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A Note on Hamiltonian Cycles in (k, n)-Factor-Critical Graphs

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Abstract. A graph G is said to be (k, n)-*factor-critical* if $G - S$ has a k-factor for any $S \subset V(G)$ with $|S| = n$. In [7], the author, Ota and Saito conjectured that if G is a 2-connected (k, n)-factor-critical graph of order p with $\sigma_3(G) \geq \frac{3}{2}(p - n - k)$, then G is hamiltonian with some exceptions. In [7], the author, Ota and Saito also characterized all those graphs which satisfy the assumption of the conjecture, but are not 1-tough and, by using this, they verified the conjecture for $k = 1$ and 2. In this paper, we verify the conjecture for $k = 3$ and 4.

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§1. Introduction

In this paper, all graphs considered are finite, undirected and without loops or multiple edges. For graph theoretic notation, we refer the reader to [4]. In particular, we denote by $\alpha(G)$ and $\delta(G)$ the independence number and the minimum degree of a graph G, respectively.

For an integer k with $k \leq \alpha(G)$, we define $\sigma_k(G)$ by

$$\sigma_k(G) = \min \left\{ \sum_{x \in S} \deg_G x \colon S \text{ is an independent set of order } k \text{ in } G \right\}.$$

For $k > \alpha(G)$, we define $\sigma_k(G) = +\infty$. We call $\sigma_k(G)$ the *minimum degree sum* of k vertices in G.

Starting with Ore's classical theorem [8] on degree sums and hamilton cycles, there are a lot of papers about degree sums and hamilton cycles. Ore's theorem is best-possible in the sense that the lower bound $p = |G|$ of $\sigma_2(G)$

cannot be replaced by $p-1$. Let $G = K_{m,m+1}$ $(m \geq 2)$. Then $p = 2m+1$, $\sigma_2(G) = 2m = p-1$ and G is not hamiltonian. However, if we put additional assumptions on G, the lower bound of $\sigma_2(G)$ in Ore's theorem may be relaxed. In fact, Faudree and van den Heuvel [5] proved that the existence of a k-factor relaxes the degree sum condition.

Theorem 1 ([5]) *Let G be a 2-connected graph of order p. If G has a k-factor and $\sigma_2(G) \geq p-k$, then G is hamiltonian.*

A graph G is said to be *n-factor-critical* if $|G| \geq n+2$ and $G-S$ has a 1-factor for each $S \subset V(G)$ with $|S| = n$. Motivated by this theorem, the author, Ota and Saito studied degree sum conditions for an n-factor-critical graph to be hamiltonian, and proved the following theorem.

Theorem 2 ([7]) *Let n be a nonnegative integer and let G be a 2-connected n-factor-critical graph of order p. Suppose $\sigma_3(G) \geq \frac{3}{2}(p-n-1)$. Then*

(1) G is hamiltonian,

(2) $\overline{K_n} + (n+1)K_2 \subset G \subset K_n + (n+1)K_2$,

(3) G is a spanning subgraph of $K_{n+1} + (K_1 \cup (n+1)K_2)$, or

(4) $n = 2$ and $\overline{K_2} + (K_4 \cup 2K_2) \subset G \subset K_2 + (K_4 \cup 2K_2)$.

They also studied the possibility of extending their results to a wider class. For a positive integer k and a nonnegative integer n, a graph G is said to be *(k,n)-factor-critical* if $|G| \geq k+n+1$ and $G-S$ has a k-factor for each $S \subset V(G)$ with $|S| = n$. Under this definition, a graph is n-factor-critical if and only if it is $(1,n)$-factor-critical, and a graph has a k-factor if and only if it is $(k,0)$-factor-critical.

They proved the following lemma.

Lemma 3 ([7]) *Let k and n be integers with $k \geq 1$ and $n \geq 0$, and let G be a 2-connected (k,n)-factor-critical graph of order p. If $\sigma_3(G) \geq \frac{3}{2}(p-n-k)$ and G is not 1-tough, then one of the following holds.*

(1) $k = 1$, $n \geq 3$ and $\overline{K_n} + (n+1)K_2 \subset G \subset K_n + (n+1)K_2$

(2) $k = 1$, $n \geq 2$ and G is a spanning subgraph of $K_{n+1} + \big((n+1)K_2 \cup K_1\big)$.

