefore $\lambda(t, x)$ is certainly defined for $(t, x) \in \Omega(R, M_0, B)$ and we have $\|\lambda\|_{\Omega(R,M_0,B)} \le \|\mu\|_{\mathcal{W}(A,R,S)} \le M_0$. Further we have

$$\partial_2\lambda(t, x) = \partial_2\mu(t, x, v(t, x)) + \partial_3\mu(t, x, v(t, x))\partial_2 v(t, x) \tag{3.8}$$

and

$$\|\partial_2\lambda\|_{\Omega(R,M_0,B)} \le \|\partial_2\mu\|_{\mathcal{W}(A,R,S)} + \|\partial_3\mu\|_{\mathcal{W}(A,R,S)} \cdot 1 \le 2M_1. \tag{3.9}$$

We have to confirm further that $\partial_2\lambda(t, x)$ satisfies the Lipschitz condition with respect to x. Take two points (t, x_1), $(t, x_2) \in \Omega(R, M_0, B)$ arbitrarily. Then we have, in virtue of (3.8),

$$\begin{aligned}
&\partial_2\lambda(t, x_1) - \partial_2\lambda(t, x_2) = \partial_2\mu(t, x_1, v(t, x_1)) - \partial_2\mu(t, x_2, v(t, x_2)) \\
&\quad + \{\partial_3\mu(t, x_1, v(t, x_1)) - \partial_3\mu(t, x_2, v(t, x_2))\}\partial_2 v(t, x_1) \\
&\quad + \partial_3\mu(t, x_2, v(t, x_2))\{\partial_2 v(t, x_1) - \partial_2 v(t, x_2)\}
\end{aligned}$$

and, in virtue of (3.3) and (3.6),

$$\begin{aligned}
&\|\partial_2\lambda(t, x_1) - \partial_2\lambda(t, x_2)\| \le M_1(\|x_1 - x_2\| + \|\partial_2 v\|_{\Omega(\cdots)} \cdot \|x_1 - x_2\|) \\
&\quad + M_1(\|x_1 - x_2\| + \|\partial_2 v\|_{\Omega(\cdots)} \cdot \|x_1 - x_2\|)\|\partial_2 v\|_{\Omega(\cdots)} \\
&\quad + \|\partial_3\mu\|_{\mathcal{W}(\cdots)}\|\partial_2 v(t, x_1) - \partial_2 v(t, x_2)\| \\
&\le 2M_1\|x_1 - x_2\| + 2M_1\|x_1 - x_2\| \cdot 1 + M_1\|x_1 - x_2\| = 5M_1\|x_1 - x_2\|.
\end{aligned}$$

As a result of the above arguments we see that, if we write $L_0 = M_0$, $L_1 = 5M_1$, then the inequalities (2.38) and (2.40) hold. Further the condition (3.4) implies the condition (2.42), if $L_1 = 5M_1$.

Therefore we can use Proposition 2.2 with $A = B$ for the IVP (2.7)-(2.2) with $\lambda(t, x)$ defined by (3.7) and conclude that the IVP (3.1)-(1.7) has a unique C^1-solution $u : \Omega(R, M_0, B) \to G^N$ satisfying (2.39) and (2.43). **QED**

Next we substitute two different functions for $v(t, x)$ in (3.1) and compare the corresponding solutions of the IVP. To be exact we now prove the following proposition.

Proposition 3.2. *Let* A, R, M_0, M_1, K_0, K_1 *and* S *be the same as in Proposition 3.1. Let* $\mu(t, x, u)$ *be an* X^N*-valued* C^1*-function of* (t, x, u) *in* $\mathcal{W}(A, R, S)$ *satisfying* (3.2) *and* (3.3). *Let* $f(t, x, u)$ *be a* G^N*-valued* C^1*-function of* (t, x, u)

in $\mathcal{W}(A, R, S)$ *satisfying* (2.9) *and* (2.41). *Let* B *be a positive constant satisfying* (3.4). *Let* $v^1 : \Omega(R, M_0, B) \to G^N$ *be a* C^1*-function such that*

$$\|v^1(t,x)\| < S, \quad \|\partial_2 v^1(t,x)\| \le 1 \tag{3.10}$$

and

$$\|\partial_2 v^1(t,x_1) - \partial_2 v^1(t,x_2)\| \le \|x_1 - x_2\|. \tag{3.11}$$

Let $u^1 : \Omega(R, M_0, B) \to G^N$ *denote the unique* C^1*-solution, whose existence is guaranteed by Proposition* 3.1, *of the IVP*

$$\partial_1 u(t,x) + \partial_2 u(t,x) \odot \mu(t,x,v^1(t,x)) = f(t,x,u(t,x)), \quad u(0,x) \equiv 0. \tag{3.12}$$

