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On Some Models of Universal Expansion in General Relativity Using Otsuki Connections

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Abstract. An Otsuki connection Γ is a cross section of the bundle $T(M) \otimes \mathcal{D}^2(M)$ and they can be undertood as generalized objects of the affine connections. Briefly, the difference between these two theories is that an Otsuki connection Γ is an affine connection if and only if the principal part $\lambda(\Gamma)$ of Γ, which is a homomorphism of the tangent bundle $T(M)$, is the identity map. We consider some special class $\Gamma(\Psi, G)$ of Γ. Using $\Gamma(\Psi, G)$, this paper presents universal expansion-like models which are exact solutions of some partial differential equations.

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§1. Basic Concepts and Preliminaries

A cross section Γ on the vector bundle $T(M) \otimes \mathcal{D}^2(M)$ is called an Otsuki connection, where $T(M)$ and $\mathcal{D}^2(M)$ are the tangent bundle and the cotangent bundle of order 2 on a smooth manifold M respectively. Using local coordinates (u^λ), Γ is written as follows:

$$\Gamma = \frac{\partial}{\partial u^\lambda} \otimes \left(P^\lambda_\mu d^2 u^\mu + \Gamma^\lambda_{\mu\nu} du^\mu \otimes du^\nu \right).$$

It is easy to see that the $P = (P^\lambda_\mu)$ tranforms as a tensor field of type $(1,1)$ under coordinates changes. The tensor field $P = (P^\lambda_\mu)$, which is denoted by $\lambda(\Gamma)$, is called the principal part of Γ. According to Otsuki [6, 7], the covariant derivative $\Gamma_X Y$ is defined by

$$\Gamma_X Y = (X(Y^\lambda) P^\mu_\lambda + \Gamma^\mu_{\lambda\nu} X^\lambda Y^\nu) \frac{\partial}{\partial u^\mu},$$

where X,Y and $\Gamma_X Y$ are tangent vector fields on M. The operator Γ_X has the following properties:

1. $\Gamma_{fX+gY}Z = f\Gamma_X Z + g\Gamma_Y Z$,

2. $\Gamma_X(Y+Z) = \Gamma_X Y + \Gamma_X Z$,

3. $\Gamma_X fY = X(f)P(Y) + f\Gamma_X Y$, where f,g are functions on M.

It is routine work to extend the covariant derivative to arbitrary tensor fields. For example, if F is a tensor field of type $(0,2)$, then $\Gamma_X F$ is defined by

$$\Gamma_X F(Y,Z) = X(F(PY,PZ)) - F(\Gamma_X Y, PZ) - F(PY, \Gamma_X Z).$$

We put

$$T(X,Y) = \Gamma_X Y - \Gamma_Y X - P[X,Y].$$

This element becomes a tensor field of type $(0,2)$ and is called the torsion tensor field of Γ. Any geodesic γ in M with an Otsuki connection Γ is given by a solution of the system of the ordinary differential equation of order 2 on M:

$$P^\lambda_\mu \frac{d^2u^\mu}{ds^2} + \Gamma^\lambda_{\mu\nu}\frac{du^\mu}{ds}\frac{du^\nu}{ds} = 0,$$

where s is an affine parameter of the connection.

Let $P = (P^\lambda_\mu)$ and $G = (g_{\lambda\mu})$ be a regular tensor field of type $(1,1)$ and a non-singular tensor field of type $(0,2)$ on M. We put

$$\tilde{G}(X,Y) = G(PX,PY).$$

Using the terminology of Otsuki connections, the Levi-Civita connection $\tilde{\nabla}$ with respect to $\tilde{G}$ is written by

$$\tilde{\nabla} = \frac{\partial}{\partial u^\lambda} \otimes \left(\delta^\lambda_\mu d^2u^\mu + \widetilde{\left\{{\lambda \atop \mu\nu}\right\}} du^\mu \otimes du^\nu\right),$$

where $\widetilde{\left\{{\lambda \atop \mu\nu}\right\}}$ are the Christoffel symbols of $\tilde{G} = (\tilde{g}_{\lambda\mu})$. We define an Otsuki connection

$$P\tilde{\nabla} = \frac{\partial}{\partial u^\lambda} \otimes \left(P^\lambda_\mu d^2u^\mu + P^\lambda_\delta \widetilde{\left\{{\delta \atop \mu\nu}\right\}} du^\mu \otimes du^\nu\right).$$

It is easy to see the following fundamental properties of $P\tilde{\nabla}$.

1 $P = \lambda(\Gamma)$.

2 $(P\tilde{\nabla})_X G = 0$.

3 $P\tilde{\nabla}$ is torsion free.

4 γ is a geodesic of $\tilde{\nabla}$ if and only if it is a geodesic of $P\tilde{\nabla}$.

Conversely $\Gamma = P\tilde{\nabla}$ is uniquely determined by the above **1**$\sim$**3**, which we write

$$\Gamma = P\tilde{\nabla} = \Gamma(P, G).$$

The Otsuki connection Γ, which we will consider in this paper, is a case $P = \Psi I$ and denoted by $\Gamma(\Psi, G)$, where Ψ is a function on M and I is the fundamental unit tensor field of type $(1, 1)$. $\Gamma(\Psi, G)$ seems to have a meaning only where Ψ does not vanish, but any function Ψ on M is available for $\Gamma(\Psi, G)$ because it can be written locally as follows:

$$\begin{aligned} \Gamma(\Psi, G) &= \frac{\partial}{\partial u^\lambda} \otimes \left(\Psi d^2 u^\lambda + \Gamma^\lambda_{\mu\nu} du^\mu \otimes du^\nu \right), \\ \Gamma^\lambda_{\mu\nu} &= \Psi \{{}^{\lambda}_{\mu\nu}\} + (\frac{\partial \Psi}{\partial u^\mu} \delta^\lambda_\nu + \frac{\partial \Psi}{\partial u^\nu} \delta^\lambda_\mu + \frac{\partial \Psi}{\partial u^\sigma} g^{\sigma\lambda} g_{\mu\nu}), \end{aligned}$$

where we use the apparaties on a Riemannian manifold (M, G). Using an affine parameter s of $\Gamma(\Psi, G)$, equations of a geodesic become

$$\Psi \frac{d^2 u^\lambda}{ds^2} + \Gamma^\lambda_{\mu\nu} \frac{du^\mu}{ds} \frac{du^\nu}{ds} = 0. \tag{1}$$

We define a set $Sing(\Gamma) \subset \mathbf{M}$ and a metric $\tilde{G}$ by

$$Sing(\Gamma) = \{ x \in \mathbf{M} \mid \Psi(x) = 0 \}, \quad \tilde{G} = \Psi^2 G.$$

The next lemma is a special case of the above property 4, which says that a space $M \backslash Sing(\Gamma)$ with an Otsuki connection $\Gamma = \Gamma(\Psi, G)$ and a Riemannian manifold $(M \backslash Sing(\Gamma), \tilde{G})$ are geodesically equivalent in the following sense.

