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Classification of Hamiltonian Cycles of a 3-Connected Graph Which Contain Five Contractible Edges

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Abstract. We classify all pairs (G, C) of a 3-connected graph G of order at least 16 and a longest cycle C of G such that C contains precisely five contractible edges of G.

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§1. Introduction

In this paper, we consider only finite, simple, undirected graphs with no loops and no multiple edges.

A graph G is called *3-connected* if $|V(G)| \geq 4$ and $G - S$ is connected for any subset S of $V(G)$ having cardinality 2. An edge e of a 3-connected graph G is called *contractible* if the graph which we obtain from G by contracting e (and replacing each of the resulting pairs of parallel edges by a simple edge) is 3-connected; otherwise e is called *noncontractible.* In [6], Tutte proved that all 3-connected graphs other than K_4 have a contractible edge. In [2], Dean, Hemminger and Ota proved that every longest cycle in a 3-connected graph other than K_4 or $K_2 \times K_3$ contains at least three contractible edges. In [3], Ellingham, Hemminger and Johnson proved that every longest cycle in a nonhamiltonian 3-connected graph contains at least six contractible edges. In view of these results, it is likely and desirable that one should obtain a complete classification of those pairs (G, C) of a 3-connected graph G and a longest cycle C of G such that C contains at most five contractible edges. The case where C contains precisely three contractible edges has already been settled by Aldred, Hemminger and Ota in [1] and by Ota in [5]. Further the case where C contains precisely four contractible edges has been settled by

Fujita in [4]. In this paper we are concerned with the case where C contains precisely five contractible edges:

Theorem 1. *Let G be a 3-connected graph of order at least 16, and let C be a longest cycle of G. Suppose that C contains precisely five contractible edges of G. Then the pair (G, C) belongs to one of the 20 types, Types 1 through 20, which are defined in Section 2.*

The organization of this paper is as follows. In Section 2, we define the type of a pair (G, C) satisfying the assumption of Theorem 1. Section 3 contains fundamental results concerning noncontractible edges lying on a hamiltonian cycle of a 3-connected graph. In Section 4, we derive basic properties of a pair (G, C) satisfying the assumption of Theorem 1, and we complete the proof of Theorem 1 in Section 5.

Our notation and terminology are standard except possibly for the following. Let G be a graph. For $U \subseteq V(G)$, we let $\langle U \rangle = \langle U \rangle_G$ denote the graph induced by U in G. For $U, V \subseteq V(G)$, we let $E(U, V)$ denote the set of edges of G which join a vertex in U and a vertex in V; if $U = \{u\}$ $(u \in V(G))$, we write $E(u, V)$ for $E(\{u\}, V)$. A subset S of $V(G)$ is called a *cutset* if $G - S$ is disconnected; thus G is 3-connected if and only if $|V(G)| \geq 4$ and G has no cutset of cardinality 2. If G is 3-connected, then for $e = uv \in E(G)$, we let $K(e) = K(u, v)$ denote the set of vertices x of G such that $\{u, v, x\}$ is a cutset; thus e is contractible if and only if $K(e) = \emptyset$. If e is noncontractible, then for each $x \in K(e)$, $\{u, v, x\}$ is called a *cutset associated with* e.

§2. Definition of the Type of a Pair (G, C)

In this section, we define the type of a pair (G, C) of a 3-connected graph G and a hamiltonian cycle C of G such that C contains precisely five contractible edges of G. Throughout this section, we let n_0, n_1, n_2, n_3 and n_4 be nonnegative integers, and let G denote a graph of order $n_0 + n_1 + n_2 + n_3 + n_4 + 5$ with vertex set $V(G) = \{a_i \,|\, 0 \leq i \leq n_0\} \cup \{b_i \,|\, 0 \leq i \leq n_1\} \cup \{c_i \,|\, 0 \leq i \leq n_2\} \cup \{d_i \,|\, 0 \leq i \leq n_3\} \cup \{e_i \,|\, 0 \leq i \leq n_4\}$ such that G contains $C = a_0 a_1 \cdots a_{n_0} b_0 b_1 \cdots b_{n_1} c_0 c_1 \cdots c_{n_2} d_0 d_1 \cdots d_{n_3} e_0 e_1 \cdots e_{n_4} a_0$ as a hamiltonian cycle. In the definition of each type, it is easy to verify that if G satisfies the required conditions, then G is 3-connected, and $a_{n_0} b_0$, $b_{n_1} c_0$, $c_{n_2} d_0$, $d_{n_3} e_0$, $e_{n_4} a_0$ are the only contractible edges of G that are on C. Further if we let $C_0 = \{a_0, a_1, \ldots, a_{n_0}\}$, $C_1 = \{b_0, b_1, \ldots, b_{n_1}\}$, $C_2 = \{c_0, c_1, \ldots, c_{n_2}\}$, $C_3 = \{d_0, d_1, \ldots, d_{n_3}\}$ and $C_4 = \{e_0, e_1, \ldots, e_{n_4}\}$, then C_0, C_1 and C_4 are nondegenerate and C_2 and C_3 are degenerate in Type 1 (see the paragraph following Lemma 4.3 for the definition of the terms "nondegenerate" and "degenerate"), C_1, C_3 and C_4 are nondegenerate and C_0 and C_2 are degenerate in Type 2, C_3 and C_4 are nondegenerate and C_0, C_1 and C_2 are degenerate in Types 3 through 7, C_1 and C_4 are nondegenerate and C_0, C_2 and C_3 are degenerate in Types 8 and 9, C_4 is nondegenerate and C_0, C_1, C_2 and C_3 are degenerate in Types 10 through 20 (in Types 10 and 11, $n_1 = n_2 = 0$; in Types 12 through 14, $n_1 = 0$ and $n_2 = 2$; in Types 15 through 20, $n_1 = n_2 = 2$).

Type 1. Let $n_0 \geq 1$, $n_1 \geq 1$, $n_2 = 0$ or 2, $n_3 = 0$ or 2, and $n_4 \geq 1$. Let

$$\begin{aligned} X &= \{a_h a_{h+2} \mid 0 \leq h \leq n_0 - 2\} \\ &\cup \{b_i b_{i+2} \mid 0 \leq i \leq n_1 - 2\} \\ &\cup \{e_x e_{x+2} \mid 0 \leq x \leq n_4 - 2\}, \end{aligned}$$

$$F_1 = \{a_1 e_{n_4-1}, a_{n_0-1} b_1\}, \qquad F_1' = \{a_0 e_{n_4-1}, a_1 e_{n_4}, a_{n_0-1} b_0, a_{n_0} b_1\},$$

$$\bar{F}_1 = \begin{cases} \{a_{n_0-1} b_0\} & (\text{if } n_1 = 1 \text{ and } n_2 = 0) \\ \{a_1 e_{n_4}\} & (\text{if } n_4 = 1 \text{ and } n_3 = 0) \\ \emptyset & (\text{otherwise}), \end{cases}$$

$$F_2 = \begin{cases} \{d_0 e_1\} & (\text{if } n_3 = 0) \\ \{d_0 d_2, d_1 e_1\} & (\text{if } n_3 = 2), \end{cases} \qquad F_2' = \begin{cases} \emptyset & (\text{if } n_3 = 0) \\ \{d_1 e_0\} & (\text{if } n_3 = 2), \end{cases}$$

$$F_3 = \begin{cases} \{c_0 b_{n_1-1}\} & (\text{if } n_2 = 0) \\ \{c_0 c_2, c_1 b_{n_1-1}\} & (\text{if } n_2 = 2), \end{cases} \quad \text{and} \quad F_3' = \begin{cases} \emptyset & (\text{if } n_2 = 0) \\ \{c_1 b_{n_1}\} & (\text{if } n_2 = 2). \end{cases}$$

Now G is said to be of Type 1, if we define X, F_1, $\bar{F}_1$, F_1', F_2, F_2', F_3, F_3' as above, then G satisfies $X \cup F_1 \cup \bar{F}_1 \cup F_2 \cup F_3 \subseteq E(G) - E(C) \subseteq X \cup F_1 \cup F_1' \cup F_2 \cup F_2' \cup F_3 \cup F_3'$. The graph depicted in Figure 1 is an example of a graph of Type 1 with $n_0 = n_1 = n_4 = 3$ and $n_2 = n_3 = 0$.

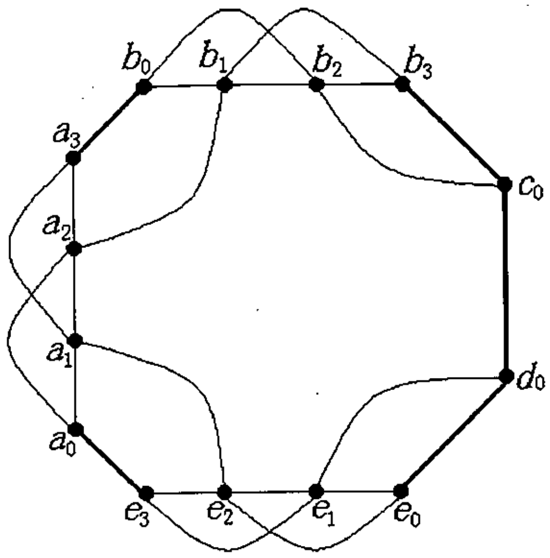

Figure 1

Type 2. Let $n_0 = 0$ or 2, $n_1 \geq 1$, $n_2 = 0$ or 2, $n_3 \geq 1$, and $n_4 \geq 1$. Let r be an integer with

(2.1) $$1 \leq r \leq \min\{n_1 + 1, n_3 + n_4 + 1\},$$

and let t' be an integer with

(2.2) $$2 \leq t' \leq r + 1,$$

and let $k_0, k_1, \ldots, k_r, k_{r+1}$ and $l_1, l_2, \ldots, l_r, l_{r+1}$ be integers such that

(2.3) $$0 = k_0 \leq k_1 < k_2 < \cdots < k_{r-1} < k_r \leq k_{r+1} = n_1,$$

(2.4) $$n_4 = l_1 > l_2 > \cdots > l_{t'-1} > 0$$

and

(2.5) $$n_3 \geq l_{t'} > l_{t'+1} > \cdots > l_r > l_{r+1} = 0.$$

Let

$$\begin{aligned} X_1 \;=\; & \Big(\bigcup_{t=1}^{r+1} \{b_i b_{i+2} \,|\, k_{t-1} \leq i \leq k_t - 2\}\Big) \\ \cup \; & \Big(\bigcup_{t=1}^{t'-2} \{e_x e_{x-2} \,|\, l_t \geq x \geq l_{t+1} + 2\}\Big) \cup \{e_x e_{x-2} \,|\, l_{t'-1} \geq x \geq 2\} \\ \cup \; & \{d_j d_{j-2} \,|\, n_3 \geq j \geq l_{t'} + 2\} \cup \Big(\bigcup_{t=t'}^{r} \{d_j d_{j-2} \,|\, l_t \geq j \geq l_{t+1} + 2\}\Big), \end{aligned}$$

$$X_2' = \begin{cases} \{d_{n_3-1}e_0, d_{n_3}e_1\} & (\text{if } l_{t'} < n_3) \\ \emptyset & (\text{if } l_{t'} = n_3), \end{cases}$$

$$Y_1 = \begin{cases} \{b_{k_t+1}e_{l_{t+1}+1} \mid 1 \le t \le t'-2\} \cup \{b_{k_t+1}d_{l_{t+1}+1} \mid t'-1 \le t \le r-1\} \\ \qquad (\text{if } l_{t'} < n_3) \\ \{b_{k_t+1}e_{l_{t+1}+1} \mid 1 \le t \le t'-2\} \cup \{b_{k_{t'-1}+1}e_1\} \\ \qquad \cup \{b_{k_t+1}d_{l_{t+1}+1} \mid t' \le t \le r-1\} (\text{if } l_{t'} = n_3), \end{cases}$$

$$\bar{Y_1} = \begin{cases} \{b_{k_r+1}d_{l_{r+1}+1}\} & (\text{if } k_r < n_1) \\ \emptyset & (\text{if } k_r = n_1), \end{cases}$$

$$Y_2 = \{b_{k_t-1}e_{l_t-1} \mid 2 \le t \le t'-1\} \cup \{b_{k_t-1}d_{l_t-1} \mid t' \le t \le r\},$$

$$\bar{Y_2} = \begin{cases} \{b_{k_1-1}e_{l_1-1}\} & (\text{if } k_1 > 0) \\ \emptyset & (\text{if } k_1 = 0), \end{cases}$$

$$Y_3' = \begin{cases} \bigcup_{t=1}^{t'-2} \{b_{k_t}e_x \mid l_t \ge x \ge l_{t+1}\} \cup \{b_{k_{t'-1}}e_x \mid l_{t'-1} \ge x \ge 0\} \\ \qquad \cup \{b_{k_{t'-1}}d_j \mid n_3 \ge j \ge l_{t'}\} \cup \bigcup_{t=t'}^{r} \{b_{k_t}d_j \mid l_t \ge j \ge l_{t+1}\} (\text{if } l_{t'} < n_3) \\ \bigcup_{t=1}^{t'-2} \{b_{k_t}e_x \mid l_t \ge x \ge l_{t+1}\} \cup \{b_{k_{t'-1}}e_x \mid l_{t'-1} \ge x \ge 0\} \\ \qquad \cup \bigcup_{t=t'}^{r} \{b_{k_t}d_j \mid l_t \ge j \ge l_{t+1}\} (\text{if } l_{t'} = n_3), \end{cases}$$