Let $v^2 : \Omega(R, M_0, B) \to G^N$ *be a* C^1*-function such that* $\|v^2(t,x)\| < S$ *for all* $(t,x) \in \Omega(R, M_0, B)$ *and assume that there is a* C^1*-solution* $u^2 : \Omega(R, M_0, B) \to G^N$ *of the IVP*

$$\partial_1 u(t,x) + \partial_2 u(t,x) \odot \mu(t,x,v^2(t,x)) = f(t,x,u(t,x)), \quad u(0,x) \equiv 0. \tag{3.13}$$

Then the inequality

$$\|u^1 - u^2\|_{\Omega(R,M_0,B)} \le 2^{-4} \|v^1 - v^2\|_{\Omega(R,M_0,B)} \tag{3.14}$$

holds. If it is further assumed that

$$\|\partial_2 v^2(t,x)\| \le 1 \tag{3.15}$$

and

$$\|\partial_2 v^2(t,x_1) - \partial_2 v^2(t,x_2)\| \le \|x_1 - x_2\|, \tag{3.16}$$

then the inequality

$$\|\partial_2 u^1 - \partial_2 u^2\|_{\Omega(R,M_0,B)} \le 2\|v^1 - v^2\|_{\Omega(\cdots)} + 2^{-3}\|\partial_2 v^1 - \partial_2 v^2\|_{\Omega(\cdots)} \tag{3.17}$$

holds.

Proof. First note that u^1 satisfies, by Proposition 3.1,

$$\|u^1\|_{\Omega(R,M_0,B)} \le \frac{S}{2}, \quad \|\partial_2 u^1\|_{\Omega(R,M_0,B)} \le 1 \tag{3.18}$$

and

$$\|\partial_2 u^1(t,x_1) - \partial_2 u^1(t,x_2)\| \le \|x_1 - x_2\|. \tag{3.19}$$

In order to compare u^1 with u^2 it is necessary for us to compare the solution of the ordinary IVP

$$\frac{d}{dt}\xi_i(t) = \mu_i(t, \xi_i(t), v^1(t, \xi_i(t))), \quad \xi_i(t_0) = x_0 \quad (1 \le j \le N) \tag{3.20}$$

with that of the IVP

$$\frac{d}{dt}\xi_i(t) = \mu_i(t, \xi_i(t), v^2(t, \xi_i(t))), \quad \xi_i(t_0) = x_0 \quad (1 \leq j \leq N). \tag{3.21}$$

In (3.20) and (3.21) the initial data (t_0, x_0) is a given point in $\Omega(R, M_0, B)$. We denote the solution of the IVP (3.20) by $\xi_i^1(t, t_0, x_0)$ and that of the IVP (3.21) by $\xi_i^2(t, t_0, x_0)$. Each $\xi_i^j(t, t_0, x_0)$ is an X-valued C^1-function of $(t, t_0, x_0) \in \tilde{\Omega}(R, M_0, B)$ [1]. Since it satisfies the integral equation

$$\xi_i^j(t, t_0, x_0) = x_0 + \int_{t_0}^{t} \mu_i(\sigma, \xi_i^j(\sigma, t_0, x_0), v^j(\sigma, \xi_i^j(\sigma, t_0, x_0)))\, d\sigma. \tag{3.22}$$

we have

$$\begin{aligned} &\xi_i^1(t, t_0, x_0) - \xi_i^2(t, t_0, x_0) \qquad (3.23)\\ &= \int_{t_0}^{t} \{\mu_i(\sigma, \xi_i^1(\sigma, t_0, x_0), v^1(\sigma, \xi_i^1(\cdots))) - \mu_i(\sigma, \xi_i^1(\sigma, t_0, x_0), v^2(\sigma, \xi_i^1(\cdots)))\} d\sigma \\ &+ \int_{t_0}^{t} \{\mu_i(\sigma, \xi_i^1(\cdots), v^2(\sigma, \xi_i^1(\cdots))) - \mu_i(\sigma, \xi_i^2(\cdots), v^2(\sigma, \xi_i^1(\cdots)))\} d\sigma \\ &+ \int_{t_0}^{t} \{\mu_i(\sigma, \xi_i^2(\cdots), v^2(\sigma, \xi_i^1(\cdots))) - \mu_i(\sigma, \xi_i^2(\cdots), v^2(\sigma, \xi_i^2(\cdots)))\} d\sigma. \end{aligned}$$

Let us denote the three integrals of the right member of (3.23) by I_1, I_2 and I_3. The norm of I_1 is estimated as follows. Since

$$\begin{aligned} &\|\mu_i(\sigma, \xi_i^1(\sigma, t_0, x_0), v^1(\sigma, \xi_i^1(\cdots))) - \mu_i(\sigma, \xi_i^1(\sigma, t_0, x_0), v^2(\sigma, \xi_i^1(\cdots)))\| \\ &\leq \|\partial_3 \mu_i\|_{\mathcal{W}(A,R,S)} \cdot \|v^1(t, \xi_i^1(\cdots)) - v^2(t, \xi_i^1(\cdots))\| \leq M_1 \|v^1 - v^2\|_{\Omega(R,M_0,B)} \end{aligned}$$

and $M_1 B \leq 10^{-1} < 2^{-3}$, we have

$$\|I_1\| \leq M_1 B \|v^1 - v^2\|_{\Omega(R,M_0,B)} \leq 2^{-3} \|v^1 - v^2\|_{\Omega(R,M_0,B)}. \tag{3.24}$$