(3) $(k,n) = (2,3)$ and $\overline{K_3} + 4K_3 \subset G \subset K_3 + 4K_3$

(4) $k \equiv 1 \pmod 2$, $n = 2$ and $\overline{K_2} + 3K_{k+1} \subset G \subset K_2 + 3K_{k+1}$

(5) $k \equiv 1 \pmod 2$, $n = 2$ and $\overline{K_2} + (2K_{k+1} \cup K_{k+3}) \subset G \subset K_2 + (2K_{k+1} \cup K_{k+3})$

(6) $k \equiv 1 \pmod 2$, $n = 1$ *and* $\overline{K_2} + (K_k \cup 2K_{k+1}) \subset G \subset K_2 + (K_k \cup 2K_{k+1})$

(7) $k \equiv 0 \pmod 2$ *and* G *is a spanning subgraph of* $K_2 + (G_1 \cup G_2 \cup G_3)$, *where* $\delta(G_i) \geq n + k - 2$ $(1 \leq i \leq 3)$ *and* $|G_1| + |G_2| + |G_3| \leq 3(n + k)$.

They also conjectured the following.

Conjecture 1 *Let* k *and* n *be integers with* $k \geq 1$ *and* $n \geq 0$, *and let* G *be a 2-connected* (k, n)*-factor-critical graph of order* p *with* $\sigma_3(G) \geq \frac{3}{2}(p - k - n)$. *Then* G *is hamiltonian or one of the graphs described in Lemma 3 (1)–(7).*

They verified the conjecture for $k = 1$ and 2 in [7].

The purpose of this paper is to prove the conjecture for $k = 3$ and 4.

Theorem 4 *Conjecture 1 is true for* $k = 3$ *and* 4.

When we consider a cycle C, we always associate with C an orientation $\overrightarrow{C}$. Then we denote the reverse orientation of C by $\overleftarrow{C}$. If $u \in V(C)$, then u^+ denotes the successor of u on $\overrightarrow{C}$ and u^- denotes its predecessor. If $A \subset V(C)$, then $A^+ = \{v^+ | v \in A\}$ and $A^- = \{v^- | v \in A\}$. For $u, v \in V(C)$, $u\overrightarrow{C}v$ denotes the set of consecutive vertices of C from u to v in the direction specified by $\overrightarrow{C}$.

In the subsequent arguments, we adopt one notation introduced in [5]. Let G be a graph and let S, $T \subset V(G)$ (possibly $S \cap T \neq \emptyset$). Furthermore, let $F \subset E(G)$. Then we define $\varepsilon_F(S, T)$ by

$$\varepsilon_F(S, T) = |\{(x, y) \colon x \in S, y \in T, xy \in F\}|.$$

Note that if $xy \in F$ with $x, y \in S \cap T$, then this edge is counted twice as (x, y) and (y, x).

First, we give basic properties of (k, n)-factor-critical graphs.

Lemma 5 ([7]) *Let* k *and* n *be integers with* $k \geq 1$ *and* $n \geq 0$, *and let* G *be a* (k, n)*-factor-critical graph of order* p *with* $\sigma_3(G) \geq \frac{3}{2}(p - k - n)$. *Then*

(1) $\delta(G) \geq n + k$,

(2) $\sigma_3(G) \geq p$, *and*

(3) *if* k *is odd, then* G *is* n*-connected.*

§2. Proof of Theorem 4

In this section, we prove Theorem 4. In the rest of the proof, we assume $3 \le k \le 4$. Also, we assume $n \ge 1$. In fact, in [5], they proved that if $\sigma_3(G) \ge \frac{3}{2}(p-k)$, then G is hamiltonian except for the graphs described in Lemma 3(7) with $k = 2$, where p, k, G are as in Conjecture 1 with $n = 0$. Hence we may assume $n \ne 0$.

By Lemma 3, we have already listed up all exceptions of Theorem 4. In order to establish Theorem 4, we have only to prove the following theorem.

Theorem 6 *Let $k \le 4$ and n be a nonnegative integer, and let G be a (k,n)-factor-critical graph of order p. If $\sigma_3(G) \ge \frac{3}{2}(p-n-k)$ and G is 1-tough, then G is hamiltonian.*

By Lemma 5, we can apply the following theorem in which, the first part was proved by Bauer, Morgana, Schmeichel and Veldman [2], and the second part was proved by Bauer, Broersma and Veldman [1].

Theorem A ([1], [2]) *Let G be a 1-tough graph on $p \ge 3$ vertices with $\sigma_3(G) \ge p$. Then every longest cycle of G has the property that $V(G) - V(C)$ is an independent set. Moreover, if G is nonhamiltonian, then G contains a longest cycle C such that $\max\{d(v) | v \in V(G) - V(C)\} \ge \frac{1}{3}\sigma_3(G)$.*

By Theorem A, we can choose a longest cycle C in G and a vertex $a \in V(G) - V(C)$ such that $N(a) \subset V(C)$ and $\deg_G(a) \ge \frac{1}{3}\sigma_3(G)$. We assume that C and a are chosen so that $\deg_G(a)$ is as large as possible. In the rest of our proof, we use several ideas of the proof of the result of Bondy and Kouider[3], and Faudree and van den Heuvel [5].