$$W_1 = \begin{cases} Y_3' & (\text{if } n_2 = 0) \\ Y_3' \cup \{b_1c_1\} & (\text{if } n_2 = 2), \end{cases} \qquad W_2 = \begin{cases} Y_3' & (\text{if } n_0 = 0) \\ Y_3' \cup \{a_1b_0\} & (\text{if } n_0 = 2), \end{cases}$$

$$Y_4' = \begin{cases} (\bigcup_{t=1}^{t'-1} \{b_ie_{l_t} \mid k_{t-1} \le i \le k_t\}) \\ \qquad \cup (\bigcup_{t=t'}^{r+1} \{b_id_{l_t} \mid k_{t-1} \le i \le k_t\}) \quad (\text{if } l_{t'} < n_3) \\ (\bigcup_{t=1}^{t'-1} \{b_ie_{l_t} \mid k_{t-1} \le i \le k_t\}) \\ \qquad \cup (\bigcup_{t=t'+1}^{r+1} \{b_id_{l_t} \mid k_{t-1} \le i \le k_t\}) \quad (\text{if } l_{t'} = n_3), \end{cases}$$

$$F_1 = \begin{cases} \{a_0 e_{n_4-1}\} & (\text{if } k_1 = 0 \text{ and } n_0 = 0) \\ \{a_0 a_2, a_1 e_{n_4-1}\} & (\text{if } k_1 = 0 \text{ and } n_0 = 2) \\ \{a_0 b_1\} & (\text{if } k_1 > 0 \text{ and } n_0 = 0) \\ \{a_0 a_2, a_1 b_1\} & (\text{if } k_1 > 0 \text{ and } n_0 = 2), \end{cases}$$

$$F_2 = \begin{cases} \{c_0 d_1\} & (\text{if } k_r = n_1 \text{ and } n_2 = 0) \\ \{c_0 c_2, c_1 d_1\} & (\text{if } k_r = n_1 \text{ and } n_2 = 2) \\ \{c_0 b_{n_1-1}\} & (\text{if } k_r < n_1 \text{ and } n_2 = 0) \\ \{c_0 c_2, c_1 b_{n_1-1}\} & (\text{if } k_r < n_1 \text{ and } n_0 = 2), \end{cases}$$

$$F_1' = \begin{cases} \emptyset & (\text{if } n_0 = 0) \\ \{a_1 b_0, a_1 e_{n_4}\} & (\text{if } n_0 = 2), \end{cases} \quad \text{and} \quad F_2' = \begin{cases} \emptyset & (\text{if } n_2 = 0) \\ \{c_1 b_{n_1}, c_1 d_0\} & (\text{if } n_2 = 2). \end{cases}$$

Further, let p and q be integers with $k_{t'-1} \le p \le q \le k_{t'}$ in the case where $l_{t'} = n_3$, and let

$$X_2 = \begin{cases} \{d_{n_3-1} e_1\} & (\text{if } l_{t'} < n_3) \\ \emptyset & (\text{if } l_{t'} = n_3,\ p \ne k_{t'} \text{ and } q \ne k_{t'-1}) \\ \{d_{n_3-1} e_0\} & (\text{if } l_{t'} = n_3, \text{ and } p = k_{t'} (\text{and then } q = k_{t'})) \\ \{d_{n_3} e_1\} & (\text{if } l_{t'} = n_3, \text{ and } q = k_{t'-1} (\text{and then } p = k_{t'-1})), \end{cases}$$

$$Y_5 = \begin{cases} \{b_{p+1} e_0\} & (\text{if } l_{t'} = n_3 \text{ and } k_{t'-1} < p < k_{t'}) \\ \emptyset & (\text{otherwise}), \end{cases}$$

$$Y_6 = \begin{cases} \{b_{q-1} d_{n_3}\} & (\text{if } l_{t'} = n_3 \text{ and } k_{t'-1} < q < k_{t'}) \\ \emptyset & (\text{otherwise}), \end{cases}$$

$$Y_5' = \begin{cases} \{b_i e_0 \mid k_{t'-1} \le i \le p\} & (\text{if } l_{t'} = n_3) \\ \emptyset & (\text{if } l_{t'} < n_3), \end{cases}$$

$$\bar{Y}_5' = \begin{cases} \{b_{p+1} e_0\} & (\text{if } l_{t'} = n_3 \text{ and } p = k_{t'-1}) \\ \emptyset & (\text{otherwise}), \end{cases}$$

$$Y_6' = \begin{cases} \{b_i d_{n_3} \mid q \le i \le k_{t'}\} & (\text{if } l_{t'} = n_3) \\ \emptyset & (\text{if } l_{t'} < n_3), \end{cases}$$

and

$$\bar{Y}_6' = \begin{cases} \{b_{q-1} d_{n_3}\} & (\text{if } l_{t'} = n_3 \text{ and } q = k_{t'}) \\ \emptyset & (\text{otherwise}). \end{cases}$$

Now G is said to be of Type 2, if there exist r and t' satisfying (2.1) and (2.2), there exist $k_0, k_1, \ldots, k_r, k_{r+1}$ satisfying (2.3), and there exist $l_1, l_2 \ldots, l_r, l_{r+1}$ satisfying (2.4) and (2.5) (and there exist p and q with $k_{t'-1} \leq p \leq q \leq k_{t'}$ if $l_{t'} = n_3$), such that G satisfies the following three conditions:

- $X_1 \cup X_2 \cup Y_1 \cup \bar{Y_1} \cup Y_2 \cup \bar{Y_2} \cup Y_5 \cup Y_6 \cup F_1 \cup F_2 \subseteq E(G) - E(C) \subseteq X_1 \cup X_2 \cup X_2' \cup Y_1 \cup \bar{Y_1} \cup Y_2 \cup \bar{Y_2} \cup Y_3' \cup Y_4' \cup Y_5 \cup Y_5' \cup \bar{Y_5}' \cup Y_6 \cup Y_6' \cup \bar{Y_6}' \cup F_1 \cup F_1' \cup F_2 \cup F_2'$,
- if $n_0 = 0$, $n_1 = 1$, $r = 1$ and $k_r (= k_1) = n_1 (= 1)$, then $W_1 \cap E(G) \neq \emptyset$,
- if $n_2 = 0$, $n_1 = 1$, $r = 1$ and $k_r (= k_1) = 0$, then $W_2 \cap E(G) \neq \emptyset$.

Type 3. Let $n_0 = 0$ or 2, $n_1 = 0$, $n_2 = 0$ or 2, $n_3 \geq 1$, and $n_4 \geq 1$. Let

$$X = \{d_j d_{j+2} \,|\, 0 \leq j \leq n_3 - 2\} \cup \{e_x e_{x+2} \,|\, 0 \leq x \leq n_4 - 2\},$$

$$Y = \{b_0 d_j \,|\, 0 \leq j \leq n_3\} \cup \{b_0 e_x \,|\, 0 \leq x \leq n_4\},$$

$$F_1 = \begin{cases} \{a_0 e_{n_4-1}\} & (\text{if } n_0 = 0) \\ \{a_0 a_2, a_1 e_{n_4-1}\} & (\text{if } n_0 = 2), \end{cases} \qquad F_1' = \begin{cases} \emptyset & (\text{if } n_0 = 0) \\ \{a_1 b_0, a_1 e_{n_4}\} & (\text{if } n_0 = 2), \end{cases}$$

$$F_2 = \begin{cases} \{c_0 d_1\} & (\text{if } n_2 = 0) \\ \{c_0 c_2, c_1 d_1\} & (\text{if } n_2 = 2), \end{cases} \qquad F_2' = \begin{cases} \emptyset & (\text{if } n_2 = 0) \\ \{c_1 b_0, c_1 d_0\} & (\text{if } n_2 = 2), \end{cases}$$

$$F_3 = \{d_{n_3-1} e_1\}, \qquad F_3' = \{d_{n_3} e_1, d_{n_3-1} e_0\},$$

$$W_1 = \begin{cases} Y & (\text{if } n_0 = 0) \\ Y \cup \{a_1 b_0\} & (\text{if } n_0 = 2), \end{cases} \qquad W_2 = \begin{cases} Y & (\text{if } n_2 = 0) \\ Y \cup \{b_0 c_1\} & (\text{if } n_2 = 2), \end{cases}$$

$$Z_1 = \begin{cases} \{b_0 d_0\} & (\text{if } n_2 = 0) \\ \emptyset & (\text{if } n_2 = 2), \end{cases} \quad \text{and} \quad Z_2 = \begin{cases} \{b_0 e_{n_4}\} & (\text{if } n_0 = 0) \\ \emptyset & (\text{if } n_0 = 2). \end{cases}$$

Under this notation, G is said to be of Type 3 if G satisfies the following conditions:

- $X \cup F_1 \cup F_2 \cup F_3 \subseteq E(G) - E(C) \subseteq X \cup Y \cup F_1 \cup F_1' \cup F_2 \cup F_2' \cup F_3 \cup F_3'$,
- for each i with $1 \leq i \leq 2$, $(W_i - Z_i) \cap E(G) \neq \emptyset$,
- if $n_0 = 0$ and $n_4 = 1$, then $\{e_0 b_0, e_0 d_{n_3-1}\} \cap E(G) \neq \emptyset$,

- if $n_2 = 0$ and $n_3 = 1$, then $\{d_{n_3}b_0, d_{n_3}e_1\} \cap E(G) \neq \emptyset$.

Type 4. Let $n_0 = 0$ or 2, $n_1 = 2$, $n_2 = 0$ or 2, $n_3 \geq 1$, and $n_4 \geq 1$. Let

$$X = \{d_j d_{j+2} \,|\, 0 \leq j \leq n_3 - 2\} \cup \{e_x e_{x+2} \,|\, 0 \leq x \leq n_4 - 2\} \cup \{b_0 b_2\},$$

$$F_1 = \begin{cases} \{a_0 b_1, a_0 e_{n_4 - 1}\} & (\text{if } n_0 = 0) \\ \{a_0 a_2, a_1 b_1, a_1 e_{n_4 - 1}\} & (\text{if } n_0 = 2), \end{cases} \quad F_1' = \begin{cases} \emptyset & (\text{if } n_0 = 0) \\ \{a_1 e_{n_4}\} & (\text{if } n_0 = 2), \end{cases}$$

$$F_2 = \begin{cases} \{c_0 d_1\} & (\text{if } n_2 = 0) \\ \{c_0 c_2, c_1 d_1\} & (\text{if } n_2 = 2), \end{cases} \quad F_2' = \begin{cases} \emptyset & (\text{if } n_2 = 0) \\ \{c_1 d_0\} & (\text{if } n_2 = 2), \end{cases}$$

$$F_3 = \{d_{n_3 - 1} e_1\}, \quad F_3' = \{d_{n_3} e_1, d_{n_3 - 1} e_0\},$$

$$F_4 = \begin{cases} \{b_1 e_{n_4 - 1}\}, & (\text{if } n_0 = 0) \\ \emptyset & (\text{if } n_0 = 2), \end{cases}$$

and

$$F_4' = \begin{cases} \{b_1 e_{n_4}\} & (\text{if } n_0 = 0) \\ \{b_1 e_{n_4 - 1}, b_1 e_{n_4}\} & (\text{if } n_0 = 2). \end{cases}$$

Under this notation, G is said to be of Type 4 if G satisfies the following three conditions:

- $X \cup F_1 \cup F_2 \cup F_3 \cup F_4 \subseteq E(G) - E(C) \subseteq X \cup F_1 \cup F_1' \cup F_2 \cup F_2' \cup F_3 \cup F_3' \cup F_4 \cup F_4'$,
- if $n_0 = 2$, then $F_4' \cap E(G) \neq \emptyset$,
- if $n_2 = 0$ and $n_3 = 1$, then $d_1 e_1 \in E(G)$.

Type 5. Let $n_0 = 0$ or 2, $n_1 = 2$, $n_2 = 0$ or 2, $n_3 \geq 1$, and $n_4 \geq 1$. Let

$$X = \{d_j d_{j+2} \,|\, 0 \leq j \leq n_3 - 2\} \cup \{e_x e_{x+2} \,|\, 0 \leq x \leq n_4 - 2\} \cup \{b_0 b_2\},$$

$$F_1 = \begin{cases} \{a_0 e_{n_4 - 1}\} & (\text{if } n_0 = 0) \\ \{a_0 a_2, a_1 e_{n_4 - 1}\} & (\text{if } n_0 = 2), \end{cases} \quad F_1' = \begin{cases} \emptyset & (\text{if } n_0 = 0) \\ \{a_1 e_{n_4}\} & (\text{if } n_0 = 2), \end{cases}$$

$$F_2 = \begin{cases} \{c_0 d_1\} & (\text{if } n_2 = 0) \\ \{c_0 c_2, c_1 d_1\} & (\text{if } n_2 = 2), \end{cases} \quad F_2' = \begin{cases} \emptyset & (\text{if } n_2 = 0) \\ \{c_1 d_0\} & (\text{if } n_2 = 2), \end{cases}$$

$$F_3 = \{d_{n_3 - 1} e_1\}, \quad \text{and} \quad F_3' = \{d_{n_3} e_1, d_{n_3 - 1} e_0\}.$$

Let p be an integer with $1 \le p \le n_4 - 1$, and set

$$Y = \{b_1 e_x \,|\, p-1 \le x \le p+1\},$$

and

$$W = \begin{cases} Y - \{b_1 e_{n_4}\} & (\text{if } n_0 = 0) \\ Y & (\text{if } n_0 = 2). \end{cases}$$

Now G is said to be of Type 5 if there exists p with $1 \le p \le n_4 - 1$ such that G satisfies the following three conditions:

- $X \cup F_1 \cup F_2 \cup F_3 \subseteq E(G) - E(C) \subseteq X \cup Y \cup F_1 \cup F_1' \cup F_2 \cup F_2' \cup F_3 \cup F_3'$,
- $W \cap E(G) \ne \emptyset$,
- if $n_2 = 0$ and $n_3 = 1$, then $d_1 e_1 \in E(G)$.