$\|I_2\|$ is estimated as follows. Since

$$\begin{aligned} &\|\mu_i(\sigma, \xi_i^1(\sigma, t_0, x_0), v^2(\sigma, \xi_i^1(\cdots))) - \mu_i(\sigma, \xi_i^2(\sigma, t_0, x_0), v^2(\sigma, \xi_i^1(\cdots)))\| \\ &\leq \|\partial_2 \mu_i\|_{\mathcal{W}(A,R,S)} \cdot \|\xi_i^1(\sigma, t_0, x_0) - \xi_i^2(\sigma, t_0, x_0)\| \leq M_1 \|\xi_i^1 - \xi_i^2\|_{\tilde{\Omega}(R,M_0,B)}, \end{aligned}$$

we have

$$\|I_2\| \leq M_1 B \|\xi_i^1 - \xi_i^2\|_{\tilde{\Omega}(R,M_0,B)} \leq 2^{-3} \|\xi_i^1 - \xi_i^2\|_{\tilde{\Omega}(R,M_0,B)}. \tag{3.25}$$

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[1] Note that this statement is true for $j = 2$, too, under the sole condition that $v^2(t, x)$ is a C^1-function such that $\|v^2(t, x)\| < S$ for all $(t, x) \in \Omega(R, M_0, B)$.

$\|I_3\|$ is estimated as follows. Since

$$\|\mu_i(\sigma, \xi_i^2(\sigma, t_0, x_0), v^2(\sigma, \xi_i^1(\cdots))) - \mu_i(\sigma, \xi_i^2(\sigma, t_0, x_0), v^2(\sigma, \xi_i^2(\cdots)))\| \\ \leq \|\partial_3 \mu_i\|_{\mathcal{W}(\cdots)} \cdot \|\partial_2 v^2\|_{\Omega(\cdots)} \cdot \|\xi_i^1(\cdots) - \xi_i^2(\cdots)\| \leq M_1 \cdot 1 \cdot \|\xi_i^1 - \xi_i^2\|_{\tilde{\Omega}(\cdots)},$$

we have

$$\|I_3\| \leq M_1 B \|\xi_i^1 - \xi_i^2\|_{\tilde{\Omega}(R,M_0,B)}. \leq 2^{-3} \|\xi_i^1 - \xi_i^2\|_{\tilde{\Omega}(R,M_0,B)}. \tag{3.26}$$

From (3.23) through (3.26) we obtain

$$\|\xi_i^1 - \xi_i^2\|_{\tilde{\Omega}(R,M_0,B)} \leq 2^{-3} \|v^1 - v^2\|_{\Omega(R,M_0,B)} + 2^{-2} \|\xi_i^1 - \xi_i^2\|_{\tilde{\Omega}(R,M_0,B)}.$$

Therefore we see that the inequality

$$\|\xi_i^1 - \xi_i^2\|_{\tilde{\Omega}(R,M_0,B)} \leq 2^{-2} \|v^1 - v^2\|_{\Omega(R,M_0,B)} \tag{3.27}$$

holds.

Using (3.27), we can estimate the difference $u^1 - u^2$ of the solutions of the IVP (3.12) and (3.13) in terms of the difference $v^1 - v^2$. By the proof of Theorem 2.1 we know that $u^j = (u_1^j, \ldots, u_N^j)$ satisfies the system of integral equations

$$u_i^j(t,x) = \int_0^t f_i(\sigma, \xi_i^j(\sigma,t,x), u^j(\sigma, \xi_i^j(\sigma,t,x))) d\sigma. \tag{3.28}$$

From (3.28) we obtain

$$\begin{aligned} &u_i^1(t,x) - u_i^2(t,x) \\ &= \int_0^t \{f_i(\sigma, \xi_i^1(\sigma,t,x), u^1(\sigma, \xi_i^1(\sigma,t,x))) - f_i(\sigma, \xi_i^2(\cdots), u^1(\sigma, \xi_i^1(\cdots)))\} d\sigma \\ &+ \int_0^t \{f_i(\sigma, \xi_i^2(\cdots), u^1(\sigma, \xi_i^1(\cdots))) - f_i(\sigma, \xi_i^2(\cdots), u^1(\sigma, \xi_i^2(\cdots)))\} d\sigma \\ &+ \int_0^t \{f_i(\sigma, \xi_i^2(\cdots), u^1(\sigma, \xi_i^2(\cdots))) - f_i(\sigma, \xi_i^2(\cdots), u^2(\sigma, \xi_i^2(\cdots)))\} d\sigma. \end{aligned} \tag{3.29}$$