Set $Y_0 = \{a\}$ and define, for $i \ge 1$,

$$X_i = N(Y_{i-1}), \quad Y_i = \{a\} \cup \{v \in V(C) | v^-, v^+ \in X_i\}.$$

Then, $N(a) = X_1 \subset X_2 \subset \dots$ and $\{a\} = Y_0 \subset Y_1 \subset \dots$. Set $X = \bigcup_{i=1}^{\infty} X_i$ and $Y = \bigcup_{i=0}^{\infty} Y_i$. Since C is a longest cycle in G and there exists no cycle C' with the same length as C satisfying $\omega(G - V(C')) < \omega(G - V(C))$, we can use the "Hopping Lemma" from Woodall [9].

Theorem B (Hopping Lemma [9]) *Let C, X and Y be defined as above. Then X and Y have the following properties.*

(1) $N(Y) = X \subset V(C)$.

(2) $X \cap X^+ = \emptyset$.

(3) $X \cap Y = \emptyset$.

Set $x = |X|$ and $y = |Y|$ and define $Z^+ = X^+ - Y$ and $Z^- = X^- - Y$, respectively. Then $|Z^+| = |Z^-| = x - y + 1$.

The subgraph $C - X$ consists of segments of the cycle C. There are two types of segments, namely,

(1) a segment consisting of an isolated vertex (the vertices in $Y - \{a\}$), and

(2) a segment consisting of two or more vertices.

The second segments can be considered as paths with one end vertex in Z^+ and the other end vertex in Z^-. We denote these "long" segments by C_0, $C_1, \cdots, C_{x-y}$. We also denote the element of $V(C_i)$ in Z^+ by p_i and the element of $V(C_i)$ in Z^- by q_i. Define $S = \bigcup_{i=0}^{x-y} V(C_i) - Z^+ - Z^-$, $R = V(G) - V(C) - \{a\}$ and $r = |R|$. And also, let $Z = Z^+ \cup Z^-$.

We will use the following lemma proved by Jackson [6].

Lemma 7 ([6]) *Let C, Z^+, Z^- and R be defined as above. Then, the following statements hold.*
(a) Z^+ and Z^- are independent sets.
(b) Every vertex of R has at most one vertex of Z^+ and at most one vertex of Z^- as a neighbor.

Since $x \geq \deg_G(a) \geq \delta(G) \geq n + k$, we can choose $X' \subset X$ with $|X'| = n$. Then, $G' = G - X'$ has a k-factor F. Since $N_G(Y) = X$, so, $N_F(Y) \subset X - X'$. Therefore,

(2.1)
$$\begin{aligned} ky &= \varepsilon_F(Y, X - X') \\ &\leq \varepsilon_F(V(G'), X - X') - \varepsilon_F(Z, X - X') = k(x - n) - \varepsilon_F(Z, X - X'). \end{aligned}$$

Hence, $x - y \geq n$. Assume $x - y = n + t$. Then by (2.1),

$$\varepsilon_F(Z, X - X') \leq k(x - y - n) = kt. \tag{2.2}$$

Since

$$x \geq \deg_G(a) \geq \frac{1}{3}\sigma_3(G) \geq \frac{1}{2}(p - k - n),$$

we have $p \leq 2x + k + n$. Assume $x = \frac{1}{2}(p - k - n) + q$, where q is a nonnegative half integer. Then, we have the following.

$$p = 2x + k + n - 2q. \tag{2.3}$$

Let s denote $|S|$. Then, we have the following.

$$p = |X| + |Y| + |Z| + |S| + |R| = x + y + 2(x - y + 1) + s + r$$

$$= 2x + (x - y) + 2 + s + r = 2x + n + t + s + r + 2.$$

Hence by (2.3), we have

$$k - 2q = t + s + r + 2. \tag{2.4}$$

Since we assume $k \leq 4$, we have $q \leq 1$.

By (2.2) and Lemma 7(b), we have the following:

$$\varepsilon_F(Z, X - X') + \varepsilon_F(Z, S) + \varepsilon_F(Z, R) \leq kt + ks + 2r. \tag{2.5}$$

Since $x \geq \deg_G(a) \geq \delta(G) \geq n+k = x-y-t+k$ and $k = t+s+r+2+2q \geq t+2$, we have $y \geq 2$, and hence $Y - \{a\} \neq \emptyset$. First, we claim the following.

Claim 1 $\deg_G(a) \geq x-q$, *and for some vertex* $v \in Y - \{a\}$, $\deg_G(v) \geq x - \frac{3}{2}q$.

Proof. The first assertion is obvious since $\deg_G(a) \geq \frac{1}{3}\sigma_3(G) \geq \frac{1}{2}(p-k-n) = x - q$.