The graph in Figure 2 is an example of a graph of Type 5 with $n_0 = n_2 = 0$, $n_1 = 2$, $n_3 = 3$, $n_4 = 6$ and $p = 3$.

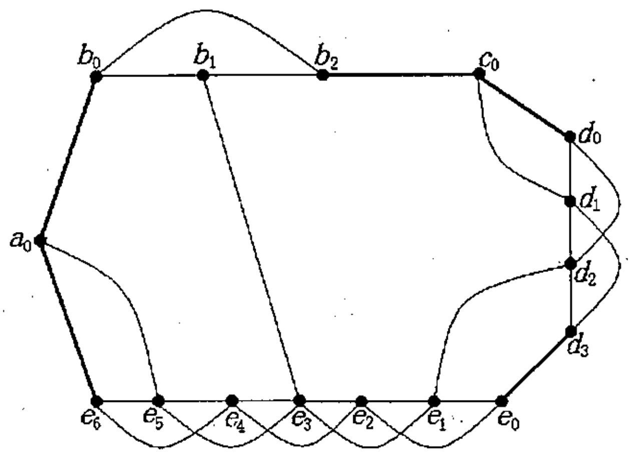

Figure 2

Type 6. Let $n_0 = 0$ or 2, $n_1 = 2$, $n_2 = 0$ or 2, $n_3 \ge 1$, and $n_4 = 1$. Let

$$X = \{d_j d_{j+2} \,|\, 0 \le j \le n_3 - 2\} \cup \{b_0 b_2\},$$

$$F_1 = \begin{cases} \{a_0 e_{n_4-1}\} & (\text{if } n_0 = 0) \\ \{a_0 a_2, a_1 e_{n_4-1}\} & (\text{if } n_0 = 2), \end{cases} \quad F_1' = \begin{cases} \emptyset & (\text{if } n_0 = 0) \\ \{a_1 e_{n_4}\} & (\text{if } n_0 = 2), \end{cases}$$

$$F_2 = \begin{cases} \{c_0d_1\} & (\text{if } n_2 = 0) \\ \{c_0c_2, c_1d_1\} & (\text{if } n_2 = 2), \end{cases} \quad F_2' = \begin{cases} \emptyset & (\text{if } n_2 = 0) \\ \{c_1d_0\} & (\text{if } n_2 = 2), \end{cases}$$

$$F_3 = \{d_{n_3-1}e_1\}, \quad F_3' = \{d_{n_3}e_1, d_{n_3-1}e_0\},$$

$$F_4 = \begin{cases} \{b_1e_0\}, & (\text{if } n_0 = 0) \\ \emptyset & (\text{if } n_0 = 2), \end{cases} \text{ and } F_4' = \begin{cases} \{b_1e_1\} & (\text{if } n_0 = 0) \\ \{b_1e_0, b_1e_1\} & (\text{if } n_0 = 2). \end{cases}$$

Under this notation, G is said to be of Type 6 if G satisfies the following three conditions:

- $X \cup F_1 \cup F_2 \cup F_3 \cup F_4 \subseteq E(G) - E(C) \subseteq X \cup F_1 \cup F_1' \cup F_2 \cup F_2' \cup F_3 \cup F_3' \cup F_4 \cup F_4'$,
- if $n_0 = 2$, then $F_4' \cap E(G) \neq \emptyset$,
- if $n_2 = 0$ and $n_3 = 1$, then $d_1e_1 \in E(G)$.

Type 7. Let $n_0 = 0$ or 2, $n_1 = 2$, $n_2 = 0$ or 2, $n_3 \geq 1$, and $n_4 \geq 1$. Let

$$X = \{d_jd_{j+2} \mid 0 \leq j \leq n_3 - 2\} \cup \{e_xe_{x+2} \mid 0 \leq x \leq n_4 - 2\} \cup \{b_0b_2\},$$

$$F_1 = \begin{cases} \{a_0e_{n_4-1}\} & (\text{if } n_0 = 0) \\ \{a_0a_2, a_1e_{n_4-1}\} & (\text{if } n_0 = 2), \end{cases} \quad F_1' = \begin{cases} \emptyset & (\text{if } n_0 = 0) \\ \{a_1e_{n_4}\} & (\text{if } n_0 = 2), \end{cases}$$

$$F_2 = \begin{cases} \{c_0d_1\} & (\text{if } n_2 = 0) \\ \{c_0c_2, c_1d_1\} & (\text{if } n_2 = 2), \end{cases} \quad F_2' = \begin{cases} \emptyset & (\text{if } n_2 = 0) \\ \{c_1d_0\} & (\text{if } n_2 = 2), \end{cases}$$

$$F_3' = \{d_{n_3}e_1, d_{n_3-1}e_0, d_{n_3-1}e_1\}, \quad F_4' = \{b_1d_{n_3-1}, b_1d_{n_3}, b_1e_0, b_1e_1\},$$

$$W_1 = \{b_1d_{n_3-1}, b_1d_{n_3}\}, \quad W_2 = \{b_1e_1, b_1e_0\},$$

$$W_3 = \{d_{n_3-1}e_1, d_{n_3-1}e_0\}, \quad W_4 = \{e_1d_{n_3-1}, e_1d_{n_3}\},$$

$$W_5 = \{d_{n_3-1}e_1, d_{n_3-1}b_1\}, \text{ and } W_6 = \{e_1d_{n_3-1}, e_1b_1\}.$$

Under this notation, G is said to be of Type 7 if G satisfies the following conditions:

- $X \cup F_1 \cup F_2 \subseteq E(G) - E(C) \subseteq X \cup F_1 \cup F_1' \cup F_2 \cup F_2' \cup F_3' \cup F_4'$,
- for each i with $1 \le i \le 6$, $W_i \cap E(G) \ne \emptyset$,
- if $n_2 = 0$ and $n_3 = 1$, then $\{d_1b_1, d_1e_1\} \cap E(G) \ne \emptyset$,
- if $n_0 = 0$ and $n_4 = 1$, then $\{e_0b_1, e_0d_{n_3-1}\} \cap E(G) \ne \emptyset$.

Type 8. Let $n_0 = 0$ or 2, $n_1 \ge 1$, $n_2 = 0$ or 2, $n_3 = 0$ or 2, and $n_4 \ge 1$. Let r be an integer with

(2.6) $$1 \le r \le \min\{n_1 + 1, n_4\},$$

and let $k_0, k_1, \ldots, k_r$ and $l_1, l_2, \ldots, l_r, l_{r+1}$ be integers such that

(2.7) $$0 = k_0 \le k_1 < \cdots < k_r \le n_1$$

and

(2.8) $$n_4 = l_1 > l_2 \cdots > l_r > l_{r+1} \ge 0.$$

Let

$$X_1 = \bigcup_{t=1}^{r} \{b_ib_{i+2} \,|\, k_{t-1} \le i \le k_t - 2\} \cup \{b_ib_{i+2} \,|\, k_r \le i \le n_1 - 2\},$$

$$X_2 = \bigcup_{t=1}^{r} \{e_xe_{x-2} \,|\, l_t \ge x \ge l_{t+1} + 2\} \cup \{e_xe_{x-2} \,|\, l_{r+1} \ge x \ge 2\},$$

$$Y_1 = \{b_{k_t+1}e_{l_{t+1}+1} \,|\, 1 \le t \le r - 1\}, \quad Y_2 = \{b_{k_t-1}e_{l_t-1} \,|\, 2 \le t \le r\},$$

$$\bar{Y_2} = \begin{cases} \{b_{k_1-1}e_{l_1-1}\} & (\text{if } k_1 > 0) \\ \emptyset & (\text{if } k_1 = 0), \end{cases}$$

$$Y_3' = \bigcup_{t=1}^{r} \{b_{k_t}e_x \,|\, l_t \ge x \ge l_{t+1}\}, \quad Y_4' = \bigcup_{t=1}^{r+1} \{b_ie_{l_t} \,|\, k_{t-1} \le i \le k_t\},$$

$$F_1 = \begin{cases} \{a_0e_{n_4-1}\} & (\text{if } n_0 = 0 \text{ and } k_1 = 0) \\ \{a_0a_2, a_1e_{n_4-1}\} & (\text{if } n_0 = 2 \text{ and } k_1 = 0) \\ \{a_0b_1\} & (\text{if } n_0 = 0 \text{ and } k_1 > 0) \\ \{a_0a_2, a_1b_1\} & (\text{if } n_0 = 2 \text{ and } k_1 > 0), \end{cases}$$

$$F_1' = \begin{cases} \emptyset & (\text{if } n_0 = 0) \\ \{a_1e_{n_4}, a_1b_0\} & (\text{if } n_0 = 2), \end{cases}$$

$$F_2 = \begin{cases} \emptyset & (\text{if } n_2 = 0 \text{ and } k_r = n_1) \\ \{c_0c_2\} & (\text{if } n_2 = 2 \text{ and } k_r = n_1) \\ \{c_0b_{n_1-1}\} & (\text{if } n_2 = 0 \text{ and } k_r < n_1) \\ \{c_0c_2, c_1b_{n_1-1}\} & (\text{if } n_2 = 2 \text{ and } k_r < n_1), \end{cases}$$

$$F_2' = \begin{cases} \{c_1b_{n_1}, c_1d_0\} \ (\text{if } n_2 = 2, k_r < n_1,\ n_3 = 0 \text{ and } l_{r+1} = 0) \\ \{c_1b_{n_1}\} \ (\text{if } n_2 = 2, \text{ and either } k_r = n_1 \text{ or } \ n_3 = 2 \text{ or } l_{r+1} > 0) \\ \emptyset \ (\text{ otherwise }), \end{cases}$$

$$F_3 = \begin{cases} \emptyset & (\text{if } n_3 = 0 \text{ and } l_{r+1} = 0) \\ \{d_0d_2\} & (\text{if } n_3 = 2 \text{ and } l_{r+1} = 0) \\ \{d_0e_1\} & (\text{if } n_3 = 0 \text{ and } l_{r+1} > 0) \\ \{d_0d_2, d_1e_1\} & (\text{if } n_3 = 2 \text{ and } l_{r+1} > 0), \end{cases}$$

and

$$F_3' = \begin{cases} \{d_1e_0, d_1c_0\} \ (\text{if } n_3 = 2, l_{r+1} > 0,\ n_2 = 0 \text{ and } k_r = n_1) \\ \{d_1e_0\} \ (\text{if } n_3 = 2, \text{ and either } l_{r+1} = 0 \text{ or } n_2 = 2 \text{ or } k_r < n_1) \\ \emptyset \ (\text{ otherwise }). \end{cases}$$