We denote the three integrals of the right member of (3.29) by I_1, I_2 and I_3. By (2.9) and (3.27) we have

$$\begin{aligned} &\|f_i(\sigma, \xi_i^1(\sigma,t,x), u^1(\sigma, \xi_i^1(\sigma,t,x))) - f_i(\sigma, \xi_i^2(\cdots), u^1(\sigma, \xi_i^1(\cdots)))\| \\ &\leq \|\partial_2 f_i\|_{\mathcal{W}(A,R,S)} \cdot \|\xi_i^1(\sigma,t,x) - \xi_i^2(\cdots)\| \leq K_1 \|\xi_i^1 - \xi_i^2\|_{\tilde{\Omega}(R,M_0,S)} \\ &\leq 2^{-2} K_1 \|v^1 - v^2\|_{\Omega(R,M_0,B)}, \end{aligned}$$

and

$$\|I_1\| \leq 2^{-2} K_1 B \|v^1 - v^2\|_{\Omega(R,M_0,B)}. \tag{3.30}$$

Similarly we have

$$\|I_2\| \le K_1 B \|\partial_2 u^1\|_{\Omega(\cdots)} \cdot \|\xi_i^1 - \xi_i^2\|_{\tilde{\Omega}(\cdots)} \le 2^{-2} K_1 B \|v^1 - v^2\|_{\Omega(\cdots)} \tag{3.31}$$

and

$$\|I_3\| \le K_1 B \|u^1 - u^2\|_{\Omega(R, M_0, B)}. \tag{3.32}$$

Using (3.29),…, (3.31), we obtain

$$\|u^1 - u^2\|_{\Omega(R,M_0,B)} \le K_1 B \|u^1 - u^2\|_{\Omega(\cdots)} + 2^{-1} K_1 B \|v^1 - v^2\|_{\Omega(\cdots)}. \tag{3.33}$$

From (3.33) we obtain, since $K_1 B \le 1/28 < 2^{-4}$, the desired inequality (3.14).

Now we add the assumptions (3.15) and (3.16) and want to estimate the difference $\partial_2 u^1 - \partial_2 u^2$. Note that v^2 now satisfies the same conditions as those for v^1 and the solution u^2 of the IVP (3.13) satisfies the same type of inequalities as (3.18) and (3.19). In particular note that the inequality $\|\partial_2 u^2(t,x)\| \le 1$ holds. Keeping this fact in mind, let us consider estimating the difference $\partial_2 u^1 - \partial_2 u^2$. To this end it is necessary to estimate the difference $\partial_3 \xi_i^1(t, t_0, x_0) - \partial_3 \xi_i^2(t, t_0, x_0)$. To this end we differentiate (3.22) with respect to x_0. Then we obtain

$$\begin{aligned} \partial_3 \xi_i^j(t, t_0, x_0) = 1_X &+ \int_{t_0}^t \partial_2 \mu_i(\sigma, \xi_i^j(\sigma, t_0, x_0), v^j(\sigma, \xi_i^j(\cdots))) \partial_3 \xi_i^j(\cdots) d\sigma \\ &+ \int_{t_0}^t \partial_3 \mu_i(\sigma, \xi_i^j(\sigma, t_0, x_0), v^j(\sigma, \xi_i^j(\cdots))) \partial_2 v^j(\cdots) \partial_3 \xi_i^j(\cdots) d\sigma \end{aligned} \tag{3.34}$$

From (3.34) we obtain

$$\begin{aligned} &\partial_3 \xi_i^1(t, t_0, x_0) - \partial_3 \xi_i^2(t, t_0, x_0) \\ &= \int_{t_0}^t \{\partial_2 \mu_i(\sigma, \xi_i^1(\cdots), v^1(\sigma, \xi_i^1(\cdots))) - \partial_2 \mu_i(\sigma, \xi_i^2(\cdots), v^2(\sigma, \xi_i^2(\cdots)))\} \\ &\quad \partial_3 \xi_i^1(\cdots) d\sigma \\ &+ \int_{t_0}^t \partial_2 \mu_i(\sigma, \xi_i^2(\cdots), v^2(\sigma, \xi_i^2(\cdots))) \{\partial_3 \xi_i^1(\cdots) - \partial_3 \xi_i^2(\cdots)\} d\sigma \\ &+ \int_{t_0}^t \{\partial_3 \mu_i(\sigma, \xi_i^1(\cdots), v^1(\sigma, \xi_i^1(\cdots))) - \partial_3 \mu_i(\sigma, \xi_i^2(\cdots), v^2(\sigma, \xi_i^2(\cdots)))\} \times \\ &\qquad \times \partial_2 v^1(\cdots) \partial_3 \xi_i^1(\cdots) d\sigma \\ &+ \int_{t_0}^t \partial_3 \mu_i(\sigma, \xi_i^2(\cdots), v^2(\sigma, \xi_i^2(\cdots))) \{\partial_2 v^1(\cdots) - \partial_2 v^2(\cdots)\} \partial_3 \xi_i^1(\cdots) d\sigma \\ &+ \int_{t_0}^t \partial_3 \mu_i(\sigma, \xi_i^2(\cdots), v^2(\sigma, \xi_i^2(\cdots))) \partial_2 v^2(\cdots) \{\partial_3 \xi_i^1(\cdots) - \partial_3 \xi_i^2(\cdots)\} d\sigma. \end{aligned}$$