Lemma 1. *A curve $\gamma(s)$ in $M \backslash Sing(\Gamma)$ is a geodesic in the sense of Otsuki geometry of $\Gamma(\Psi, G)$ if and only if it is a geodesic in the sense of Riemannian geometry of $(M \backslash Sing(\Gamma), \tilde{G})$.*

Lemma 1 shows that $\tilde{G} = \Psi^2 G$ has an important meaning in $\Gamma(\Psi, G)$ geometry, which we call the essential metric of an Otsuki connection $\Gamma(\Psi, G)$.

In the paper [4] we define a function $S_{\Gamma(\Psi, G)}$ and the condition (A) as follows:

$$(A) \qquad \delta \int_{\mathbf{M}} S_{\Gamma(\Psi, G)} dV_G = 0.$$

Using local coordinates (u^λ) and the apparaties on the Riemannian manifold (M, G), the condition (A) becomes as follows:

$$\begin{aligned} \Psi(R^{\mu\nu} &- \frac{1}{2} g^{\mu\nu} S) - \frac{1}{2} \nabla_\lambda \nabla_\kappa (\Psi^3)(g^{\mu\lambda} g^{\nu\kappa} - g^{\mu\nu} g^{\lambda\kappa}) \\ &+ 12 (\nabla_\lambda \Psi)(\nabla_\kappa \Psi)(g^{\mu\lambda} g^{\nu\kappa} - \frac{1}{2} g^{\mu\nu} g^{\lambda\kappa}) \Psi = 0, \\ \Psi(\triangle &- \frac{1}{8} S) + \frac{1}{2} g^{\mu\nu} (\nabla_\mu \Psi)(\nabla_\nu \Psi) = 0, \end{aligned}$$

where S, $\triangle$ are the scalar curvature, the Laplace-Beltrami operator of (M, G). It appears very difficult to find non-trivial solutions, which means solutions with Ψ not being constant, of the above equations. However, using a function h and a metric $\overline{G}$, which are defined by

$$h = \frac{\sqrt{3}}{2} \log \Psi, \overline{G} = \Psi^3 G,$$

the equations become the following simpler forms [5]:

$$\begin{aligned} &\overline{R}^{\mu\nu} - \frac{1}{2}\overline{g}^{\mu\nu}\overline{S} + (\overline{g}^{\mu\nu}\overline{g}^{\alpha\beta} - \overline{g}^{\mu\alpha}\overline{g}^{\nu\beta})(\overline{\nabla}_\alpha h)(\overline{\nabla}_\beta h) = 0, \\ &\overline{\triangle}(h) = 0, \end{aligned}$$

where we use the apparaties on Riemannian manifold $(M, \overline{G})$. Rewriting these equations to the covariant forms, we have

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}S = (2\delta^\alpha_\mu \delta^\beta_\nu - g_{\mu\nu}g^{\alpha\beta})(\nabla_\alpha h)(\nabla_\beta h), \tag{2}$$

$$\triangle(h) = 0, \tag{3}$$

where we use the apparaties on $(M \backslash Sing(\Gamma), \overline{G})$ but abbreviate the bar for convenience, which we do not think causes any confusion and we will use these notation from now on. (2) and (3) are the Euler-Lagrange equations of a Lagrange density $L(h, \overline{G})$ which is defined as follows [5]:

$$L(h, \overline{G}) = g^{\mu\nu}\left[\left\{{}^{\alpha}_{\mu\beta}\right\}\left\{{}^{\beta}_{\nu\alpha}\right\} - \left\{{}^{\alpha}_{\mu\nu}\right\}\left\{{}^{\beta}_{\alpha\beta}\right\} + 2(\nabla_\mu h)(\nabla_\nu h)\right]\sqrt{-|\overline{G}|},$$

where $(M \backslash Sing(\Gamma), \overline{G})$ is a 4-dimensional Lorentz manifold and $|\overline{G}| = det(g_{\lambda\mu})$. In the same paper [5] we look for solutions $\Gamma(\Psi, G)$ of the above equations under the condition that $\Gamma(\Psi, G)$ has the spherical symmetry and find two interesting families of Otsuki connections, one of which is the schwartzshild spacetime and the other is peculiar to the theory of Otsuki connections $\Gamma(\Psi, G)$.

In Section 2 we will find exact solutions $\Gamma(\Psi, G)$ of the equations, whose essential metric has the following form:

$$\overline{G} = -dw^2 + R^2(w)D(r)(dx^2 + dy^2 + dz^2).$$

Taking the results in advance, $D(r)$ becomes:

$$D(r) = \left(1 + \frac{\varepsilon r^2}{4}\right)^{-2},$$

where $\varepsilon = -1, 0, 1$. Some elementary properties of the functions $R(w)$ will be discussed in Section 3.

The following ranges of indices are used throughout this paper:

$$1 \le i, j, k, \cdots \le 3, \quad 0 \le \alpha, \beta, \gamma, \cdots \le 3.$$

§2. Universal Expansion Models by $\Gamma(\Psi, G)$

We consider a metric $\overline{G}$ and a function h of the following forms:

$$\begin{aligned}\overline{G} &= g_{\lambda\mu} dx^\lambda dx^\mu \\ &= -B(t,r)dt^2 + A(t,r)(dx^2+dy^2+dz^2), \\ h &= h(t),\end{aligned}$$

where $r^2 = x^2+y^2+z^2$ and we often use t,x,y,z instead of x^0,x^1,x^2,x^3. Using these forms, the Christoffel symbols $\{{}^{\lambda}_{\mu\nu}\} = \{{}^{\lambda}_{\nu\mu}\}$ of $\overline{G}$ become as follows:

$$\begin{aligned}&\{{}^{t}_{tt}\} = \{{}^{0}_{00}\} = \frac{1}{2}\frac{B_t}{B}, \{{}^{0}_{0i}\} = \{{}^{0}_{i0}\} = \frac{1}{2}\frac{B_r}{B}\frac{x^i}{r}, \\ &\{{}^{0}_{ij}\} = \{{}^{0}_{ji}\} = \frac{1}{2}\frac{A_t}{B}\delta_{ij}, \{{}^{i}_{00}\} = \frac{1}{2}\frac{B_r}{A}\frac{x^i}{r}, \\ &\{{}^{i}_{0j}\} = \{{}^{i}_{j0}\} = \frac{1}{2}\frac{A_t}{A}\delta_{ij}, \\ &\{{}^{i}_{jk}\} = \{{}^{i}_{kj}\} = \frac{1}{2}\frac{A_r}{A}\left(\frac{x^k}{r}\delta_{ij} + \frac{x^j}{r}\delta_{ik} - \frac{x^i}{r}\delta_{jk}\right),\end{aligned}$$

where $A_t = \frac{\partial A}{\partial t}$, $A_r = \frac{\partial A}{\partial r}$, etc.. Components of Ricci tensor and the scalar curvature S of $\overline{G}$ are given as follows:

$$\begin{aligned}R_{00}(\equiv R_{tt}) &= -\frac{3}{2}\left(\frac{A_t}{A}\right)_t - \frac{3}{4}\left(\frac{A_t}{A}\right)^2 + \frac{3}{4}\left(\frac{A_t}{A}\right)\left(\frac{B_t}{B}\right) \\ &\quad +\frac{1}{2}\frac{B}{A}\left\{\left(\frac{B_1}{B}\right)_1 + \left(\frac{B_2}{B}\right)_2 + \left(\frac{B_3}{B}\right)_3\right\} \\ &\quad +\frac{1}{4}\frac{B}{A}\left\{\left(\frac{A_1}{A}\right)\left(\frac{B_1}{B}\right) + \left(\frac{A_2}{A}\right)\left(\frac{B_2}{B}\right) + \left(\frac{A_3}{A}\right)\left(\frac{B_3}{B}\right)\right\} \\ &\quad +\frac{1}{4}\frac{B}{A}\left\{\left(\frac{B_1}{B}\right)^2 + \left(\frac{B_2}{B}\right)^2 + \left(\frac{B_3}{B}\right)^2\right\}, \\ R_{0k}\,(= R_{k0}) &= \left(\frac{A_t}{A}\right)_k + \frac{1}{2}\left(\frac{A_t}{A}\right)\left(\frac{B_k}{B}\right), \\ R_{kk} &= \frac{1}{2}\frac{A}{B}\left(\frac{A_t}{A}\right)_t + \frac{3}{4}\frac{A}{B}\left(\frac{A_t}{A}\right)^2 - \frac{3}{4}\frac{A}{B}\left(\frac{A_t}{A}\right)\left(\frac{B_t}{B}\right) \\ &\quad -\frac{1}{2}\left\{\left(\frac{A_1}{A}\right)_1 + \left(\frac{A_2}{A}\right)_2 + \left(\frac{A_3}{A}\right)_3\right\} \\ &\quad -\frac{1}{4}\left\{\left(\frac{A_1}{A}\right)^2 + \left(\frac{A_2}{A}\right)^2 + \left(\frac{A_3}{A}\right)^2\right\} \\ &\quad -\frac{1}{2}\left\{\left(\frac{A_k}{A}\right)_k + \left(\frac{B_k}{B}\right)_k\right\} + \frac{1}{4}\left(\frac{A_k}{A}\right)^2 - \frac{1}{4}\left(\frac{B_k}{B}\right)^2,\end{aligned}$$

$$
\begin{aligned}
R_{km}\,(=R_{mk}) &= -\frac{1}{2}\left(\frac{A_m}{A}\right)_k - \frac{1}{2}\left(\frac{B_m}{B}\right)_k + \frac{1}{4}\left(\frac{A_k}{A}\right)\left(\frac{A_m}{A}\right) \\
&\quad + \frac{1}{4}\left(\frac{A_k}{A}\right)\left(\frac{B_m}{B}\right) + \frac{1}{4}\left(\frac{A_m}{A}\right)\left(\frac{B_k}{B}\right) - \frac{1}{4}\left(\frac{B_k}{B}\right)\left(\frac{B_m}{B}\right),
\end{aligned}
$$

where $k \neq m$,

$$
\begin{aligned}
S &= \frac{3}{B}\left(\frac{A_t}{A}\right)_t + \frac{3}{B}\left(\frac{A_t}{A}\right)^2 - \frac{3}{2}\frac{1}{B}\left(\frac{A_t}{A}\right)\left(\frac{B_t}{B}\right) \\
&\quad - \frac{2}{A}\left\{\left(\frac{A_1}{A}\right)_1 + \left(\frac{A_2}{A}\right)_2 + \left(\frac{A_3}{A}\right)_3\right\} \\
&\quad - \frac{1}{A}\left\{\left(\frac{B_1}{B}\right)_1 + \left(\frac{B_2}{B}\right)_2 + \left(\frac{B_3}{B}\right)_3\right\} \\
&\quad - \frac{1}{2}\frac{1}{A}\left\{\left(\frac{A_1}{A}\right)^2 + \left(\frac{A_2}{A}\right)^2 + \left(\frac{A_3}{A}\right)^2\right\} \\
&\quad - \frac{1}{4}\left\{\left(\frac{A_1}{A}\right)\left(\frac{B_1}{B}\right) + \left(\frac{A_2}{A}\right)\left(\frac{B_2}{B}\right) + \left(\frac{A_3}{A}\right)\left(\frac{B_3}{B}\right)\right\} \\
&\quad - \frac{1}{2}\frac{1}{A}\left\{\left(\frac{B_1}{B}\right)^2 + \left(\frac{B_2}{B}\right)^2 + \left(\frac{B_3}{B}\right)^2\right\},
\end{aligned}
$$

where we use the following formulas and notations:

$$
\begin{aligned}
R_{\mu\nu} &= R_{\nu\mu} = \frac{\partial}{\partial x^\lambda}\left\{{\lambda \atop \mu\nu}\right\} - \frac{\partial}{\partial x^\nu}\left\{{\lambda \atop \mu\lambda}\right\} + \left\{{\lambda \atop \kappa\lambda}\right\}\left\{{\kappa \atop \mu\nu}\right\} - \left\{{\lambda \atop \kappa\nu}\right\}\left\{{\kappa \atop \mu\lambda}\right\}, \\
S &= g^{\mu\nu} R_{\mu\nu}, \\
A_1 &= \frac{\partial A}{\partial x^1}, \quad A_2 = \frac{\partial A}{\partial x^2}, \quad A_3 = \frac{\partial A}{\partial x^3} \quad etc..
\end{aligned}
$$

Using these equalities, the left side of (2), which are denoted by $G_{\mu\nu}$ become as follows:

$$
\begin{aligned}
G_{tt} &= R_{tt} - \frac{1}{2} g_{tt} S = R_{tt} + \frac{1}{2} B S \\
&= \frac{3}{4}\left(\frac{A_t}{A}\right)^2 - \frac{A}{B}\left\{\left(\frac{A_1}{A}\right)_1 + \left(\frac{A_2}{A}\right)_2 + \left(\frac{A_3}{A}\right)_3\right\} \\
&\quad - \frac{1}{4}\frac{B}{A}\left\{\left(\frac{A_1}{A}\right)^2 + \left(\frac{A_2}{A}\right)^2 + \left(\frac{A_3}{A}\right)^2\right\} \\
&\quad + \frac{1}{8}\frac{B}{A}\left\{\left(\frac{A_1}{A}\right)\left(\frac{B_1}{B}\right) + \left(\frac{A_2}{A}\right)\left(\frac{B_2}{B}\right) + \left(\frac{A_3}{A}\right)\left(\frac{B_3}{B}\right)\right\}, \\
G_{kk} &= R_{kk} - \frac{1}{2} g_{kk} S = R_{kk} - \frac{1}{2} A S \\
&= -\frac{A}{B}\left(\frac{A_t}{A}\right)_t - \frac{3}{4}\frac{A}{B}\left(\frac{A_t}{A}\right)^2 + \frac{1}{2}\frac{A}{B}\left(\frac{A_t}{A}\right)\left(\frac{B_t}{B}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}\left\{\left(\frac{A_1}{A}\right)_1+\left(\frac{A_2}{A}\right)_2+\left(\frac{A_3}{A}\right)_3\right\} \\
&+\frac{1}{2}\left\{\left(\frac{B_1}{B}\right)_1+\left(\frac{B_2}{B}\right)_2+\left(\frac{B_3}{B}\right)_3\right\} \\
&+\frac{1}{8}\left\{\left(\frac{A_1}{A}\right)\left(\frac{B_1}{B}\right)+\left(\frac{A_2}{A}\right)\left(\frac{B_2}{B}\right)+\left(\frac{A_3}{A}\right)\left(\frac{B_3}{B}\right)\right\}, \\
&+\frac{1}{4}\left\{\left(\frac{B_1}{B}\right)^2+\left(\frac{B_2}{B}\right)^2+\left(\frac{B_3}{B}\right)^2\right\} \\
&-\frac{1}{2}\left(\frac{A_k}{A}\right)_k-\frac{1}{2}\left(\frac{B_k}{B}\right)_k+\frac{1}{4}\left(\frac{A_k}{A}\right)^2-\frac{1}{4}\left(\frac{B_k}{B}\right)^2,
\end{aligned}
$$