Set

$$u = \begin{cases} b_{k_r+1} & (\text{if } k_r < n_1) \\ c_0 & (\text{if } k_r = n_1 \text{ and } n_2 = 0) \\ c_1 & (\text{if } k_r = n_1 \text{ and } n_2 = 2) \end{cases}$$

and

$$w = \begin{cases} e_{l_{r+1}-1} & (\text{if } l_{r+1} > 0) \\ d_0 & (\text{if } l_{r+1} = 0 \text{ and } n_3 = 0) \\ d_1 & (\text{if } l_{r+1} = 0 \text{ and } n_3 = 2), \end{cases}$$

and let

$$D = \begin{cases} \{ue_x \,|\, l_{r+1} - 2 \geq x \geq 0\} & (\text{if } k_r = n_1, l_{r+1} > 0 \text{ and } n_2 = 0) \\ \emptyset & (\text{otherwise}), \end{cases}$$

$$F = \begin{cases} \{b_iw \,|\, k_r + 2 \leq i \leq n_1\} & (\text{if } k_r < n_1, l_{r+1} = 0 \text{ and } n_3 = 0) \\ \emptyset & (\text{otherwise}), \end{cases}$$

$$I = \begin{cases} \{b_{k_r}w, ue_{l_{r+1}}, ue_{l_{r+1}+1}, we_{l_{r+1}+1}\} \\ \qquad (\text{if } k_r = n_1, l_{r+1} = 0, n_2 = 0 \text{ and } n_3 = 0) \\ \{b_{k_r}w, uw, ue_{l_{r+1}}, ue_{l_{r+1}+1}, we_{l_{r+1}+1}\} \ (\text{otherwise}), \end{cases}$$

$$W_1 = \{ue_{l_{r+1}+1}, we_{l_{r+1}+1}\},$$

$$W_2 = \begin{cases} \{ud_1, uw, ue_{l_{r+1}}, ue_{l_{r+1}+1}\} \cup D \\ \qquad \text{(if } k_r = n_1, l_{r+1} > 0, n_2 = 0 \text{ and } n_3 = 2) \\ \{uw, ue_{l_{r+1}}, ue_{l_{r+1}+1}\} \cup D \\ \qquad \text{(otherwise),} \end{cases}$$

$$W_3 = \begin{cases} \{b_{k_r}w, uw, c_1w, we_{l_{r+1}+1}\} \cup F \\ \qquad \text{(if } k_r < n_1, l_{r+1} = 0, n_2 = 2 \text{ and } n_3 = 0) \\ \{b_{k_r}w, uw, we_{l_{r+1}+1}\} \cup F \\ \qquad \text{(otherwise),} \end{cases}$$

$$W_4 = \{b_{k_r}e_x \mid l_r \geq x \geq l_{r+1}\} \cup \{b_{k_r}w\},$$

$$W_5 = \begin{cases} \{b_{k_r}w, b_{k_r}e_{l_{r+1}}, uw, ue_{l_{r+1}}\} & \text{(if } l_{r+1} > 0) \\ \{b_{k_r}e_{l_{r+1}}, ue_{l_{r+1}}\} & \text{(if } l_{r+1} = 0 \text{ and } n_3 = 0) \\ \{b_{k_r}e_{l_{r+1}}, ue_{l_{r+1}}, we_{l_{r+1}}\} & \text{(if } l_{r+1} = 0 \text{ and } n_3 = 2), \end{cases}$$

$$Z_2 = \begin{cases} \{ue_0\} & \text{(if } k_r = n_1, l_{r+1} \geq 2, n_2 = 0 \text{ and } n_3 = 0) \\ \{uw\} & \text{(if } l_{r+1} = 1 \text{ and } n_3 = 0) \\ \{uw, ue_{l_{r+1}}\} & \text{(if } l_{r+1} = 0 \text{ and } n_3 = 0) \\ \emptyset & \text{(otherwise),} \end{cases}$$

$$Z_3 = \begin{cases} \{b_{n_1}w\} & \text{(if } k_r \leq n_1 - 2, l_{r+1} = 0, n_2 = 0 \text{ and } n_3 = 0) \\ \{uw\} & \text{(if } k_r = n_1 - 1 \text{ and } n_2 = 0) \\ \{b_{k_r}w, uw\} & \text{(if } k_r = n_1 \text{ and } n_2 = 0) \\ \emptyset & \text{(otherwise),} \end{cases}$$

$$Z_4 = \begin{cases} \{b_{k_r}u\} & \text{(if } n_2 = 2) \\ \emptyset & \text{(if } n_2 = 0), \end{cases} \quad \text{and} \quad Z_4' = \begin{cases} \{a_1b_0\} & \text{(if } n_0 = 2) \\ \emptyset & \text{(if } n_0 = 0). \end{cases}$$

Now G is said to be of Type 8, if there exists r satisfying (2.6), there exist $k_0, k_1, \ldots, k_r$ satisfying (2.7), and there exist $l_1, l_2 \ldots, l_r, l_{r+1}$ satisfying (2.8), such that G satisfies the following conditions:

- $X_1 \cup X_2 \cup Y_1 \cup Y_2 \cup \bar{Y_2} \cup F_1 \cup F_2 \cup F_3 \subsetneq E(G) - E(C)$
  $\subsetneq X_1 \cup X_2 \cup Y_1 \cup Y_2 \cup \bar{Y_2} \cup Y_3' \cup Y_4' \cup I \cup D \cup F \cup F_1 \cup F_1' \cup F_2 \cup F_2' \cup F_3 \cup F_3'$,
- $W_1 \cap E(G) \neq \emptyset$,
- $(W_2 - Z_2) \cap E(G) \neq \emptyset$,

- $(W_3 - Z_3) \cap E(G) \neq \emptyset$,
- in the case where $n_1 = 1$ and $r = 1$, we have $(W_4 \cup Z_4) \cap E(G) \neq \emptyset$ if $k_1 = 1$ and $n_0 = 0$, and we have $(W_4 \cup Z_4') \cap E(G) \neq \emptyset$ if $k_1 = 0$ and $n_2 = 0$,
- if $n_0 = 0$, $r = 1$, $k_r = 0$ and $l_{r+1} = n_4 - 1$, then $W_5 \cap E(G) \neq \emptyset$.

Type 9. Let $n_0 = 0$ or 2, $n_1 \geq 1$, $n_2 = 0$ or 2, $n_3 = 0$ or 2, and $n_4 \geq 1$. Let

$$X = \{b_i b_{i+2} \mid 0 \leq i \leq n_1 - 2\} \cup \{e_x e_{x-2} \mid n_4 \geq x \geq 2\},$$

$$F_1 = \begin{cases} \emptyset & (\text{if } n_0 = 0) \\ \{a_0 a_2\} & (\text{if } n_0 = 2), \end{cases}$$

$$F_1' = \begin{cases} \{a_0 b_1, a_0 e_{n_4 - 1}, b_0 e_{n_4}, b_0 e_{n_4 - 1}, b_1 e_{n_4}, b_1 e_{n_4 - 1}\} & (\text{if } n_0 = 0) \\ \{a_1 b_0, a_1 b_1, a_1 e_{n_4}, a_1 e_{n_4 - 1}, b_0 e_{n_4}, b_0 e_{n_4 - 1}, b_1 e_{n_4}, b_1 e_{n_4 - 1}\} & (\text{if } n_0 = 2), \end{cases}$$

$$F_2 = \begin{cases} \{c_0 b_{n_1 - 1}\} & (\text{if } n_2 = 0) \\ \{c_0 c_2, c_1 b_{n_1 - 1}\} & (\text{if } n_2 = 2), \end{cases} \qquad F_2' = \begin{cases} \emptyset & (\text{if } n_2 = 0) \\ \{c_1 b_{n_1}\} & (\text{if } n_2 = 2), \end{cases}$$

$$F_3 = \begin{cases} \{d_0 e_1\} & (\text{if } n_3 = 0) \\ \{d_0 d_2, d_1 e_1\} & (\text{if } n_3 = 2), \end{cases} \qquad F_3' = \begin{cases} \emptyset & (\text{if } n_3 = 0) \\ \{d_1 e_0\} & (\text{if } n_3 = 2), \end{cases}$$

$$W_1 = \begin{cases} \{b_1 e_{n_4}, b_1 e_{n_4 - 1}\} & (\text{if } n_0 = 0) \\ \{b_1 a_1, b_1 e_{n_4}, b_1 e_{n_4 - 1}\} & (\text{if } n_0 = 2), \end{cases}$$

$$W_2 = \begin{cases} \{b_0 e_{n_4 - 1}, b_1 e_{n_4 - 1}\} & (\text{if } n_0 = 0) \\ \{a_1 e_{n_4 - 1}, b_0 e_{n_4 - 1}, b_1 e_{n_4 - 1}\} & (\text{if } n_0 = 2), \end{cases}$$

$$W_3 = \begin{cases} \{b_0 e_{n_4}, b_0 e_{n_4 - 1}\} & (\text{if } n_0 = 0) \\ \{a_1 b_0, b_0 e_{n_4}, b_0 e_{n_4 - 1}\} & (\text{if } n_0 = 2), \end{cases}$$

$$W_4 = \begin{cases} \{a_0 e_{n_4 - 1}, b_0 e_{n_4 - 1}\} & (\text{if } n_0 = 0) \\ \{a_1 e_{n_4 - 1}, b_0 e_{n_4 - 1}\} & (\text{if } n_0 = 2), \end{cases}$$

$$W_5 = \begin{cases} \{b_0 e_{n_4}, b_1 e_{n_4}\} & (\text{if } n_0 = 0) \\ \{a_1 e_{n_4}, b_0 e_{n_4}, b_1 e_{n_4}\} & (\text{if } n_0 = 2), \end{cases}$$

and

$$W_6 = \begin{cases} \{a_0 b_1, b_1 e_{n_4}\} & (\text{if } n_0 = 0) \\ \{a_1 b_1, b_1 e_{n_4}\} & (\text{if } n_0 = 2). \end{cases}$$

Under this notation, G is said to be of Type 9 if G satisfies the following five conditions:

- $X \cup F_1 \cup F_2 \cup F_3 \subseteq E(G) - E(C) \subseteq X \cup F_1 \cup F_1' \cup F_2 \cup F_2' \cup F_3 \cup F_3'$,
- for each i with $1 \leq i \leq 2$, $W_i \cap E(G) \neq \emptyset$,
- if $n_0 = 2$, then $\{a_1b_1, a_1e_{n_4-1}\} \cap E(G) \neq \emptyset$, and if $n_0 = 0$, then $\{a_0b_1, a_0e_{n_4-1}\} \cap E(G) \neq \emptyset$,
- if $n_1 = 1$ and $n_2 = 0$, then $W_3 \cap E(G) \neq \emptyset$ and $W_4 \cap E(G) \neq \emptyset$,
- if $n_4 = 1$ and $n_3 = 0$, then $W_5 \cap E(G) \neq \emptyset$ and $W_6 \cap E(G) \neq \emptyset$.

The graph in Figure 3 is an example of a graph of Type 9 with $n_0 = n_2 = n_3 = 0$ and $n_1 = n_4 = 3$.

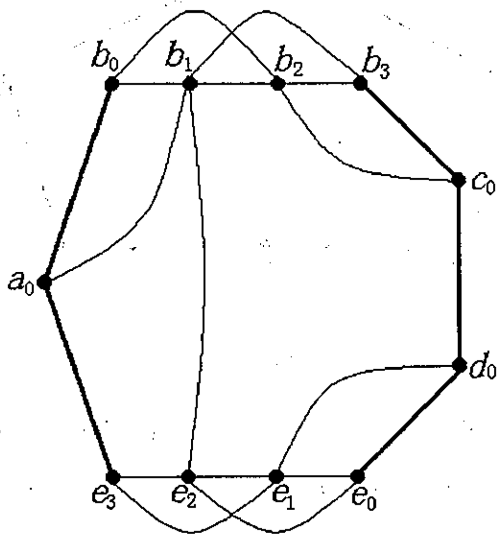

Figure 3

Type 10. Let $n_0 = 0$ or 2, $n_1 = 0$, $n_2 = 0$, $n_3 = 0$ or 2, and $n_4 \geq 3$. Let

$$X = \{e_x e_{x+2} \mid 0 \leq x \leq n_4 - 2\},$$

$$F_1 = \begin{cases} \{a_0e_{n_4-1}\} & (\text{if } n_0 = 0) \\ \{a_0a_2, a_1e_{n_4-1}\} & (\text{if } n_0 = 2), \end{cases} \qquad F_1' = \begin{cases} \emptyset & (\text{if } n_0 = 0) \\ \{a_1b_0, a_1e_{n_4}\} & (\text{if } n_0 = 2), \end{cases}$$

$$F_2 = \begin{cases} \{d_0e_1\} & (\text{if } n_3 = 0) \\ \{d_0d_2, d_1e_1\} & (\text{if } n_3 = 2), \end{cases} \qquad F_2' = \begin{cases} \emptyset & (\text{if } n_3 = 0) \\ \{d_1c_0, d_1e_0\} & (\text{if } n_3 = 2), \end{cases}$$

$$Z_1 = \begin{cases} \{b_0e_{n_4}\} & (\text{if } n_0 = 0) \\ \emptyset & (\text{if } n_0 = 2), \end{cases} \quad \text{and} \quad Z_2 = \begin{cases} \{c_0e_0\} & (\text{if } n_3 = 0) \\ \emptyset & (\text{if } n_3 = 2). \end{cases}$$

Let p be an integer with $1 \le p \le n_4 - 1$, and let

$$Y_1 = \{b_0 e_x \mid p-1 \le x \le n_4\}, \quad \text{and} \quad Y_2 = \{c_0 e_x \mid 0 \le x \le p+1\}.$$

Now G is said to be of Type 10 if there exists p with $1 \le p \le n_4 - 1$ such that G satisfies the following two conditions:

- $X \cup F_1 \cup F_2 \subseteq E(G) - E(C) \subseteq X \cup F_1 \cup F_1' \cup F_2 \cup F_2' \cup Y_1 \cup Y_2$,
- for each i with $1 \le i \le 2$, $(Y_i - Z_i) \cap E(G) \ne \emptyset$.

The graph in Figure 4 is an example of a graph of Type 10 with $n_0 = n_1 = n_2 = n_3 = 0$, $n_4 = 6$ and $p = 3$.