We denote the five integrals in the right member of the last equality by $I_1, \ldots, I_5$. We want to estimate the norms of these integrals. In order to estimate I_1 we can use the Lipschitz condition (3.3) for $\partial_2\mu$. By (3.3) we have

(3.35)
$$\begin{aligned}
&\|\partial_2\mu_i(\sigma,\xi_i^1(\sigma,t_0,x_0),v^1(\sigma,\xi_i^1(\cdots)))-\partial_2\mu_i(\sigma,\xi_i^2(\sigma,t_0,x_0),v^2(\sigma,\xi_i^2(\cdots)))\|\\
&\le M_1(\|\xi_i^1(\cdots)-\xi_i^2(\cdots)\|\\
&\quad+\|v^1(\sigma,\xi_i^1(\cdots))-v^2(\sigma,\xi_i^1(\cdots))\|+\|v^2(\sigma,\xi_i^1(\cdots))-v^2(\sigma,\xi_i^2(\cdots))\|)\\
&\le M_1(2^{-2}+1+\|\partial_2v^2\|_{\Omega(R,M_0,B)}\cdot 2^{-2})\|v^1-v^2\|_{\Omega(R,M_0,B)}\\
&\le 2M_1\|v^1-v^2\|_{\Omega(R,M_0,B)}.
\end{aligned}$$

From (3.35) and the inequality $\|\partial_3\xi_i^1(\cdots)\|\le 2$ we obtain

(3.36) $$\|I_1\|\le 4M_1B\|v^1-v^2\|_{\Omega(R,M_0,B)}\le 2^{-1}\|v^1-v^2\|_{\Omega(R,M_0,B)}.$$

Since $\|\partial_2\mu_i(\cdots)\|\le M_1$, we have

(3.37) $$\|I_2\|\le M_1B\|\partial_3\xi_i^1-\partial_3\xi_i^2\|_{\tilde\Omega(R,M_0,B)}\le 2^{-3}\|\partial_3\xi_i^1-\partial_3\xi_i^2\|_{\tilde\Omega(R,M_0,B)}.$$

As for I_3 we use the Lipschitz condition (3.3) for $\partial_3\mu_i$ and obtain, just like (3.35),

(3.38)
$$\begin{aligned}
&\|\partial_3\mu_i(\sigma,\xi_i^1(\sigma,t_0,x_0),v^1(\sigma,\xi_i^1(\cdots)))-\partial_3\mu_i(\sigma,\xi_i^2(\sigma,t_0,x_0),v^2(\sigma,\xi_i^2(\cdots)))\|\\
&\le 2M_1\|v^1-v^2\|_{\Omega(R,M_0,B)}.
\end{aligned}$$

By means of (3.38) and the inequalities $\|\partial_2v^1(\cdots)\|\le 1$, $\|\partial_3\xi_i^1(\cdots)\|\le 2$ we obtain

(3.39) $$\|I_3\|\le 4M_1B\|v^1-v^2\|_{\Omega(R,M_0,B)}\le 2^{-1}\|v^1-v^2\|_{\Omega(R,M_0,B)}.$$

Since $\|\partial_3\mu_i(\cdots)\|\le M_1$ and $\|\partial_3\xi_i^1(\cdots)\|\le 2$, we have

$$\|I_4\|\le 2M_1B\|\partial_2v^1-\partial_2v^2\|_{\Omega(R,M_0,B)}\le 2^{-2}\|\partial_2v^1-\partial_2v^2\|_{\Omega(R,M_0,B)}.$$

Since $\|\partial_2v^2\|_{\Omega(\ldots)}\le 1$, we have

$$\|I_5\|\le M_1B\|\partial_3\xi_i^1-\partial_3\xi_i^2\|_{\tilde\Omega(R,M_0,B)}\le 2^{-3}\|\partial_3\xi_i^1-\partial_3\xi_i^2\|_{\tilde\Omega(R,M_0,B)}.$$

In conclusion we know now that

$$\begin{aligned}
&\|\partial_3\xi_i^1(t,t_0,x_0)-\partial_3\xi_i^2(t,t_0,x_0)\|\le\|I_1\|+\|I_2\|+\|I_3\|+\|I_4\|+\|I_5\|\\
&\le 2\|v^1-v^2\|_{\Omega(R,M_0,B)}+2^{-2}\|\partial_2v^1-\partial_2v^2\|_{\Omega(R,M_0,B)}\\
&\quad+2^{-2}\|\partial_3\xi_i^1-\partial_3\xi_i^2\|_{\tilde\Omega(R,M_0,B)}
\end{aligned}$$

and

$$\|\partial_3\xi_i^1 - \partial_3\xi_i^2\|_{\tilde{\Omega}(R,M_0,B)} \le 2\|v^1 - v^2\|_{\Omega(\cdots)} + 2^{-1}\|\partial_2 v^1 - \partial_2 v^2\|_{\Omega(\cdots)}. \tag{3.40}$$