$$
\begin{aligned}
G_{km} &= R_{km}-\frac{1}{2}g_{km}S = R_{km} \\
&= -\frac{1}{2}\left(\frac{A_m}{A}\right)_k-\frac{1}{2}\left(\frac{B_m}{B}\right)_k+\frac{1}{4}\left(\frac{A_k}{A}\right)\left(\frac{A_m}{A}\right) \\
&\quad +\frac{1}{4}\left(\frac{A_k}{A}\right)\left(\frac{B_m}{B}\right)+\frac{1}{4}\left(\frac{A_m}{A}\right)\left(\frac{B_k}{B}\right)-\frac{1}{4}\left(\frac{B_k}{B}\right)\left(\frac{B_m}{B}\right),
\end{aligned}
$$

where $k \neq m$. On the other hand, the right sides of (2), which are denoted by $H_{\mu\nu}$, become as follows:

$$
\begin{aligned}
H_{tt} &= (2\delta_t^\alpha\delta_t^\beta - g_{tt}g^{\alpha\beta})\nabla_\alpha(h)\nabla_\beta(h) = (h_t)^2, \\
H_{kk} &= (2\delta_k^\alpha\delta_k^\beta - g_{kk}g^{\alpha\beta})\nabla_\alpha(h)\nabla_\beta(h) = \frac{A}{B}(h_t)^2, \\
\textit{the others} &= 0,
\end{aligned}
$$

where $h_t = \frac{\partial h}{\partial t}$. (2) becomes as follows:

$$G_{00} = (h_t)^2, \tag{4}$$

$$G_{kk} = \frac{A}{B}(h_t)^2, \tag{5}$$

$$G_{0k} = G_{k0} = 0, \tag{6}$$

$$G_{km} = G_{mk} = 0, \tag{7}$$

where $k \neq m$. Now we assume that $A(t,r)$ has a form of $A(t,r) = C(t)D(r)$, then (6) becomes $B_rC_t = 0$. We assume $B_r = 0$ i.e. $B = B(t)$ and rewrite $\overline{G}$ and h using the following new variable $\overline{t}$ such that $\overline{t} = \int^t \sqrt{B(s)}ds$ for t:. Now $\overline{G}$ and h become as follows:

$$
\begin{aligned}
\overline{G} &= -d\overline{t}^2 + \tilde{A}(\overline{t},r)(dx^2+dy^2+dz^2), \\
h(t) &= \tilde{h}(\overline{t}),
\end{aligned}
$$

where $\tilde{A}(\overline{t},r) = C(t)D(r) = \tilde{C}(\overline{t})D(r)$. Without loss of generality, we can assume as follows:

$$\overline{G} = -dt^2 + A(t,r)(dx^2+dy^2+dz^2),$$

$$h = h(t),$$

where $A(t,r) = C(t)D(r)$. Using these forms of $\overline{G}$ and h, (4), (5) and (7) become as follows:

$$\frac{3}{4}\left(\frac{C_t}{C}\right)^2 + \frac{1}{CD^2}\left\{D_{rr} + \frac{2}{r}D_r - \frac{3}{4D}(D_r)^2\right\} = (h_t)^2, \tag{8}$$

$$-\left\{C_{tt} - \frac{1}{4C}(C_t)^2\right\} + \frac{1}{2D}\left\{D_{rr} + \frac{1}{r}D_r - \frac{1}{D}(D_r)^2\right\} = CD(h_t)^2, \tag{9}$$

$$-\frac{1}{2D}\left\{D_{rr} - \frac{1}{r}D_r - \frac{3}{2D}(D_r)^2\right\} = 0. \tag{10}$$

Equality (6) becomes as follows:

$$h_{tt} + \frac{3}{2C}(C_t h_t) = \frac{1}{2h_t C^3}\left\{(h_t)^2 C^3\right\}_t = 0. \tag{11}$$

(8)$\sim$(11) are the fundamental equations of this paper.

Using a new function: $\frac{1}{D} = f^2$, (10) becomes as follows:

$$-\frac{2}{f^3}\left(f_{rr} - \frac{1}{r}f_r\right) = 0,$$

where $f_r = \frac{\partial f}{\partial r}$, $f_{rr} = \frac{\partial^2 f}{\partial r^2}$. Integrating this equality, we have

$$f(r) = 1 + \frac{\varepsilon r^2}{4}, \qquad \varepsilon = -1, 0, 1,$$

and $D(r)$ becomes as follows:

$$D(r) = \left(1 + \frac{\varepsilon r^2}{4}\right)^{-2}, \qquad \varepsilon = -1, 0, 1,$$

where we use a boundary condition such that

$$\lim_{r \to +\infty} D(r) = 1.$$

Putting these $D(r)$ into (8),(9) and (11), we have the following:

$$\frac{3}{4}\left(\frac{C_t}{C}\right)^2 + \frac{3\varepsilon}{C} = (h_t)^2, \tag{12}$$

$$\frac{C_{tt}}{C} - \frac{1}{4}\left(\frac{C_t}{C}\right)^2 + \frac{\varepsilon}{C} = -(h_t)^2, \tag{13}$$

$$\left\{(h_t)^2 C^3\right\}_t = 0. \tag{14}$$

Differentiating (12) by t, we have

$$\frac{3}{2}\left(\frac{C_t}{C}\right)\left(\frac{C_{tt}}{C}\right)-\frac{3}{2}\left(\frac{C_t}{C}\right)^3-\frac{3\varepsilon}{C}\left(\frac{C_t}{C}\right)=2(h_t)(h_{tt}).$$

Using (14) to the right side of the above equality, we have as follows:

$$\begin{aligned}&-\frac{3}{2}\left(\frac{C_t}{C}\right)\left\{\frac{3}{4}\left(\frac{C_t}{C}\right)^2+\frac{3\varepsilon}{C}-(h_t)^2\right\}\\&+\frac{3}{2}\left(\frac{C_t}{C}\right)\left\{\frac{C_{tt}}{C}-\frac{1}{4}\left(\frac{C_t}{C}\right)^2+\frac{\varepsilon}{C}+(h_t)^2\right\}=0.\end{aligned}$$

This equality shows that (12) and (14) imply (13). Integrating (14), we have

$$(h_t)^2=\frac{3\eta^2}{C^3},\qquad \eta\geq 0.$$

Substituting this equality into (12), we have

$$\frac{1}{4}\left(\frac{C_t}{C}\right)^2+\frac{\varepsilon}{C}-\frac{\eta^2}{C^3}=0.$$