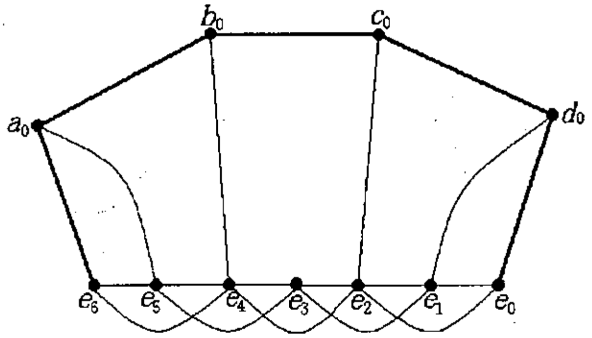

Figure 4

Type 11. Let $n_0 = 0$ or 2, $n_1 = 0$, $n_2 = 0$, $n_3 = 0$ or 2, and $n_4 \ge 3$. Let

$$X = \{e_x e_{x+2} \mid 0 \le x \le n_4 - 2\}, \quad Y = \{c_0 e_x \mid 0 \le x \le n_4\},$$

$$F_1 = \begin{cases} \{a_0 c_0, a_0 e_{n_4-1}\} & (\text{if } n_0 = 0) \\ \{a_0 a_2, a_1 c_0, a_1 e_{n_4-1}\} & (\text{if } n_0 = 2), \end{cases}$$

$$F_1' = \begin{cases} \emptyset & (\text{if } n_0 = 0) \\ \{a_1 b_0, a_1 e_{n_4}\} & (\text{if } n_0 = 2), \end{cases}$$

$$F_2 = \begin{cases} \{d_0 e_1\} & (\text{if } n_3 = 0) \\ \{d_0 d_2, d_1 e_1\} & (\text{if } n_3 = 2), \end{cases} \qquad F_2' = \begin{cases} \emptyset & (\text{if } n_3 = 0) \\ \{d_1 c_0, d_1 e_0\} & (\text{if } n_3 = 2), \end{cases}$$

$$F_3 = \begin{cases} \{b_0 e_{n_4-1}\}, & (\text{if } n_0 = 0) \\ \emptyset & (\text{if } n_0 = 2), \end{cases}$$

and

$$F_3' = \begin{cases} \{b_0 e_{n_4}\} & (\text{if } n_0 = 0) \\ \{b_0 e_{n_4-1}, b_0 e_{n_4}\} & (\text{if } n_0 = 2). \end{cases}$$

Under this notation, G is said to be of Type 11 if G satisfies the following three conditions:

- $X \cup F_1 \cup F_2 \cup F_3 \subseteq E(G) - E(C) \subseteq X \cup Y \cup F_1 \cup F_1' \cup F_2 \cup F_2' \cup F_3 \cup F_3'$,
- if $n_0 = 2$, then $F_3' \cap E(G) \neq \emptyset$,
- in the case where $n_0 = 0$, we have $Y \cap E(G) \neq \emptyset$ if $n_3 = 0$, and we have $(Y \cup \{c_0 d_1\}) \cap E(G) \neq \emptyset$ if $n_3 = 2$.

Type 12. Let $n_0 = 0$ or 2, $n_1 = 0$, $n_2 = 2$, $n_3 = 0$ or 2, and $n_4 \geq 3$. Let

$$X = \{e_x e_{x+2} \mid 0 \leq x \leq n_4 - 2\},$$

$$F_1 = \begin{cases} \{a_0 c_1\} & (\text{if } n_0 = 0) \\ \{a_0 a_2, a_1 c_1\} & (\text{if } n_0 = 2), \end{cases} \qquad F_1' = \begin{cases} \{a_0 e_{n_4 - 1}\} & (\text{if } n_0 = 0) \\ \{a_1 e_{n_4}, a_1 e_{n_4 - 1}\} & (\text{if } n_0 = 2), \end{cases}$$

$$F_2 = \begin{cases} \{d_0 e_1\} & (\text{if } n_3 = 0) \\ \{d_0 d_2, d_1 e_1\} & (\text{if } n_3 = 2), \end{cases} \qquad F_2' = \begin{cases} \emptyset & (\text{if } n_3 = 0) \\ \{d_1 e_0\} & (\text{if } n_3 = 2), \end{cases}$$

$$F_3' = \begin{cases} \{b_0 c_1, b_0 e_{n_4 - 1}, b_0 e_{n_4}\} & (\text{if } n_0 = 0) \\ \{b_0 a_1, b_0 c_1, b_0 e_{n_4 - 1}, b_0 e_{n_4}\} & (\text{if } n_0 = 2), \end{cases}$$

$$F_4 = \{c_0 c_2\}, \qquad F_4' = \{c_1 e_{n_4 - 1}, c_1 e_{n_4}\},$$

$$W_1 = \begin{cases} \{e_{n_4 - 1} a_0, e_{n_4 - 1} b_0\} & (\text{if } n_0 = 0) \\ \{e_{n_4 - 1} a_1, e_{n_4 - 1} b_0\} & (\text{if } n_0 = 2), \end{cases}$$

$$W_2 = \begin{cases} \{e_{n_4 - 1} b_0, e_{n_4 - 1} c_1\} & (\text{if } n_0 = 0) \\ \{e_{n_4 - 1} b_0, e_{n_4} b_0\} \cup F_4' & (\text{if } n_0 = 2), \end{cases}$$

$$Z_1 = \{b_0 c_1\}, \quad \text{and} \quad Z_2 = \begin{cases} \{b_0 e_{n_4}\} & (\text{if } n_0 = 0) \\ \{b_0 a_1\} & (\text{if } n_3 = 2). \end{cases}$$

Under this notation, G is said to be of Type 12 if G satisfies the following conditions:

- $X \cup F_1 \cup F_2 \cup F_4 \subseteq E(G) - E(C) \subseteq X \cup F_1 \cup F_1' \cup F_2 \cup F_2' \cup F_3' \cup F_4 \cup F_4'$,
- for each i with $1 \leq i \leq 2$, $W_i \cap E(G) \neq \emptyset$,
- $(F_3' - Z_1) \cap E(G) \neq \emptyset$,
- $(F_3' - Z_2) \cap E(G) \neq \emptyset$,
- if $n_0 = 0$, then $F_4' \cap E(G) \neq \emptyset$.

Type 13. Let $n_0 = 0$ or 2, $n_1 = 0$, $n_2 = 2$, $n_3 = 0$ or 2, and $n_4 \geq 3$. Let p be an integer with $1 \leq p \leq n_4 - 1$, and let

$$X = \{e_x e_{x+2} \mid 0 \leq x \leq p-2,\ p \leq x \leq n_4 - 2\},$$

$$X' = \{e_{p+1} e_{p-1}\},$$

$$Y = \{b_0 e_x \mid p-1 \leq x \leq n_4\},$$

$$F_1 = \begin{cases} \{a_0 e_{n_4-1}\} & (\text{if } n_0 = 0) \\ \{a_0 a_2, a_1 e_{n_4-1}\} & (\text{if } n_0 = 2), \end{cases} \quad F_1' = \begin{cases} \emptyset & (\text{if } n_0 = 0) \\ \{a_1 b_0, a_1 e_{n_4}\} & (\text{if } n_0 = 2), \end{cases}$$

$$F_2 = \begin{cases} \{d_0 e_1\} & (\text{if } n_3 = 0) \\ \{d_0 d_2, d_1 e_1\} & (\text{if } n_3 = 2), \end{cases} \quad F_2' = \begin{cases} \emptyset & (\text{if } n_3 = 0) \\ \{d_1 e_0\} & (\text{if } n_3 = 2), \end{cases}$$

$$F_3' = \begin{cases} \{b_0 c_1\} & (\text{if } n_0 = 0) \\ \{b_0 a_1, b_0 c_1\} & (\text{if } n_0 = 2), \end{cases}$$

$$F_4 = \{c_0 c_2\}, \quad F_4' = \{c_1 e_{p-1}, c_1 e_p, c_1 e_{p+1}\},$$

$$W_1 = X' \cup \{e_{p+1} c_1\}, \quad W_2 = X' \cup \{e_{p-1} b_0\},$$

$$W_3 = \begin{cases} Y & (\text{if } n_0 = 0) \\ \{b_0 a_1\} \cup Y & (\text{if } n_0 = 2), \end{cases}$$

$$W_4 = \begin{cases} \{b_0 c_1\} \cup (Y - \{b_0 e_{n_4}\}) & (\text{if } n_0 = 0) \\ \{b_0 c_1\} \cup Y & (\text{if } n_0 = 2), \end{cases}$$

and

$$W_5 = \begin{cases} F_4' & (\text{if } n_3 = 2 \text{ or } p \geq 2) \\ F_4' - \{c_1 e_{p-1}\} & (\text{if } n_3 = 0 \text{ and } p = 1). \end{cases}$$

Now G is said to be of Type 13 if there exists p with $1 \leq p \leq n_4 - 1$ such that G satisfies the following two conditions:

- $X \cup F_1 \cup F_2 \cup F_4 \subseteq E(G) - E(C) \subseteq X \cup X' \cup Y \cup F_1 \cup F_1' \cup F_2 \cup F_2' \cup F_3' \cup F_4 \cup F_4'$,
- for each i with $1 \leq i \leq 5$, $W_i \cap E(G) \neq \emptyset$.

The graph in Figure 5 is an example of a graph of Type 13 with $n_0 = n_1 = n_3 = 0$, $n_2 = 2$, $n_4 = 6$ and $p = 3$.

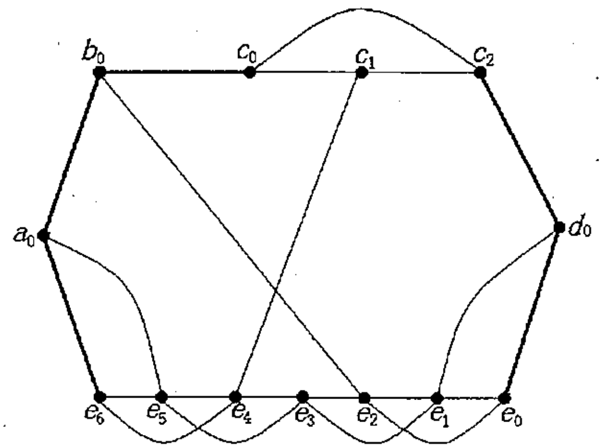

Figure 5

Type 14. Let $n_0 = 0$ or 2, $n_1 = 0$, $n_2 = 2$, $n_3 = 0$ or 2, and $n_4 \geq 3$. Let

$$X = \{e_x e_{x+2} \mid 0 \leq x \leq n_4 - 2\},$$

$$Y = \{b_0 e_x \mid 0 \leq x \leq n_4\},$$

$$F_1 = \begin{cases} \{a_0 e_{n_4-1}\} & (\text{if } n_0 = 0) \\ \{a_0 a_2, a_1 e_{n_4-1}\} & (\text{if } n_0 = 2), \end{cases} \qquad F_1' = \begin{cases} \emptyset & (\text{if } n_0 = 0) \\ \{a_1 b_0, a_1 e_{n_4}\} & (\text{if } n_0 = 2), \end{cases}$$

$$F_2 = \begin{cases} \emptyset & (\text{if } n_3 = 0) \\ \{d_0 d_2\} & (\text{if } n_3 = 2), \end{cases} \qquad F_2' = \begin{cases} \{d_0 b_0, d_0 c_1, d_0 e_1\} & (\text{if } n_3 = 0) \\ \{d_1 b_0, d_1 c_1, d_1 e_0, d_1 e_1\} & (\text{if } n_3 = 2), \end{cases}$$

$$F_3 = \begin{cases} \{c_1 e_1, c_0 c_2\} & (\text{if } n_3 = 0) \\ \{c_0 c_2\} & (\text{if } n_3 = 2), \end{cases}$$

and

$$F_3' = \begin{cases} \{c_1 b_0, c_1 e_0\} & (\text{if } n_3 = 0) \\ \{c_1 b_0, c_1 e_0, c_1 e_1\} & (\text{if } n_3 = 2). \end{cases}$$

Let $d = d_0$ if $n_3 = 0$, and let $d = d_1$ if $n_3 = 2$, and set

$$W_1 = \{db_0, dc_1\}, \qquad W_2 = \{db_0, de_1\},$$

$$W_3 = \begin{cases} \{b_0 d\} \cup Y & (\text{if } n_0 = 0) \\ \{b_0 a_1, b_0 d\} \cup Y & (\text{if } n_0 = 2), \end{cases}$$

$$W_4 = \begin{cases} \{b_0 c_1, b_0 d\} \cup (Y - \{b_0 e_{n_4}\}) & (\text{if } n_0 = 0) \\ \{b_0 c_1, b_0 d\} \cup Y & (\text{if } n_0 = 2). \end{cases}$$

Under this notation, G is said to be of Type 14 if G satisfies the following three conditions:

- $X \cup F_1 \cup F_2 \cup F_3 \subseteq E(G) - E(C) \subseteq X \cup Y \cup F_1 \cup F_1' \cup F_2 \cup F_2' \cup F_3 \cup F_3'$,
- if $n_3 = 2$, then $\{e_1c_1, e_1d_1\} \cap E(G) \neq \emptyset$ and $\{c_1e_0, c_1e_1\} \cap E(G) \neq \emptyset$,
- for each i with $1 \leq i \leq 4$, $W_i \cap E(G) \neq \emptyset$.