Using (3.40), we can estimate the difference $\partial_2 u^1 - \partial_2 u^2$. To this end we differentiate (3.28) in x. Then we obtain

$$\begin{aligned}\partial_2 u_i^j(t,x) &= \int_0^t \partial_2 f_i(\sigma, \xi_i^j(\sigma,t,x), u^j(\sigma,\xi_i^j(\cdots)))\partial_3\xi_i^j(\sigma,t,x)d\sigma \\ &+ \int_0^t \partial_3 f_i(\sigma,\xi_i^j(\sigma,t,x),u^j(\sigma,\xi_i^j(\cdots)))\partial_2 u^j(\sigma,\xi_i^j(\cdots))\partial_3\xi_i^j(\cdots)d\sigma.\end{aligned} \tag{3.41}$$

From (3.41) we obtain

$$\begin{aligned}&\partial_2 u_i^1(t,x) - \partial_2 u_i^2(t,x) \\ &= \int_0^t \{\partial_2 f_i(\sigma,\xi_i^1(\sigma,t,x),u^1(\sigma,\xi_i^1(\cdots))) - \partial_2 f_i(\sigma,\xi_i^2(\cdots),u^2(\sigma,\xi_i^2(\cdots)))\} \\ &\qquad \partial_3\xi_i^1(\cdots)d\sigma \\ &+ \int_0^t \partial_2 f_i(\sigma,\xi_i^2(\cdots),u^2(\sigma,\xi_i^2(\cdots)))\{\partial_3\xi_i^1(\cdots) - \partial_3\xi_i^2(\cdots)\}d\sigma \\ &+ \int_0^t \{\partial_3 f_i(\sigma,\xi_i^1(\cdots),u^1(\sigma,\xi_i^1(\cdots))) - \partial_3 f_i(\sigma,\xi_i^2(\cdots),u^2(\sigma,\xi_i^2(\cdots)))\} \\ &\qquad \partial_2 u^1(\sigma,\xi_i^1(\cdots))\partial_3\xi_i^1(\cdots)d\sigma \\ &+ \int_0^t \partial_3 f_i(\sigma,\xi_i^2(\sigma,t,x),u^2(\sigma,\xi_i^2(\cdots))) \\ &\qquad \{\partial_2 u^1(\sigma,\xi_i^1(\cdots)) - \partial_2 u^1(\sigma,\xi_i^2(\cdots))\}\partial_3\xi_i^1(\cdots)d\sigma \\ &+ \int_0^t \partial_3 f_i(\sigma,\xi_i^2(\cdots),u^2(\sigma,\xi_i^2(\cdots))) \\ &\qquad \{\partial_2 u^1(\sigma,\xi_i^2(\cdots)) - \partial_2 u^2(\sigma,\xi_i^2(\cdots))\}\partial_3\xi_i^1(\cdots)d\sigma \\ &+ \int_0^t \partial_3 f_i(\sigma,\xi_i^2(\sigma,t,x),u^2(\sigma,\xi_i^2(\cdots))) \\ &\qquad \partial_2 u^2(\sigma,\xi_i^2(\cdots))\{\partial_3\xi_i^1(\cdots) - \partial_3\xi_i^2(\cdots)\}d\sigma.\end{aligned} \tag{3.42}$$

We denote the six integrals in the right member of (3.42) by $I_1, \ldots, I_6$. By (2.41), (2.17), (3.27) and (3.14) we have

$$\begin{aligned}\|I_1\| &\le K_1\{\|\xi_i^1(\cdots) - \xi_i^2(\cdots)\| + \|u^1(\sigma,\xi_i^1(\cdots)) - u^2(\sigma,\xi_i^1(\cdots))\| \\ &\qquad + \|u^2(\sigma,\xi_i^1(\cdots)) - u^2(\sigma,\xi_i^2(\cdots))\|\} \cdot \|\partial_3\xi_i^1(\cdots)\| \\ &\le 2K_1 B\{\|\xi_i^1 - \xi_i^2\|_{\tilde{\Omega}(\cdots)} + \|u^1 - u^2\|_{\Omega(\cdots)} + \|\partial_2 u\|_{\Omega(\cdots)}\|\xi_i^1 - \xi_i^2\|_{\tilde{\Omega}(\cdots)}\} \\ &\le 2K_1 B(2^{-1} + 2^{-4} + 2^{-2})\|v^1 - v^2\|_{\Omega(\cdots)} \\ &\le 2K_1 B\|v^1 - v^2\|_{\Omega(R,M_0,B)} \le 2^{-3}\|v^1 - v^2\|_{\Omega(R,M_0,B)}.\end{aligned} \tag{3.43}$$