Summerizing the preceding results of this section, we have

Lemma 2. *Let $\Gamma(\Psi, G)$ be an Otsuki connection with the condition (A) and $\overline{G}$ and h have forms such that*

$$\begin{aligned}\overline{G}&=-B(t,r)dt^2+A(t,r)(dx^2+dy^2+dz^2),\\h&=h(t),\end{aligned}$$

where $B_t\neq 0$ and $A(t,r)=C(t)D(r)$. Then $\overline{G}$ becomes

$$\overline{G}=-dt^2+C(t)D(r)(dx^2+dy^2+dz^2)$$

and $h(t)$, $C(t)$ and $D(r)$ satisfy the following equalities:

$$(h_t)^2=\frac{3\eta^2}{C^3},\qquad \eta\geq 0, \tag{15}$$

$$D(r)=\left(1+\frac{\varepsilon r^2}{4}\right)^{-2},\qquad \varepsilon=-1,0,1, \tag{16}$$

$$\frac{1}{4}\left(\frac{C_t}{C}\right)^2+\frac{\varepsilon}{C}-\frac{\eta^2}{C^3}=0. \tag{17}$$

A case $\eta = 0$ in Lemma 2 is easy to treat. Then, (15) implies $h \equiv 0$, which means that $\Gamma(\Psi, G)$ is the Levi-Civita connection of Riemannian manifold (M, G), where $M \subseteq \mathbf{R^4}$. (17) implies that $C(t) = t^2$ for $\varepsilon = -1$, $C(t) \equiv 1$ for $\varepsilon = 0$ and no solution for $\varepsilon = 1$. Now we have

Lemma 3. *For a case $\eta = 0$ in Lemma 2, $\Gamma(\Psi, G)$ becomes the Levi-Civita connection of Riemannian manifold (M, G), $M \subseteq \mathbf{R^4}$, such that G is either of the following two:*

$$\begin{aligned} G &= -dt^2 + t^2\left(1 - \frac{r^2}{4}\right)^{-2}(dx^2 + dy^2 + dz^2), \\ G &= -dt^2 + dx^2 + dy^2 + dz^2. \end{aligned}$$

From now on we suppose $\eta > 0$ and we define a function $\zeta(t)$ by

$$C(t) = \eta^2\zeta^2(t).$$

It is tedious to display every detail of calculations for all the cases $\varepsilon = -1, 0, 1$, but we give details of calculations for one case $\varepsilon = -1$ and only the results for other two. Using $\zeta(t)$, (17) is written as follows:

$$(\zeta_t)^2 = \frac{1}{\eta^2}\left(\frac{1 + \zeta^4}{\zeta^4}\right).$$

Defining a non-negative function $\vartheta(t)$ by

$$\zeta^2(t) = \sinh(\vartheta(t))$$

and inserting it into the above equality, we have

$$\cosh^2(\vartheta)(\vartheta_t)^2 = \frac{4}{\eta^2}\left(\frac{1 + \zeta^4}{\zeta^2}\right).$$

This equality is rewritten as follows:

$$dt = \frac{1}{2}\eta\sqrt{\sinh(\vartheta)}d\vartheta. \tag{18}$$

(18) shows that $\vartheta(t)$ is the inverse function of

$$t(\vartheta) = \frac{1}{2}\int_0^{\vartheta}\sqrt{\sinh(s)}ds.$$

Using

$$C(t) = \eta^2\zeta^2(t) = \eta^2\sinh(\vartheta(t))$$

and (18) in (15), we have

$$(h_t)^2 = \frac{3}{\eta^2 \sinh^3(\vartheta)},$$
$$h_\vartheta = h_t \frac{dt}{d\vartheta} = \pm\frac{\sqrt{3}}{2}\left(\frac{1}{\sinh(\vartheta)}\right),$$

where $h_\vartheta = \frac{\partial h}{\partial \vartheta}$. Integrating the second equality of the aboves, we have

$$h(t) = \mp\frac{\sqrt{3}}{2}\int_{\vartheta(t)}^{\infty}\frac{ds}{\sinh(s)}.$$

Now we find $\overline{G}$ and h, i.e. Otsuki connections $\Gamma(\Psi, G)$, as follows:

$$\begin{aligned}\overline{G} &= -dt^2 + \eta^2 \sinh(\vartheta(t))\left(1 - \frac{r^2}{4}\right)^{-2}(dx^2 + dy^2 + dz^2),\\ h(t) &= \pm\frac{\sqrt{3}}{2}\int_{\vartheta(t)}^{\infty}\frac{ds}{\sinh(s)}.\end{aligned}$$

The other cases are similar to the above calculations.

Theorem 1. *Under the same conditions as Lemma 2, $\overline{G}$ and $h(t)$ are given as follows:*

(Riemannian Type)

1. $\varepsilon = 0$

$$\overline{G} = -dt^2 + dx^2 + dy^2 + dz^2.$$

2. $\varepsilon = -1$

$$\overline{G} = -dt^2 + t^2\left(1 - \frac{r^2}{4}\right)^{-2}(dx^2 + dy^2 + dz^2).$$

(Otsuki Type)

1. $\varepsilon = -1$

$$\begin{aligned}\overline{G} &= -dt^2 + \eta^2 \sinh(\vartheta(t))\left(1 - \frac{r^2}{4}\right)^{-2}(dx^2 + dy^2 + dz^2),\\ h(t) &= \pm\frac{\sqrt{3}}{2}\int_{\vartheta(t)}^{\infty}\frac{ds}{\sinh(s)}.\end{aligned}$$

2. $\varepsilon = 0 \quad (t > 0)$

$$\begin{aligned} \overline{G} &= -dt^2 + \mu^2 t^{\frac{2}{3}} \left(1 - \frac{r^2}{4}\right)^{-2} (dx^2 + dy^2 + dz^2), \quad \mu^3 = 3\eta^2, \\ h(t) &= \pm \frac{1}{\sqrt{3}} \log(t). \end{aligned}$$

3. $\varepsilon = 1 \quad (0 \leq \vartheta \leq \pi)$

$$\begin{aligned} \overline{G} &= -dt^2 + \eta^2 \sin(\vartheta(t)) \left(1 + \frac{r^2}{4}\right)^{-2} (dx^2 + dy^2 + dz^2), \\ h(t) &= \pm \frac{\sqrt{3}}{2} \int_{\vartheta(t)}^{\frac{\pi}{2}} \frac{ds}{\sin(s)}, \end{aligned}$$

where $\vartheta(t)$ *is the inverse function of*

$$t(\vartheta) = \frac{1}{2} \int_0^{\vartheta} \sqrt{\sinh(s)} ds.$$

§3. The Essential Metrics, the Variable w and the Function $R(w)$

As discussed in Section 1, neither a metric G nor $\overline{G}$ but $\tilde{G} \equiv \Psi^2 G = \Psi^{-1}\overline{G}$, which is called the essential metric of $\Gamma(\Psi, G)$, has an important meaning on a manifold M with an Otsuki connection $\Gamma(\Psi, G)$. Lemma 1 says that any geodesic in $M \backslash Sing(\Gamma)$ with Otsuki connection $\Gamma(\Psi, G)$ is a geodesic in a Riemannian manifold $(M \backslash Sing(\Gamma), \tilde{G})$ and vice versa, where $Sing(\Gamma) \subseteq \mathbf{M}$ is defined as follows:

$$Sing(\Gamma) = \{x \in \mathbf{M} \mid \Psi(x) = 0\}.$$

Using a new variable w, which is defined by

$$w(t) = \int_0^t \Psi^{-\frac{1}{2}}(s) ds,$$

the essential metric $\tilde{G}$ i.e.