Type 15. Let $n_0 = 0$ or 2, $n_1 = 2$, $n_2 = 2$, $n_3 = 0$ or 2, and $n_4 \geq 3$. Let

$$X = \{e_x e_{x+2} \mid 0 \leq x \leq n_4 - 2\} \cup \{b_0b_2, c_0c_2\},$$

$$F_1 = \begin{cases} \{a_0e_{n_4-1}\} & (\text{if } n_0 = 0) \\ \{a_0a_2, a_1e_{n_4-1}\} & (\text{if } n_0 = 2), \end{cases} \quad F_1' = \begin{cases} \emptyset & (\text{if } n_0 = 0) \\ \{a_1e_{n_4}\} & (\text{if } n_0 = 2), \end{cases}$$

$$F_2 = \begin{cases} \{d_0e_1\} & (\text{if } n_3 = 0) \\ \{d_0d_2, d_1e_1\} & (\text{if } n_3 = 2), \end{cases} \quad \text{and} \quad F_2' = \begin{cases} \emptyset & (\text{if } n_3 = 0) \\ \{d_1e_0\} & (\text{if } n_3 = 2). \end{cases}$$

Let p and q be integers with $1 \leq p < q \leq n_4 - 1$, and let

$$Y_1 = \{b_1e_{q-1}, b_1e_q, b_1e_{q+1}\},$$

$$Y_2 = \{c_1e_{p-1}, c_1e_p, c_1e_{p+1}\},$$

$$W_1 = \begin{cases} Y_1 - \{b_1e_{n_4}\} & (\text{if } n_0 = 0 \text{ and } q = n_4 - 1) \\ Y_1 & (\text{ otherwise }), \end{cases}$$

and

$$W_2 = \begin{cases} Y_2 - \{c_1e_0\} & (\text{if } n_3 = 0 \text{ and } p = 1) \\ Y_2 & (\text{ otherwise }). \end{cases}$$

Now G is said to be of Type 15 if there exist p and q with $1 \leq p < q \leq n_4 - 1$ such that G satisfies the following two conditions:

- $X \cup F_1 \cup F_2 \subseteq E(G) - E(C) \subseteq X \cup Y_1 \cup Y_2 \cup F_1 \cup F_1' \cup F_2 \cup F_2'$,
- for each i with $1 \leq i \leq 2$, $W_i \cap E(G) \neq \emptyset$.

The graph in Figure 6 is an example of a graph of Type 15 with $n_0 = n_3 = 0$, $n_1 = n_2 = 2$, $n_4 = 9$, $p = 3$ and $q = 6$.

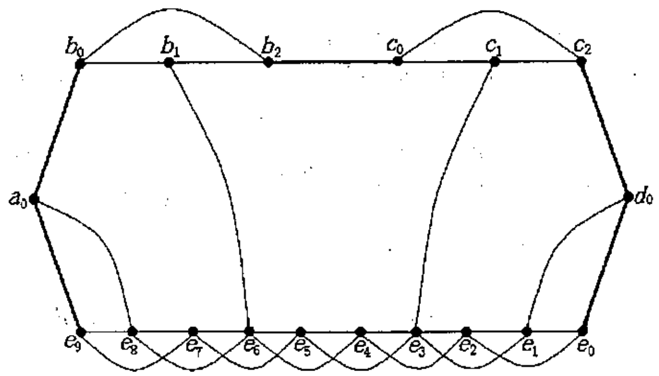

Figure 6

Type 16. Let $n_0 = 0$ or 2, $n_1 = 2$, $n_2 = 2$, $n_3 = 0$ or 2, and $n_4 \geq 3$. Let

$$X = \{e_x e_{x+2} \,|\, 0 \leq x \leq n_4 - 2\} \cup \{b_0 b_2, c_0 c_2\},$$

$$F_1 = \begin{cases} \{a_0 b_1, a_0 e_{n_4 - 1}\} & (\text{if } n_0 = 0) \\ \{a_0 a_2, a_1 b_1, a_1 e_{n_4 - 1}\} & (\text{if } n_0 = 2), \end{cases} \qquad F_1' = \begin{cases} \emptyset & (\text{if } n_0 = 0) \\ \{a_1 e_{n_4}\} & (\text{if } n_0 = 2), \end{cases}$$

$$F_2 = \begin{cases} \{d_0 e_1\} & (\text{if } n_3 = 0) \\ \{d_0 d_2, d_1 e_1\} & (\text{if } n_3 = 2), \end{cases} \qquad F_2' = \begin{cases} \emptyset & (\text{if } n_3 = 0) \\ \{d_1 e_0\} & (\text{if } n_3 = 2), \end{cases}$$

$$F_3 = \begin{cases} \{b_1 e_{n_4 - 1}\}, & (\text{if } n_0 = 0) \\ \emptyset & (\text{if } n_0 = 2), \end{cases} \qquad F_3' = \begin{cases} \{b_1 e_{n_4}\} & (\text{if } n_0 = 0) \\ \{b_1 e_{n_4 - 1}, b_1 e_{n_4}\} & (\text{if } n_0 = 2). \end{cases}$$

Let p be an integer with $1 \leq p \leq n_4 - 1$, and let

$$Y = \{c_1 e_{p-1}, c_1 e_p, c_1 e_{p+1}\},$$

and

$$W = \begin{cases} Y - \{c_1 e_0\} & (\text{if } n_3 = 0 \text{ and } p = 1) \\ Y & (\text{ otherwise }). \end{cases}$$

Now G is said to be of Type 16 if there exists p with $1 \leq p \leq n_4 - 1$ such that G satisfies the following three conditions:

- $X \cup F_1 \cup F_2 \cup F_3 \subseteq E(G) - E(C) \subseteq X \cup Y \cup F_1 \cup F_1' \cup F_2 \cup F_2' \cup F_3 \cup F_3'$,
- if $n_0 = 2$, then $F_3' \cap E(G) \neq \emptyset$,
- $W \cap E(G) \neq \emptyset$.

Type 17. Let $n_0 = 0$ or 2, $n_1 = 2$, $n_2 = 2$, $n_3 = 0$ or 2, and $n_4 \geq 3$. Let

$$F_1 = \begin{cases} \{a_0e_{n_4-1}\} & (\text{if } n_0 = 0) \\ \{a_0a_2, a_1e_{n_4-1}\} & (\text{if } n_0 = 2), \end{cases} \qquad F_1' = \begin{cases} \emptyset & (\text{if } n_0 = 0) \\ \{a_1e_{n_4}\} & (\text{if } n_0 = 2), \end{cases}$$

$$F_2 = \begin{cases} \{d_0e_1\} & (\text{if } n_3 = 0) \\ \{d_0d_2, d_1e_1\} & (\text{if } n_3 = 2), \end{cases} \quad \text{and} \quad F_2' = \begin{cases} \emptyset & (\text{if } n_3 = 0) \\ \{d_1e_0\} & (\text{if } n_3 = 2). \end{cases}$$

Let p be an integer with $1 \leq p \leq n_4 - 1$, and let

$$X = \{e_xe_{x+2} \mid 0 \leq x \leq p-2, p \leq x \leq n_4 - 2\} \cup \{b_0b_2, c_0c_2\},$$

$$X_1' = \{e_{p+1}e_{p-1}\}, \qquad X_2' = \{b_1c_1\},$$

$$Y_1 = \{b_1e_{p-1}, b_1e_p, b_1e_{p+1}\},$$

$$Y_2 = \{c_1e_{p-1}, c_1e_p, c_1e_{p+1}\},$$

$$W_1 = X_1' \cup \{c_1e_{p+1}\}, \qquad W_2 = X_1' \cup \{b_1e_{p-1}\},$$

$$W_3 = X_2' \cup \{c_1e_{p+1}\}, \quad \text{and} \quad W_4 = X_2' \cup \{b_1e_{p-1}\}.$$

Now G is said to be of Type 17 if there exists p with $1 \leq p \leq n_4 - 1$ such that G satisfies the following five conditions:

- $X \cup F_1 \cup F_2 \subseteq E(G) - E(C) \subseteq X \cup X_1' \cup X_2' \cup Y_1 \cup Y_2 \cup F_1 \cup F_1' \cup F_2 \cup F_2'$,
- for each i with $1 \leq i \leq 2$, $Y_i \cap E(G) \neq \emptyset$,
- for each i with $1 \leq i \leq 4$, $W_i \cap E(G) \neq \emptyset$,
- if $n_0 = 0$ and $p = n_4 - 1$, then $\{b_1e_{n_4-1}, b_1e_{n_4-2}, c_1e_{n_4-1}, c_1e_{n_4-2}\} \cap E(G) \neq \emptyset$,
- if $n_3 = 0$ and $p = 1$, then $\{b_1e_1, b_1e_2, c_1e_1, c_1e_2\} \cap E(G) \neq \emptyset$.

Type 18. Let $n_0 = 0$ or 2, $n_1 = 2$, $n_2 = 2$, $n_3 = 0$ or 2, and $n_4 \geq 3$. Let

$$X = \{e_xe_{x+2} \mid 0 \leq x \leq n_4 - 2\} \cup \{b_0b_2, c_0c_2\},$$

$$X' = \{b_1c_1\},$$

$$F_1 = \begin{cases} \emptyset & (\text{if } n_0 = 0) \\ \{a_0a_2\} & (\text{if } n_0 = 2), \end{cases}$$

$$F_1' = \begin{cases} \{a_0b_1, a_0c_1, a_0e_{n_4-1}\} & (\text{if } n_0 = 0) \\ \{a_1b_1, a_1c_1, a_1e_{n_4-1}, a_1e_{n_4}\} & (\text{if } n_0 = 2), \end{cases}$$

$$F_2 = \begin{cases} \{d_0e_1\} & (\text{if } n_3 = 0) \\ \{d_0d_2, d_1e_1\} & (\text{if } n_3 = 2), \end{cases} \qquad F_2' = \begin{cases} \emptyset & (\text{if } n_3 = 0) \\ \{d_1e_0\} & (\text{if } n_3 = 2), \end{cases}$$

$$F_3' = \{b_1e_{n_4-1}, b_1e_{n_4}\}, \text{ and } F_4' = \{c_1e_{n_4-1}, c_1e_{n_4}\}.$$

Let $a = a_0$ if $n_0 = 0$, and let $a = a_1$ if $n_0 = 2$, and set

$$W_1 = \{ab_1, ac_1\}, \qquad W_2 = \{c_1a, c_1b_1\},$$

$$W_3 = \{ac_1, ae_{n_4-1}\}, \qquad W_4 = \{e_{n_4-1}a, e_{n_4-1}b_1\},$$

$$W_5 = \{c_1a\} \cup F_4',$$

$$W_6 = \begin{cases} F_3' & (\text{if } n_0 = 0) \\ \{b_1a\} \cup F_3' & (\text{if } n_0 = 2), \end{cases}$$

$$W_7 = \begin{cases} \{b_1c_1\} \cup (F_3' - \{b_1e_{n_4}\}) & (\text{if } n_0 = 0) \\ \{b_1c_1\} \cup F_3' & (\text{if } n_0 = 2), \end{cases}$$

and

$$W_8 = \begin{cases} (F_3' - \{b_1e_{n_4}\}) \cup (F_4' - \{c_1e_{n_4}\}) & (\text{if } n_0 = 0) \\ F_3' \cup F_4' & (\text{if } n_0 = 2). \end{cases}$$

Under this notation, G is said to be of Type 18 if G satisfies the following two conditions:

- $X \cup F_1 \cup F_2 \subseteq E(G) - E(C) \subseteq X \cup X' \cup F_1 \cup F_1' \cup F_2 \cup F_2' \cup F_3' \cup F_4'$,
- for each i with $1 \le i \le 8$, $W_i \cap E(G) \ne \emptyset$.

Type 19. Let $n_0 = 0$ or 2, $n_1 = 2$, $n_2 = 2$, $n_3 = 0$ or 2, and $n_4 \ge 3$. Let

$$X = \{e_xe_{x+2} \mid 0 \le x \le n_4 - 2\} \cup \{b_0b_2, c_0c_2\},$$

$$F_1 = \begin{cases} \{a_0b_1, a_0e_{n_4-1}\} & (\text{if } n_0 = 0) \\ \{a_0a_2, a_1b_1, a_1e_{n_4-1}\} & (\text{if } n_0 = 2), \end{cases} \qquad F_1' = \begin{cases} \emptyset & (\text{if } n_0 = 0) \\ \{a_1e_{n_4}\} & (\text{if } n_0 = 2), \end{cases}$$

$$F_2 = \begin{cases} \{d_0c_1, d_0e_1\} & (\text{if } n_3 = 0) \\ \{d_0d_2, d_1c_1, d_1e_1\} & (\text{if } n_3 = 2), \end{cases} \qquad F_2' = \begin{cases} \emptyset & (\text{if } n_3 = 0) \\ \{d_1e_0\} & (\text{if } n_3 = 2), \end{cases}$$

$$F_3 = \begin{cases} \{b_1e_{n_4-1}\} & (\text{if } n_0 = 0) \\ \emptyset & (\text{if } n_0 = 2), \end{cases} \qquad F_3' = \begin{cases} \{b_1e_{n_4}\} & (\text{if } n_0 = 0) \\ \{b_1e_{n_4-1}, b_1e_{n_4}\} & (\text{if } n_0 = 2), \end{cases}$$

$$F_4 = \begin{cases} \{c_1e_1\} & (\text{if } n_3 = 0) \\ \emptyset & (\text{if } n_3 = 2), \end{cases} \quad \text{and } F_4' = \begin{cases} \{c_1e_0\} & (\text{if } n_3 = 0) \\ \{c_1e_0, c_1e_1\} & (\text{if } n_3 = 2). \end{cases}$$

Under this notation, G is said to be of Type 19 if G satisfies the following three conditions:

- $X \cup F_1 \cup F_2 \cup F_3 \cup F_4 \subseteq E(G) - E(C) \subseteq X \cup F_1 \cup F_1' \cup F_2 \cup F_2' \cup F_3 \cup F_3' \cup F_4 \cup F_4'$,
- if $n_0 = 2$, then $F_3' \cap E(G) \neq \emptyset$,
- if $n_3 = 2$, then $F_4' \cap E(G) \neq \emptyset$.