By (2.9) and (3.40) we have

$$(3.44)\quad \|I_2\| \le 2K_1B\|v^1-v^2\|_{\Omega(R,M_0,B)} + 2^{-1}K_1B\|\partial_2 v^1-\partial_2 v^2\|_{\Omega(R,M_0,B)} \le 2^{-3}\|v^1-v^2\|_{\Omega(R,M_0,B)} + 2^{-5}\|\partial_2 v^1-\partial_2 v^2\|_{\Omega(R,M_0,B)}$$

Like (3.43) we have

$$(3.45)\qquad \|I_3\| \le 2K_1B\|v^1-v^2\|_{\Omega(R,M_0,B)} \le 2^{-3}\|v^1-v^2\|_{\Omega(R,M_0,B)}.$$

By (2.9), (3.19), (2.17) and (3.27) we have

$$(3.46)\quad \|I_4\| \le K_1\|\xi_i^1-\xi_i^2\|_{\tilde\Omega(R,M_0,B)}\cdot 2\cdot B \le 2^{-1}K_1B\|v^1-v^2\|_{\Omega(R,M_0,B)} \le 2^{-5}\|v^1-v^2\|_{\Omega(R,M_0,B)}.$$

By (2.9) and (2.17) we have

$$(3.47)\quad \|I_5\| \le 2K_1B\|\partial_2 u^1-\partial_2 u^2\|_{\Omega(R,M_0,B)} \le 2^{-1}\|\partial_2 u^1-\partial_2 u^2\|_{\Omega(R,M_0,B)}.$$

By (2.9), (2.39) and (3.40) we have

$$(3.48)\quad \|I_6\| \le K_1B(2\|v^1-v^2\|_{\Omega(R,M_0,B)} + 2^{-1}\|\partial_2 v^1-\partial_2 v^2\|_{\Omega(R,M_0,B)}) \le 2^{-3}\|v^1-v^2\|_{\Omega(R,M_0,B)} + 2^{-5}\|\partial_2 v^1-\partial_2 v^2\|_{\Omega(R,M_0,R)}.$$

From (3.42),...,(3.48) we obtain

$$\begin{aligned} &\|\partial_2 u^1-\partial_2 u^2\|_{\Omega(R,M_0,B)} \le 2^{-1}\|\partial_2 u^1-\partial_2 u^2\|_{\Omega(R,M_0,B)} \\ &+(2^{-3}+2^{-3}+2^{-3}+2^{-5}+2^{-3})\|v^1-v^2\|_{\Omega(R,M_0,B)} \\ &+(2^{-5}+2^{-5})\|\partial_2 v^1-\partial_2 v^2\|_{\Omega(R,M_0,R)} \\ \le\ &\|v^1-v^2\|_{\Omega(R,M_0,B)} + 2^{-4}\|\partial_2 v^1-\partial_2 v^2\|_{\Omega(R,M_0,R)} \end{aligned}$$

and the desired inequality (3.17). **QED**

§4. Solution of the Quasi-linear IVP

Using Propositions 3.1 and 3.2 we can now easily solve the quasi-linear IVP (1.6)-(1.7). Our result is stated in the following theorem.

Theorem 4.1. *Let X and G be real Banach spaces. Let A, R, S, M_0, M_1, K_0 and K_1 be positive constants. Let N be a natural number. Let $\mu(t,x,u)$ be an X^N-valued C^1-function of $(t,x,u)\in\mathcal{W}(A,R,S)$ satisfying* (3.2) *and* (3.3). *Let $f(t,x,u)$ be a G^N-valued C^1-function of $(t,x,u)\in\mathcal{W}(A,R,S)$ satisfying* (2.9) *and* (2.41). *Let B be a positive constant satisfying* (3.4).

Then the IVP (1.6)-(1.7) *has a unique C^1-solution $u:\Omega(R,M_0,B)\to G^N$.*

Proof. In order to show the existence of a solution of the IVP we construct a sequence of successive approximations of the solution as follows. By Proposition 3.1 we know that there is a unique C^1-solution $u^1 : \Omega(R, M_0, B) \to G^N$ of the IVP

$$\partial_1 u(t,x) + \partial_2 u(t,x) \odot \mu(t,x,0) = f(t,x,u(t,x)), \quad u(0,x) = 0. \tag{4.1}$$