$$\tilde{G} \equiv \Psi^{-1} G = -\Psi^{-1}(t) dt^2 + \Psi^{-1}(t) C(t) \left(1 + \frac{\varepsilon r^2}{4}\right)^{-2} (dx^2 + dy^2 + dz^2),$$

becomes as follows:

$$\tilde{G} = -dw^2 + R^2(w)\left(1 + \frac{\varepsilon r^2}{4}\right)^{-2}(dx^2 + dy^2 + dz^2),$$

where $R^2(w) = \Psi^{-1}(t)C(t)$. The induced metric of $\tilde{G}$ on a hyperplane $\mathbf{H_w} \subset \mathbf{M} \subset \mathbf{R^4}$ such that

$$\mathbf{H_w} = \{(w, x, y, z) \in \mathbf{M} \mid \quad w = const.\} \hookrightarrow_i \mathbf{M}$$

is given as follows:

$$\begin{aligned} i^*(\tilde{G}) &= R^2\left(1 + \frac{\varepsilon r^2}{4}\right)^{-2}(dx^2 + dy^2 + dz^2) \\ &= \left(1 + \frac{\varepsilon \tilde{r}^2}{4R^2}\right)^{-2}(d\tilde{x}^2 + d\tilde{y}^2 + d\tilde{z}^2), \end{aligned}$$

where $R = R(w)$, $\tilde{x} = Rx$, $\tilde{y} = Ry$, $\tilde{z} = Rz$, $\tilde{r}^2 = \tilde{x}^2 + \tilde{y}^2 + \tilde{z}^2$. Since this metric is just that of a sphere with a radius $R = R(w)$ for $\varepsilon = 1$, $R = R(w)$, which can be understood as the radius of the model at w. Rewriting Theorem 1 using the variable w, we have

Theorem 2. *Let $\Gamma(\Psi, G)$ be a Otsuki connection with the condition (A) which has the forms as follows:*

$$\begin{aligned} \overline{G} &= -dt^2 + C(t)D(r)(dx^2 + dy^2 + dz^2), \\ h &= h(t) \end{aligned}$$

and $\Gamma(\Psi, G)$ be an Otsuki type i.e. $h \neq const.$, then the coresponding essential metric $\tilde{G}$ and the function $R(w)$ become as follows:

I

$$\begin{aligned} \overline{G} &= -dt^2 + \eta^2 \sinh(\vartheta(t))\left(1 - \frac{r^2}{4}\right)^{-2}(dx^2 + dy^2 + dz^2), \\ h(t) &= \frac{\sqrt{3}}{2}\int_{\vartheta(t)}^{\infty} \frac{ds}{\sinh(s)}, \\ \tilde{G} &= -dw^2 + R^2(w)\left(1 - \frac{r^2}{4}\right)^{-2}(dx^2 + dy^2 + dz^2), \\ R^2(w) &= \eta^2 \exp\left(+\int_{\vartheta(t)}^{\infty} \frac{ds}{\sinh(s)}\right)\sinh(\vartheta(t)), \end{aligned}$$

where $w(t) = \int_0^t \Psi^{-\frac{1}{2}}(s)ds = \int_0^t ds \exp\left(-\int_{\vartheta(s)}^{\infty} \frac{du}{\sinh(u)}\right)$ and $\vartheta(t)$ is the inverse function of $t(\vartheta) = \frac{1}{2}\eta\int_0^{\vartheta} \sinh^{\frac{1}{2}}(s)ds$.

II

$$\begin{aligned}
\overline{G} &= -dt^2 + \eta^2 \sinh(\vartheta(t)) \left(1 - \frac{r^2}{4}\right)^{-2} (dx^2 + dy^2 + dz^2), \\
h(t) &= -\frac{\sqrt{3}}{2} \int_{\vartheta(t)}^{\infty} \frac{ds}{\sinh(s)}, \\
\tilde{G} &= -dw^2 + R^2(w) \left(1 - \frac{r^2}{4}\right)^{-2} (dx^2 + dy^2 + dz^2), \\
R^2(w) &= \eta^2 \exp\left(-\int_{\vartheta(t)}^{\infty} \frac{ds}{\sinh(s)}\right) \sinh(\vartheta(t)),
\end{aligned}$$

where $w(t) = \int_0^t \Psi^{-\frac{1}{2}}(s)ds = \int_0^t ds \exp\left(+\int_{\vartheta(s)}^{\infty} \frac{du}{\sinh(u)}\right)$ *and* $\vartheta(t)$ *is the inverse function of* $t(\vartheta) = \frac{1}{2}\eta \int_0^{\vartheta} \sinh^{\frac{1}{2}}(s)ds$.

III

$$\begin{aligned}
\overline{G} &= -dt^2 + \sqrt{3}\eta t^{\frac{2}{3}} (dx^2 + dy^2 + dz^2), \\
h(t) &= \frac{1}{\sqrt{3}} \log(t), \\
\tilde{G} &= -dw^2 + (3\eta^2)^{\frac{2}{3}} (dx^2 + dy^2 + dz^2), \\
R^2(w) &= (3\eta^2)^{\frac{2}{3}} = const.,
\end{aligned}$$

where $w(t) = \int_0^t \Psi^{-\frac{1}{2}}(s)ds = \frac{3}{2} t^{\frac{2}{3}}$.

IV

$$\begin{aligned}
\overline{G} &= -dt^2 + (3\eta^2)^{\frac{2}{3}} t^{\frac{2}{3}} (dx^2 + dy^2 + dz^2), \\
h(t) &= -\frac{1}{\sqrt{3}} \log(t), \\
\tilde{G} &= -dw^2 + \frac{4}{3}(3\eta^2)^{\frac{2}{3}} w (dx^2 + dy^2 + dz^2), \\
R^2(w) &= \frac{4}{3}(3\eta^2)^{\frac{2}{3}} w,
\end{aligned}$$

where $w(t) = \int_0^t \Psi^{-\frac{1}{2}}(s)ds = \frac{3}{4} t^{\frac{4}{3}}$.

V

$$\begin{aligned}
\overline{G} &= -dt^2 + \eta^2 \sin(\vartheta(t)) \left(1 + \frac{r^2}{4}\right)^{-2} (dx^2 + dy^2 + dz^2), \\
h(t) &= \frac{\sqrt{3}}{2} \int_{\vartheta(t)}^{\frac{\pi}{2}} \frac{ds}{\sin(s)},
\end{aligned}$$

$$\begin{aligned}\tilde{G} &= -dw^2 + R^2(w)\left(1+\frac{r^2}{4}\right)^{-2}(dx^2+dy^2+dz^2),\\ R^2(w) &= \eta^2 \exp\left(+\int_{\vartheta(t)}^{\frac{\pi}{2}}\frac{ds}{\sin(s)}\right)\sin(\vartheta(t)),\end{aligned}$$

where $w(t) = \int_0^t \Psi^{-\frac{1}{2}}(s)ds = \int_0^t ds \exp\left(-\int_{\vartheta(s)}^{\frac{\pi}{2}}\frac{du}{\sin(u)}\right)$ *and* $\vartheta(t)$ *is the inverse function of* $t(\vartheta) = \frac{1}{2}\eta\int_0^{\vartheta}\sin^{\frac{1}{2}}(s)ds$.