Type 20. Let $n_0 = 0$ or 2, $n_1 = 2$, $n_2 = 2$, $n_3 = 0$ or 2, and $n_4 \geq 3$. Let

$$X = \{e_x e_{x+2} \mid 0 \leq x \leq n_4 - 2\} \cup \{b_0 b_2, c_0 c_2\},$$

$$F_1 = \begin{cases} \{a_0 e_{n_4-1}\} & (\text{if } n_0 = 0) \\ \{a_0 a_2, a_1 e_{n_4-1}\} & (\text{if } n_0 = 2), \end{cases} \qquad F_1' = \begin{cases} \emptyset & (\text{if } n_0 = 0) \\ \{a_1 e_{n_4}\} & (\text{if } n_0 = 2), \end{cases}$$

$$F_2 = \begin{cases} \{d_0 e_1\} & (\text{if } n_3 = 0) \\ \{d_0 d_2, d_1 e_1\} & (\text{if } n_3 = 2), \end{cases} \qquad F_2' = \begin{cases} \emptyset & (\text{if } n_3 = 0) \\ \{d_1 e_0\} & (\text{if } n_3 = 2), \end{cases}$$

$$F_3 = \begin{cases} \{b_1 e_{n_4-1}\} & (\text{if } n_0 = 0) \\ \emptyset & (\text{if } n_0 = 2), \end{cases} \qquad F_3' = \begin{cases} \{b_1 e_{n_4}\} & (\text{if } n_0 = 0) \\ \{b_1 e_{n_4-1}, b_1 e_{n_4}\} & (\text{if } n_0 = 2), \end{cases}$$

$$F_4 = \{c_1 e_{n_4}\}, \quad \text{and} \quad F_4' = \{c_1 e_{n_4-2}, c_1 e_{n_4-1}\}.$$

Under this notation, G is said to be of Type 20 if G satisfies the following two conditions:

- $X \cup F_1 \cup F_2 \cup F_3 \cup F_4 \subseteq E(G) - E(C) \subseteq X \cup F_1 \cup F_1' \cup F_2 \cup F_2' \cup F_3 \cup F_3' \cup F_4 \cup F_4'$,
- if $n_0 = 2$, then $F_3' \cap E(G) \neq \emptyset$.

§3. Preliminaries

In this section, we prove fundamental results concerning noncontractible edges lying on a hamiltonian cycle of a 3-connected graph.

Throughout this section, we let G denote a 3-connected graph of order $n+1$ ($n \geq 4$), and let $C = v_0 v_1 \cdots v_n v_0$ denote a hamiltonian cycle of G. Lemmas 3.1 through 3.8 are proved in Section 3 of [4] (and also in Ota [5]), and we omit their proofs (in Lemmas 3.1 through 3.8, we assume that the edge $v_n v_0$ is noncontractible, and let $\{v_n, v_0, v_a\}$ be a cutset associated with it).

Lemma 3.1.

(i) *No edge of G joins a vertex in $\{v_k \mid 1 \leq k \leq a-1\}$ and a vertex in $\{v_k \mid a+1 \leq k \leq n-1\}$.*

(ii) *There exists k with $1 \leq k \leq a-1$ such that $v_n v_k \in E(G)$.*

Lemma 3.2. *If $a = 2$, then $E(v_1, V(G)) - E(C) = \{v_1 v_n\}$.*

Lemma 3.3. *Suppose that v_0v_1 is noncontractible and $v_a \in K(v_0, v_1)$. Then $v_nv_1 \in E(G)$.*

Lemma 3.4. *Suppose that v_av_{a+1} is noncontractible, and let $\{v_a, v_{a+1}, v_j\}$ be a cutset associated with it. Then $a+3 \leq j \leq n$ (and hence $a \leq n-3$). Further, if $j = n$, then $v_0v_{a+1} \in E(G)$.*

Lemma 3.5. *Let $1 \leq j \leq a-2$. Suppose that v_jv_{j+1} is noncontractible, and let $\{v_j, v_{j+1}, v_l\}$ be a cutset associated with it, and suppose that $a+1 \leq l \leq n-1$. Then $l = a+1$, v_av_l is contractible and, unless $l = n-1$, we have $v_l \in K(v_n, v_0)$.*

Lemma 3.6. *Suppose that v_0v_1 is noncontractible, and let $\{v_0, v_1, v_j\}$ be a cutset associated with it, and suppose that $a+1 \leq j \leq n-2$. Then $v_j \in K(v_n, v_0)$.*

Lemma 3.7. *Suppose that $K(v_n, v_0) = \{v_2\}$, and that v_0v_1 is noncontractible. Then $K(v_0, v_1) = \{v_{n-1}\}$.*

Lemma 3.8.

(i) *If $a = 2$, then v_1v_2 is contractible.*

(ii) *If $a \geq 2$, then there exists j with $0 \leq j \leq a-1$ such that v_jv_{j+1} is contractible.*

(iii) *If $a \geq 3$ and there exists only one j with $0 \leq j \leq a-1$ such that v_jv_{j+1} is contractible, then v_av_{a+1} is contractible.*

Lemma 3.9. *Let l be an integer with $3 \leq l \leq n-1$.*

(i) *Suppose that for each j with $l+1 \leq j \leq n$, $v_{j-1}v_j$ is noncontractible and $K(v_{j-1}, v_j) \cap \{v_i \mid 1 \leq i \leq l-2\} \neq \emptyset$. Then G has no edge $v_{j_1}v_{j_2}$ such that $l \leq j_1 < j_1 + 3 \leq j_2 \leq n$.*

(ii) *Suppose that $l \leq n-3$, let h be an integer with $l+2 \leq h \leq n-1$, and suppose that for each j with $l+1 \leq j \leq n$ and $j \neq h$, $v_{j-1}v_j$ is noncontractible and $K(v_{j-1}, v_j) \cap \{v_i \mid 1 \leq i \leq l-2\} \neq \emptyset$. Further let $v_{j_1}v_{j_2} \in E(G)$ be an edge such that $l \leq j_1 < j_1 + 3 \leq j_2 \leq n$. Then $j_1 = h-2$ and $j_2 = h+1$.*

Proof. Let $v_{j_1}v_{j_2} \in E(G)$ be an edge such that $l \leq j_1 < j_1+3 \leq j_2 \leq n$. Let j be an integer with $j_1 + 2 \leq j \leq j_2 - 1$. If the assumption of (i) holds, or if the assumption of (ii) holds and $j \neq h$, then $K(v_{j-1}, v_j) \cap \{v_i \mid 1 \leq i \leq l-2\} \neq \emptyset$, which contradicts Lemma 3.1(i). Thus the assumption of (ii) holds and $j = h$. Since j was arbitrary, this means that $j_1 + 2 = j_2 - 1 = h$, as desired. ∎

Lemma 3.10. *Let $1 \leq i_1$ and $i_1 + 2 \leq i_2 < i_3 \leq n-1$, and suppose that $v_i v_{i+1}$ is noncontractible for all $0 \leq i \leq i_1 - 1$.*

(I) *Suppose that $K(v_i, v_{i+1}) \cap \{v_j \mid i_2 \leq j \leq i_3\} \neq \emptyset$ for all $0 \leq i \leq i_1 - 1$.*

(i) *Suppose that $v_{i_2} \in K(v_0, v_1)$. Then for each $0 \leq i \leq i_1 - 1$, $v_{i_2} \in K(v_i, v_{i+1})$.*

(ii) *Suppose that $v_{i_3} \notin K(v_0, v_1)$. Then $v_{i_3} \notin K(v_i, v_{i+1})$ for each $0 \leq i \leq i_1 - 1$.*

(iii) *Let $i_2 < l < i_3$, and suppose that $v_l \in K(v_0, v_1)$ and $v_l \in K(v_{i_1-1}, v_{i_1})$. Then $v_l \in K(v_i, v_{i+1})$ for each $0 \leq i \leq i_1 - 1$.*

(II) *Suppose that $v_{i_2} \in K(v_0, v_1)$, and $v_{i_2} \notin K(v_i, v_{i+1})$ for each $1 \leq i \leq i_1 - 1$. Then $K(v_i, v_{i+1}) \cap \{v_j \mid i_2 \leq j \leq i_3\} = \emptyset$ for each $1 \leq i \leq i_1 - 1$.*

Proof.

(I) (i) Let $1 \leq i \leq i_1 - 1$, and take $v_k \in K(v_i, v_{i+1}) \cap \{v_j \mid i_2 \leq j \leq i_3\}$. We may assume that $k \neq i_2$. But then, applying Lemma 3.5 or 3.6 to $\{v_0, v_1, v_{i_2}\}$ and $\{v_i, v_{i+1}, v_k\}$ according as $i \geq 2$ or $i = 1$, we obtain $v_{i_2} \in K(v_i, v_{i+1})$, as desired.

(ii) Take $v_k \in K(v_0, v_1) \cap \{v_j \mid i_2 \leq j \leq i_3\}$. We have $k \neq i_3$ by assumption. Let $1 \leq i \leq i_1 - 1$, and suppose that $v_{i_3} \in K(v_i, v_{i+1})$. Then applying Lemma 3.5 or 3.6 to $\{v_0, v_1, v_k\}$ and $\{v_i, v_{i+1}, v_{i_3}\}$, we get $v_{i_3} \in K(v_0, v_1)$, a contrdiction.

(iii) Let $1 \leq i \leq i_1 - 2$, and take $v_k \in K(v_i, v_{i+1}) \cap \{v_j \mid i_2 \leq j \leq i_3\}$. We may assume $k \neq l$. If $l < k \leq i_3$, then we get $v_l \in K(v_i, v_{i+1})$ by applying Lemma 3.5 or 3.6 to $\{v_0, v_1, v_l\}$ and $\{v_i, v_{i+1}, v_k\}$; if $i_2 \leq k < l$, then we get $v_l \in K(v_i, v_{i+1})$ by applying Lemma 3.5 or 3.6 to $\{v_{i_1-1}, v_{i_1}, v_l\}$ and $\{v_i, v_{i+1}, v_k\}$.

(II) Let $1 \leq i \leq i_1 - 1$, and suppse that there exists $v_k \in K(v_i, v_{i+1})$ with $i_2 \leq k \leq i_3$. We have $k \neq i_2$ by assumption. But then applying Lemma 3.5 or 3.6 to $\{v_0, v_1, v_{i_2}\}$ and $\{v_i, v_{i+1}, v_k\}$, we get $v_{i_2} \in K(v_i, v_{i+1})$, a contradiction. ∎

Lemma 3.11. *Let m, u, z be integers with $0 \leq m < u$ and $u+2 \leq z < n$, and let $A = \{v_i \mid 0 \leq i \leq m\}$, $P = \{v_i \mid m+1 \leq i \leq u\}$, $B = \{v_i \mid u+1 \leq i \leq z\}$, $Q = \{v_i \mid z+1 \leq i \leq n\}$ (thus A, P, $Q \neq \emptyset$ and, $|B| \geq 2$). For convenience, let $s = z - (u+1)$ and let $w_j = u_{j+u+1}$ for each $0 \leq j \leq s$. Thus $s > 0$ and $B = \{w_j \mid 0 \leq j \leq s\}$. Suppose that $v_i v_{i+1}$ is noncontractible for all $0 \leq i \leq m-1$. Also suppose that one of the following two situations occurs:*

(11a) $w_{j-1}w_j$ *is noncontractible for all* $1 \le j \le s$; *or*

(11b) *there exists* h *with* $2 \le h \le s-1$ *such that* $w_{h-1}w_h$ *is contractible and* $w_{j-1}w_j$ *is noncontractible for all* $1 \le j \le s$ *with* $j \ne h$.

For convenience, set

$$N = \begin{cases} \{j \mid 1 \le j \le s\} & (\textit{if } (11a) \textit{ holds}) \\ \{j \mid 1 \le j \le s\} - \{h\} & (\textit{if } (11b) \textit{ holds}). \end{cases}$$

Moreover, suppose that $K(v_i, v_{i+1}) \cap B \ne \emptyset$ *for all* $0 \le i \le m-1$ *and* $K(w_{j-1}, w_j) \cap A \ne \emptyset$ *for all* $j \in N$. *Then there exists an integer* r *with* $1 \le r \le \min\{m+1, s\}$, *and there exist integers* $k_0, k_1, k_2, \ldots, k_r, k_{r+1}$ *and* $l_1, l_2, \ldots, l_r, l_{r+1}$ *with* $0 = k_0 \le k_1 < k_2 < \cdots < k_{r-1} < k_r \le k_{r+1} = m$ *and* $s = l_1 > l_2 > \cdots > l_r > l_{r+1} = 0$ *such that the following hold.*

(I) *If* (11b) *holds, then* $l_t \ne h$ *for all* $1 \le t \le r+1$, *and* $l_1, l_{r+1} \ne h-1$, *and one of the following holds*:

(11b-1) $l_t \ne h-1$ *for all* $1 \le t \le r+1$; *or*

(11b-2) *there exists an integer* t' *with* $2 \le t' \le r$ *such that* $l_{t'} = h-1$.