Set $v(t,x) = u^1(t,x)$. Then Proposition 3.1 tells us that this v satisfies (3.5) and (3.6). Therefore we can use Proposition 3.1 again and know that the IVP

$$\partial_1 u(t,x) + \partial_2 u(t,x) \odot \mu(t,x,u^1(t,x)) = f(t,x,u(t,x)), \quad u(0,x) = 0 \tag{4.2}$$

has a unique C^1-solution $u^2 : \Omega(R, M_0, B) \to G^N$. In this way we successively obtain a sequence $\{u^j(t,x)\}$ of G^N-valued functions of $(t,x) \in \Omega(R, M_0, B)$. Each fuction u^j is the unique C^1-solution of the IVP

$$\partial_1 u(t,x) + \partial_2 u(t,x) \odot \mu(t,x,u^{j-1}(t,x)) = f(t,x,u(t,x)), \quad u(0,x) = 0. \tag{4.3}$$

It satisfies the inequalities

$$\|u^j\|_{\Omega(R,L_0,B)} \le \frac{S}{2}, \quad \|\partial_2 u^j\|_{\Omega(R,L_0,B)} \le 1 \tag{4.4}$$

and

$$\|\partial_2 u^j(t,x_1) - \partial_2 u^j(t,x_2)\| \le \|x_1 - x_2\|. \tag{4.5}$$

For the above obtained sequence $\{u^j\}$ we can now apply Proposition 3.2. Take two consecutive members u^k and u^{k+1} of the sequence arbitralily. These two functions satisfy the conditions imposed in Proposition 3.2 on the functions v^1 and v^2. Moreover, u^{k+1} and u^{k+2} are, respectively, the solutions of the IVP (3.12), (3.13) with $v^1 = u^k$ and $v^2 = u^{k+1}$. Therefore we see that the inequalities

$$\|u^{k+1} - u^{k+2}\|_{\Omega(R,M_0,B)} \le 2^{-4} \|u^k - u^{k+1}\|_{\Omega(R,M_0,B)} \tag{4.6}$$

and

$$\begin{aligned} &\|\partial_2 u^{k+1} - \partial_2 u^{k+2}\|_{\Omega(R,M_0,B)} \\ &\le 2\|u^k - u^{k+1}\|_{\Omega(R,M_0,B)} + 2^{-3} \|\partial_2 u^k - \partial_2 u^{k+1}\|_{\Omega(R,M_0,R)} \end{aligned} \tag{4.7}$$

hold.

From (4.6) it immediately follows that the sequence of functions $\{u^j\}$ converges uniformly in $\Omega(R, M_0, B)$. Exactly speaking, there is a continuous function $u^\infty : \Omega(R, M_0, B) \to G^N$ such that

$$\lim_{j \to \infty} \|u^j - u^\infty\|_{\Omega(R,M_0,B)} = 0.$$

From this fact and the inequality (4.7) it follows that the sequence of functions $\{\partial_2 u^j\}$, too, converges uniformly in $\Omega(R, M_0, B)$. Exactly speaking, there is a continuous function $w : \Omega(R, M_0, B) \to \mathcal{L}(X; G^N)$ such that

$$\lim_{j\to\infty} \|\partial_2 u^j - w\|_{\Omega(R,M_0,B)} = 0.$$

From these facts it follows further that $u^\infty(t, x)$ is continuously differentiable with respect to x and the equality $\partial_2 u^\infty(t, x) = w(t, x)$ holds. Further, letting $j \to \infty$, in the relation

$$\partial_1 u^j(t, x) + \partial_2 u^j(t, x) \odot \mu(t, x, u^{j-1}(t, x)) = f(t, x, u^j(t, x)),$$

we know that $u^\infty(t, x)$ is continuously differentiable in t, too, and satisfies the equation

$$\partial_1 u^\infty(t, x) + \partial_2 u^\infty(t, x) \odot \mu(t, x, u^\infty(t, x)) = f(t, x, u^\infty(t, x))$$

for $(t, x) \in \Omega(R, M_0, B)$. Thus we have shown the existence of a C^1-solution in $\Omega(R, M_0, B)$ of the IVP (1.6)-(1.7).

The uniqueness of the solution of the IVP is proved as follows. If we set $v^1 = u^\infty$, then we can regard u^∞ as a solution of the semi-linear IVP (3.12). Note that $v^1 = u^\infty$ satisfies all the conditions imposed on v^1 in Proposition 3.2. Now suppose that there is another C^1-solution $u : \Omega(R, M_0, B) \to G^N$ of the IVP (1.6)-(1.7). Then, by setting $v^2 = u$, we can regard u as a C^1-solution of the semi-linear IVP (3.13). Therefore we see, by Proposition 3.2, that the inequality

$$\|u^\infty - u\|_{\Omega(R,M_0,B)} \le 2^{-4}\|v^1 - v^2\|_{\Omega(\cdots)} = 2^{-4}\|u^\infty - u\|_{\Omega(\cdots)}$$

holds. It follows that $\|u^\infty - u\|_{\Omega(R,M_0,B)} = 0$ and $u = u^\infty$. This completes the proof of the uniqueness of the solution of the IVP. **QED**

Remark. The author has some plans to extend the above result further. One of them is to replace the real variable t by a vector variable as in Yamanaka[8], [9]. Another plan is to consider the case where the number of unknown functions is infinite as in Zaitov[10].

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