VI

$$\begin{aligned}\overline{G} &= -dt^2 + \eta^2\sin(\vartheta(t))\left(1+\frac{r^2}{4}\right)^{-2}(dx^2+dy^2+dz^2),\\ h(t) &= -\frac{\sqrt{3}}{2}\int_{\vartheta(t)}^{\frac{\pi}{2}}\frac{ds}{\sin(s)},\\ \tilde{G} &= -dw^2 + R^2(w)\left(1+\frac{r^2}{4}\right)^{-2}(dx^2+dy^2+dz^2),\\ R^2(w) &= \eta^2 \exp\left(-\int_{\vartheta(t)}^{\frac{\pi}{2}}\frac{ds}{\sin(s)}\right)\sinh(\vartheta(t)),\end{aligned}$$

where $w(t) = \int_0^t \Psi^{-\frac{1}{2}}(s)ds = \int_0^t ds \exp\left(+\int_{\vartheta(s)}^{\frac{\pi}{2}}\frac{du}{\sin(u)}\right)$ *and* $\vartheta(t)$ *is the inverse function of* $t(\vartheta) = \frac{1}{2}\eta\int_0^{\vartheta}\sin^{\frac{1}{2}}(s)ds$.

The functions $R(w)$ in Theorem 2, which are measured by the variables w, have some elementary properties. Especially for type I in Theorem 2, we have

Lemma 4. *The range of* w *becomes* $0 \le w < +\infty$ *and there exists positive constants* α, β *such that*

1.
$$\lim_{w\to+\infty} R(w) = \alpha.$$

2.
$$\frac{1}{\beta} \le \lim_{w\to+\infty}\frac{R(w)}{w^2} \le \beta.$$

3.
$$\frac{dR(w)}{dw} > 0$$

for any $0 < w < +\infty$.

4.

$$\lim_{w\to 0}\frac{dR(w)}{dw}=0.$$

Proof. Since

$$s \le \sinh(s) \le (e+e^{-1})s, \tag{19}$$

$$\left(\frac{1-e^{-1}}{2}\right)e^{s} \le \sinh(s) \le \frac{1}{2}e^{s} \tag{20}$$

for any $0 \le s \le 1$, $1 \le s < +\infty$ respectively, we have

$$\left(\frac{1}{\vartheta(t)}\right)^{\frac{1}{e+e^{-1}}} \le \exp\left(\int_{\vartheta(t)}^{1}\frac{ds}{\sinh(s)}\right) \le \frac{1}{\vartheta(t)} \tag{21}$$

for any $0 \le \vartheta(t) \le 1$ and

$$\exp(2e^{-\vartheta(t)}) \le \exp\left(\int_{\vartheta(t)}^{\infty}\frac{ds}{\sinh(s)}\right) \le \exp\left(\left(\frac{2}{1-e^{-1}}\right)e^{-\vartheta(t)}\right) \tag{22}$$

for any $1 \le \vartheta(t) < +\infty$. By (19), (20) and an explicit form of $w(t)$:

$$w(t)=\frac{1}{2}\eta\int_{0}^{\vartheta(t)}d\xi\,\sinh^{\frac{1}{2}}(\xi)\exp\left(\frac{1}{2}\int_{\xi}^{\infty}\frac{ds}{\sinh(s)}\right), \tag{23}$$

there exist $0 < \lambda_1 < \delta_1$ such that

$$\lambda_1\xi^{\frac{1}{2}\left(1-\frac{1}{e+e^{-1}}\right)} \le \sinh^{\frac{1}{2}}(\xi)\exp\left(\frac{1}{2}\int_{\xi}^{1}\frac{ds}{\sinh(s)}\right) \le \delta_1 \tag{24}$$

for any $0 \le \xi \le 1$ and by (20) and (22), there exist $0 < \lambda_2 < \delta_2$ such that

$$\lambda_2 e^{\frac{\xi}{2}} \le \sinh^{\frac{1}{2}}(\xi)\exp\left(\frac{1}{2}\int_{\xi}^{\infty}\frac{ds}{\sinh(s)}\right) \le \delta_2 e^{\frac{\xi}{2}} \tag{25}$$

for any $1 \le \xi < +\infty$. Using (23) $\sim$ (25) and

$$\frac{dw}{dt}=\Psi^{-\frac{1}{2}}(t)=\exp\left(-\frac{1}{2}\int_{\vartheta(t)}^{\infty}\frac{ds}{\sinh(s)}\right),$$

we have $\frac{dw}{dt} > 0$ for any $0 \le t < +\infty$, $lim_{t\to+0}w(t)=0$ and $lim_{t\to+\infty}w(t)=+\infty$. Under these preparations, we will prove 1 $\sim$ 4. By (24), (25) and the explicit form such that

$$R^2(w)=\eta^2\sinh(\vartheta(t))\exp\left(+\int_{\vartheta(t)}^{\infty}\frac{ds}{\sinh(s)}\right), \tag{26}$$

there exist λ_3,$\delta_3 > 0$ such that

$$\eta^2((\lambda_2)^2 e^{\vartheta} + \lambda_3) \leq R^2(w) \leq \eta^2((\delta_2)^2 e^{\vartheta} + \delta_3) \tag{27}$$

for any $1 \leq \vartheta < +\infty$. On the other hand by (23) $\sim$ (25), there exist λ_4, $\delta_4 > 0$ such that

$$\eta \left(\lambda_2 e^{\frac{\vartheta}{2}} + \lambda_4\right) \leq w(t) \leq \eta \left(\delta_2 e^{\frac{\vartheta}{2}} + \delta_4\right) \tag{28}$$

for any $0 \leq t < +\infty$. By (26) $\sim$ (28), we have

$$\lim_{w \to +\infty} R(w) = \eta\sqrt{\mu_1}, \qquad \frac{\lambda_2}{\delta_2} \leq \lim_{w \to +\infty} \frac{R(w)}{w^2} \leq \frac{\delta_2}{\lambda_2},$$

where $\mu_1 = \lim_{\vartheta \to +0} \sinh(\vartheta) \exp\left(\int_{\vartheta}^{\infty} \frac{ds}{\sinh(s)}\right)$. Using an equality:

$$\frac{dR(w)}{dw} = \frac{1}{\sinh(\vartheta)} \{\cosh(\vartheta) - 1\} \exp\left(\frac{1}{2}\int_{\vartheta}^{\infty} \frac{ds}{\sinh(s)}\right),$$

we have $\frac{dR(w)}{dw} > 0$ for any $0 < w < +\infty$ and $lim_{w \to +0} \frac{dR(w)}{dw} = 0$.

Next we discuss the functions $R(w)$ of type V in Theorem 2.