For any vertex $u \in Y$, $\deg_G(u) \geq \delta(G) \geq n + k = (x - y - t) + (t + s + r + 2 + 2q) \geq x - y + 2$. Hence, if $|Y| = 2$, then the second assertion is also obvious. Suppose $|Y| \geq 3$. Take arbitrary distinct vertices $u, v \in Y - \{a\}$. We may assume $\deg_G(v) \geq \deg_G(u)$. Since Y is independent, we have

$$\deg_G(v) + \deg_G(u) + \deg_G(a) \geq \sigma_3(G) \geq 3x - 3q. \tag{2.6}$$

Since $\deg_G(a) \leq x$, (2.6) implies that $2\deg_G(v) + x \geq 3x - 3q$, or equivalently $\deg_G(v) \geq x - \frac{3}{2}q$. □

By Claim 1, we can choose a vertex $v \in Y - \{a\}$ such that $\deg_G(v) \geq x - \frac{3}{2}q$. Since $|Z^+| = |Z^-| = n + t + 1 \geq 2$, there exist at least two long segments. We may assume that the long segments $C_0, C_1, \ldots, C_{n+t}$ appear in this order along $v^+ \overrightarrow{C} v^-$.

We prove the following claim.

Claim 2 $t \geq 1$.

Proof. Assume $t = 0$. Then by (2.2), we have $\varepsilon_F(Z, X - X') = 0$. Suppose first that $q_0^+ \neq p_{n+t}^-$. Since $q \leq 1$, we have $\deg_G(a) \geq x - 1$ by Claim 1. It follows that $\{v^+, q_0^+\} \subset N_G(a)$ or $\{v^-, p_{n+t}^-\} \subset N_G(a)$. By symmetry, we may assume the latter. Since $\varepsilon_F(\{p_{n+t}\}, X - X') = 0$ and $\varepsilon_F(\{p_{n+t}\}, S \cup R) \leq s + r$, we have $\varepsilon_F(\{p_{n+t}\}, Z^-) \geq k - (s+r) = 2 + 2q$ by Lemma 7(a) and (2.4). Since $\deg_G(v) \geq x - \frac{3}{2}q$ by Claim 1, there exists a vertex $q_i \in Z^-$ with $i \neq n + t$ such that $p_{n+t}q_i, vq_i^+ \in E(G)$. Then, we have a cycle

$$ap_{n+t}^- \overleftarrow{C} q_i^+ v \overrightarrow{C} q_i p_{n+t} \overrightarrow{C} v^- a,$$

which is longer than C, a contradiction.

Suppose $q_0^+ = p_{n+t}^-$. Then $|Z^+| = |Z^-| = 2$. On the other hand, since $\varepsilon_F(\{p_{n+t}\}, X - X') = 0$ and $\varepsilon_F(\{p_{n+t}\}, S \cup R) \leq s + r$, we have $\varepsilon_F(\{p_{n+t}\}, Z^-) \geq 2 + 2q$. Hence we have $q = 0$. Thus by Claim 1, we have $N_G(a) = X$. In particular, this implies that $\{v^-, p_{n+t}^-\} \subset N_G(a)$, and hence the same argument as in the previous paragraph leads us to a contradiction. □

By Claim 2 and (2.4), since $k \leq 4$, we have $q \leq \frac{1}{2}$. Hence by Claim 1, we have $N_G(a) = N_G(v) = X$.

Claim 3 *For any i, j with $0 \leq i < j \leq n + t$, $q_i p_j \notin E(G)$.*

Proof. If $q_i p_j \in E(G)$ for some i, j with $0 \leq i < j \leq n + t$, then, we have a cycle

$$a p_j^- \overleftarrow{C} q_i^+ v \overrightarrow{C} q_i p_j \overrightarrow{C} v^- a,$$

which is longer than C, a contradiction. □

By Claim 3, we have $\varepsilon_F(\{q_0\}, Z^+) \leq 1$, $\varepsilon_F(\{p_{n+t}\}, Z^-) \leq 1$, $\varepsilon_F(\{q_1\}, Z^+) \leq 2$ and $\varepsilon_F(\{p_{n+t-1}\}, Z^-) \leq 2$. Let $Z' = \{q_0, q_1, p_{n+t}, p_{n+t-1}\}$. Then by Lemma 7(a), we have

$$\varepsilon_F(Z', X - X') + \varepsilon_F(Z', S) + \varepsilon_F(Z', R) \geq 4k - 6. \tag{2.7}$$

On the other hand, by (2.5), the left hand side of (2.7) is at most $kt + ks + 2r$. Since $k \geq s + t + r + 2$ by (2.4), we have $4(s + t + r + 2) - 6 \leq k(s + t) + 2r$. Hence $k > 4$, a contradiction. This completes the proof. □

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