Lemma 5. *For the type V in Theorem 2, there exist δ, $\nu > 0$ such that*

1. $\frac{dR(w)}{dw} < 0$ *for any* $0 < w < \nu$,
2. $\lim_{w \to +0} \frac{dR(w)}{dw} = 0$,
3. $\lim_{w \to \nu - 0} \frac{dR(w)}{dw} = 0$,
4. $\lim_{w \to +0} R(w) = \delta$,
5. $\lim_{w \to \nu - 0} R(w) = 0$.

Proof. Since

$$\frac{2}{\pi} s \leq \sin(s) \leq s, \tag{29}$$

$$\frac{2}{\pi}(\pi - s) \leq \sinh(s) \leq \pi - s \tag{30}$$

for any $0 \leq s \leq \frac{\pi}{2}$, $\frac{\pi}{2} \leq s \leq \pi$ respectively, we have

$$\frac{\pi}{2\vartheta} \leq \exp\left(\int_0^{\frac{\pi}{2}} \frac{ds}{\sin(s)}\right) \leq \left(\frac{\pi}{2\vartheta}\right)^{\frac{\pi}{2}} \tag{31}$$

for any $0 \le \vartheta \le \frac{\pi}{2}$ and

$$\left\{\frac{2}{\pi}(\pi-\vartheta)\right\}^{\frac{\pi}{2}} \le \exp\left(\int_{\vartheta}^{\frac{\pi}{2}} \frac{ds}{\sin(s)}\right) \le \frac{2}{\pi}(\pi-\vartheta) \tag{32}$$

for any $\frac{\pi}{2} \le \vartheta \le \pi$. By (29), (31) and an explicit form of $w(t)$ such that

$$w(t) = \frac{1}{2}\eta \int_0^{\vartheta} d\xi \sin^{\frac{1}{2}}(\xi) \exp\left(\frac{1}{2}\int_{\xi}^{\frac{\pi}{2}} \frac{ds}{\sin(s)}\right), \tag{33}$$

we have

$$1 \le \sin^{\frac{1}{2}}(\xi) \exp\left(\frac{1}{2}\int_{\xi}^{\frac{\pi}{2}} \frac{ds}{\sin(s)}\right) \le \left(\frac{\pi}{2}\right)^{\frac{\pi}{4}} \left(\frac{1}{\xi}\right)^{\frac{\pi-2}{4}} \tag{34}$$

for any $0 < \xi \le \frac{\pi}{2}$ and by (30) and (32), we have

$$\left\{\frac{\pi}{2}(\pi-\xi)\right\}^{\frac{\pi+2}{4}} \le \sin^{\frac{1}{2}}(\xi) \exp\left(\frac{1}{2}\int_{\xi}^{\frac{\pi}{2}} \frac{ds}{\sin(s)}\right) \le \sqrt{\frac{2}{\pi}}(\pi-\xi) \tag{35}$$

for any $\frac{\pi}{2} \le \xi < \pi$. Using the explicit form of $\vartheta(t)$, (33) $\sim$ (35) and

$$\frac{dw}{dt} = \Psi^{-\frac{1}{2}}(t) = \exp\left(-\frac{1}{2}\int_{\vartheta}^{\frac{\pi}{2}} \frac{ds}{\sin(s)}\right),$$

we have $\frac{dw}{dt} > 0$ for any $0 < \vartheta < \pi$, $\lim_{\vartheta\to+0} w(t) = 0$ and $\lim_{\vartheta\to\pi-0} w(t) = \nu < +\infty$, where

$$\nu = \frac{1}{2}\eta \int_0^{\pi} d\xi \sin^{\frac{1}{2}}(\xi) \exp\left(\int_{\xi}^{\frac{\pi}{2}} \frac{ds}{\sin(s)}\right).$$

Under these preparations we will prove 1 $\sim$ 5. By (33) $\sim$ (35) and the formula:

$$R^2(w) = \eta^2 \sin(\vartheta) \exp\left(\int_{\vartheta}^{\frac{\pi}{2}} \frac{ds}{\sin(s)}\right),$$

we have

$$\frac{dR(w)}{dw} = \left(\frac{\cos(\vartheta)-1}{\sin(\vartheta)}\right) \exp\left(\frac{1}{2}\int_{\vartheta}^{\frac{\pi}{2}} \frac{ds}{\sin(s)}\right) \tag{36}$$

The right side of (36) is negative on $0 < w < \nu$ and tends to zero when $\vartheta \to +0$ i.e. $w \to +0$, so is a bounded function on $0 \le w \le \nu_1$, where ν_1 is a small positive number. Thus $R(w)$ is a bounded function on $0 \le w \le \nu$ and has the following properties:

$\frac{dR(w)}{dw} < 0$ for any $0 < w < \nu$,

$\lim_{w\to+0} R(w) = \eta\sqrt{\lambda_1} \equiv \delta < +\infty$,

$\lim_{w\to\nu-0} R(w) = 0, \qquad \lim_{w\to+0} \frac{dR(w)}{dw} = 0$,

$\lim_{w\to\nu-0} \frac{dR(w)}{dw} = -\infty$,

where

$\lambda_1 = \lim_{\vartheta\to+0} \sin(\vartheta) \exp\left(\int_{\vartheta}^{\frac{\pi}{2}} \frac{ds}{\sin(s)}\right) < +\infty.$

Regarding properties of the functions $R(w)$, the other cases in Theorem 2 are trivial or almost the same as Lemma 4 or Lemma 5. Now we have the following

Theorem 3. *The functions $R(w)$ in Theorem 2, which are measured by the variables w, have the following properties:*

I The range of w becomes $0 \leq w < +\infty$ and there exist α, $\beta > 0$ such that

1. $\lim_{w\to+0} R(w) = \alpha$,
2. $\frac{1}{\beta} \leq \lim_{w\to+\infty} \frac{R(w)}{w^2} \leq \beta$,
3. $\frac{dR(w)}{dw} > 0$ *for any* $0 < w < +\infty$,
4. $\lim_{w\to+0} \frac{dR(w)}{dw} = 0$.

II The range of w becomes $0 \leq w < +\infty$ and there exists $\delta > 0$ such that

1. $\lim_{w\to+0} R(w) = 0$,
2. $\frac{1}{\delta} \leq \lim_{w\to+\infty} \frac{R(w)}{w^2} \leq \delta$,
3. $\frac{dR(w)}{dw} > 0$ *for any* $0 < w < +\infty$,
4. $\lim_{w\to+0} \frac{dR(w)}{dw} = +\infty$.

III $R(w) = const.$. Thus, this case is trivial.

IV $R(w) = const.\sqrt{w}$. Thus, this case is trivial.

V There exist δ, $\nu > 0$ such that

1. *$R(w)$ is defined on $0 \leq w \leq \nu$,*
2. $\frac{dR(w)}{dw} < 0$ *for any* $0 < w < \nu$,
3. $\lim_{w\to+0} \frac{dR(w)}{dw} = 0$,

4. $\lim_{w \to \nu - 0} \frac{dR(w)}{dw} = -\infty$,
5. $\lim_{w \to +0} R(w) = \delta$,
6. $\lim_{w \to \nu - 0} R(w) = 0$.

VI The functions $R(w)$ of these models are the same as $R(\nu - w)$ in V.

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