(II) (i) (a) *If* $k_1 = 0$, *then* $v_{k_1} = v_0 \in K(w_{l_1}, w_{l_1-1}) = K(w_s, w_{s-1})$.

(b) *If* $k_1 > 0$, *then* $m > 0$ *and* $w_{l_1} = w_s \in K(v_{k_0}, v_{k_0+1}) = K(v_0, v_1)$.

(ii) (a) *If* $k_r = m$, *then* $v_m = v_{k_r} \in K(w_{l_{r+1}+1}, w_{l_{r+1}}) = K(w_0, w_1)$.

(b) *If* $k_r < m$, *then* $m > 0$ *and* $w_{l_{r+1}} = w_0 \in K(v_{k_{r+1}-1}, v_{k_{r+1}}) = K(v_{m-1}, v_m)$.

(III) *Set*

$$X_1 = \bigcup_{t=1}^{r+1} \{v_i v_{i+2} \mid k_{t-1} \le i \le k_t - 2\},$$

$$X_3 = \begin{cases} \emptyset & (\textit{if } (11a) \textit{ or } (11b\text{-}2) \textit{ holds}) \\ \{w_{h-2}w_h, w_{h-1}w_{h+1}\} & (\textit{if } (11b\text{-}1) \textit{ holds}), \end{cases}$$

$$Y_1 = \begin{cases} \{v_{k_t+1}w_{l_{t+1}+1} \mid 1 \le t \le r-1\} & (\textit{if } (11a) \textit{ or } (11b\text{-}1) \textit{ holds}) \\ \Big(\{v_{k_t+1}w_{l_{t+1}+1} \mid 1 \le t \le r-1\} - \{v_{k_{t'-1}+1}w_{l_{t'}+1}\}\Big) & \\ \qquad \cup \{v_{k_{t'-1}+1}w_{l_{t'}+2}\} & (\textit{if } (11b\text{-}2) \textit{ holds}), \end{cases}$$

$$\bar{Y}_1 = \begin{cases} \{v_{k_r+1}w_{l_{r+1}+1}\} & (\textit{if } k_r < m) \\ \emptyset & (\textit{if } k_r = m), \end{cases}$$

$$Y_2 = \{v_{k_t-1}w_{l_t-1} \,|\, 2 \le t \le r\},$$

$$\bar{Y}_2 = \begin{cases} \{v_{k_1-1}w_{l_1-1}\} & (\textit{if } k_1 > 0) \\ \emptyset & (\textit{if } k_1 = 0), \end{cases}$$

$$Y_3' = \begin{cases} \bigcup_{t=1}^{r} \{v_{k_t}w_j \,|\, l_t \ge j \ge l_{t+1}\} & (\textit{if (11a) holds or (11b-1) holds}) \\ \bigcup_{t=1}^{r} \{v_{k_t}w_j \,|\, l_t \ge j \ge l_{t+1}\} - \{v_{k_{t'-1}}w_{l_{t'}}\} & (\textit{if (11b-2) holds}), \end{cases}$$

$$Y_4' = \begin{cases} \bigcup_{t=1}^{r+1} \{v_i w_{l_t} \,|\, k_{t-1} \le i \le k_t\} & (\textit{if (11a) or (11b-1) holds}) \\ \bigcup_{t=1}^{r+1} \{v_i w_{l_t} \,|\, k_{t-1} \le i \le k_t\} - \{v_i w_{l_{t'}} \,|\, k_{t'-1} \le i \le k_{t'}\} & \\ \qquad\qquad (\textit{if (11b-2) holds}). \end{cases}$$

(i) *Suppose that* (11b-2) *holds. Then there exist integers* p *and* q *with* $k_{t'-1} \le p \le q \le k_{t'}$ *such that if we set*

$$X_2 = \begin{cases} \bigcup_{t=1}^{r} \{w_j w_{j-2} \,|\, l_t \ge j \ge l_{t+1}+2\} \\ \qquad\qquad (\textit{if } q = k_{t'-1} (\textit{so } p = k_{t'-1})) \\ \bigcup_{t=1}^{r} \{w_j w_{j-2} \,|\, l_t \ge j \ge l_{t+1}+2\} - \{w_{l_{t'}} w_{l_{t'+2}}\} \\ \qquad\qquad (\textit{if } q \ne k_{t'-1} \textit{and } p \ne k_{t'}) \\ \left(\bigcup_{t=1}^{r} \{w_j w_{j-2} \,|\, l_t \ge j \ge l_{t+1}+2\} - \{w_{l_{t'}} w_{l_{t'+2}}\}\right) \\ \qquad \cup \{w_{l_{t'}-1} w_{l_{t'}+1}\} \quad (\textit{if } p = k_{t'} (\textit{so } q = k_{t'})), \end{cases}$$

$$Z_1 = \begin{cases} \{v_{p+1}w_{l_{t'}+1}\} & (\textit{if } k_{t'-1} < p < k_{t'}) \\ \emptyset & (\textit{ otherwise }), \end{cases}$$

$$Z_2 = \begin{cases} \{v_{q-1}w_{l_{t'}}\} & (\text{\textit{if} } k_{t'-1} < q < k_{t'}) \\ \emptyset & (\text{ \textit{otherwise} }), \end{cases}$$

$$Z'_1 = \{v_i w_{l_{t'}+1} \mid k_{t'-1} \leq i \leq p\},$$

$$\bar{Z}'_1 = \begin{cases} \{v_{p+1}w_{l_{t'}+1}\} & (\text{\textit{if} } p = k_{t'-1}) \\ \emptyset & (\text{ \textit{otherwise} }), \end{cases}$$

$$Z'_2 = \{v_i w_{l_{t'}} \mid q \leq i \leq k_{t'}\},$$

$$\bar{Z}'_2 = \begin{cases} \{v_{q-1}w_{l_{t'}}\} & (\text{\textit{if} } q = k_{t'}) \\ \emptyset & (\text{ \textit{otherwise} }), \end{cases}$$

then we have

$$E(\langle A\rangle) - E(C) = X_1,$$

$$X_2 \subseteq E(\langle B\rangle) - E(C) \subseteq X_2 \cup X_3,$$

and

$$\begin{aligned} & Y_1 \cup \bar{Y_1} \cup Y_2 \cup \bar{Y_2} \cup Z_1 \cup Z_2 \\ \subseteq\ & E(A, B) \\ \subseteq\ & Y_1 \cup \bar{Y_1} \cup Y_2 \cup \bar{Y_2} \cup Z_1 \cup Z_2 \cup Y'_3 \cup Y'_4 \cup Z'_1 \cup \bar{Z_1}' \cup Z'_2 \cup \bar{Z_2}'. \end{aligned}$$

(ii) *If* (11a) *or* (11b-1) *holds, then the same conclusion as in* (i) *holds with*

$$X_2 = \begin{cases} \bigcup_{t=1}^{r} \{w_j w_{j-2} \mid l_t \geq j \geq l_{t+1} + 2\} & (\text{\textit{if} (11a) \textit{holds}}) \\ \left(\bigcup_{t=1}^{r} \{w_j w_{j-2} \mid l_t \geq j \geq l_{t+1} + 2\} - \{w_{h-2}w_h, w_{h-1}w_{h+1}\}\right) & \\ \qquad \cup \{w_{h-2}w_{h+1}\} & (\text{\textit{if} (11b-1) \textit{holds}}), \end{cases}$$

and

$$Z_1 = Z_2 = Z'_1 = Z'_2 = \bar{Z}'_1 = \bar{Z}'_2 = \emptyset.$$

Proof. For the sake of clarity, we sepatate some points of the proof and present them as claims. Arguing exactly as in Claims 5.1 and 5.2 in the proof of Proposition 1 of [4], we obtain the following two claims:

Claim 3.1. *Let $0 \leq i \leq m-1$ and $j \in N$, and take $w_l \in K(v_i, v_{i+1}) \cap B$ and $v_k \in K(w_{j-1}, w_j) \cap A$. Then the following hold.*

(i) *If $l \geq j$, then $k \geq i+1$.*

(ii) *If $l \leq j-1$, then $k \leq i$.*

Claim 3.2. *For each $j \in N$, $|K(w_j, w_{j-1}) \cap A| = 1$.*

For each $j \in N$, write $K(w_j, w_{j-1}) \cap A = \{v_{i_j}\}$. Note that if (11b) holds, then i_h is not defined. Arguing as in the proof of Claims 5.4 of [4], we obtain:

Claim 3.3.

(i) *If* (11a) *holds, then $i_{j+1} \leq i_j$ for each $j \in N - \{s\}$.*

(ii) *If* (11b) *holds, then $i_{j+1} \leq i_j$ for each $j \in N - \{h-1, s\}$, and $i_{h+1} \leq i_{h-1}$.*

Write $\{i_j \mid j \in N\} = \{k_1, k_2, \ldots, k_r\}$ with $k_1 < k_2 < \cdots < k_r$. By Claim 3.3, there exist $l_1, l_2, \ldots l_{r+1}$ with $s = l_1 > l_2 > \cdots > l_{r+1} = 0$ such that for each $1 \leq t \leq r$, we have

$$i_j = k_t \text{ for all } j \in N \text{ with } l_t \geq j \geq l_{t+1} + 1, \tag{3.1}$$

and such that in the case where (11b) holds, we have

$$l_t \neq h \text{ for all } 1 \leq t \leq r+1. \tag{3.2}$$

In the case where (11b) holds, we have

$$l_1, l_{r+1} \neq h - 1 \tag{3.3}$$

because $2 \leq h \leq s-1$ by the definition of h. By (3.3), it is clear that if (11b) holds, then (11b-1) or (11b-2) holds. This proves (I).

For convenience, let $k_0 = 0$ (even if $k_1 = 0$) and $k_{r+1} = m$ (even if $k_r = m$).

Claim 3.4.

(I) *For each $1 \leq t \leq r$, we have $K(w_j, w_{j-1}) \cap A = \{v_{k_t}\}$ for all j such that $j \in N$ and $l_t \geq j \geq l_{t+1} + 1$.*

(II) (i) *If* (11a) *or* (11b-1) *holds, then for each $1 \leq t \leq r+1$, we have $K(v_i, v_{i+1}) \cap B = \{w_{l_t}\}$ for all $k_{t-1} \leq i \leq k_t - 1$.*

(ii) *If* (11b-2) *holds, then the following hold.*

(a) *For each* $1 \le t \le r+1$ *with* $t \ne t'$, *we have* $K(v_i, v_{i+1}) \cap B = \{w_{l_t}\}$ *for all* $k_{t-1} \le i \le k_t - 1$.

(b) *There exist* p *and* q *with* $k_{t'-1} \le p \le q \le k_{t'}$ *such that the following hold:*

(b-1) *for each* $k_{t'-1} \le i \le p-1$, $K(v_i, v_{i+1}) \cap B = \{w_{l_{t'}+1}\}$;

(b-2) *for each* $p \le i \le q-1$, $K(v_i, v_{i+1}) \cap B = \{w_{l_{t'}+1}, w_{l_{t'}}\}$;

(b-3) *for each* $q \le i \le k_{t'} - 1$, $K(v_i, v_{i+1}) \cap B = \{w_{l_{t'}}\}$.

Proof. We can prove (I), (II)(i) and (II)(ii)(a) by arguing exactly as in Claim 5.5 of [4]. To prove (II)(ii)(b), suppose that (11b-2) holds. Let $k_{t'-1} \le i \le k_{t'} - 1$, and take $w_l \in K(v_i, v_{i+1}) \cap B$. Since $l_{t'} = h - 1 \ge 1$, we get $v_{k_{t'}} \in K(w_{l_{t'}}, w_{l_{t'}-1})$ by (I), and hence it follows from Claim 3.1(ii) that $l \ge l_{t'}$. Similarity, since $l_{t'} = h-1 \le s-2$ and $l_{t'-1} \ge l_{t'} + 2$ by (3-2), we get $v_{k_{t'-1}} \in K(w_{l_{t'}+2}, w_{l_{t'}+1})$ by (I), and hence it follows from Claim 3.1(i) that $l \le l_{t'} + 1$. Thus

$$K(v_i, v_{i+1}) \cap B \subseteq \{w_{l_{t'}+1}, w_{l_{t'}}\} \text{ for all } k_{t'-1} \le i \le k_{t'} - 1. \tag{3.4}$$

By the assumption that $K(v_i, v_{i+1}) \cap B \ne \emptyset$ for all $0 \le i \le m-1$, this in particular implies that

$$K(v_i, v_{i+1}) \cap \{w_{l_{t'}+1}, w_{l_{t'}}\} \ne \emptyset \text{ for all } k_{t'-1} \le i \le k_{t'} - 1. \tag{3.5}$$

Let

$$p = \begin{cases} \min\{i \mid w_{l_{t'}} \in K(v_i, v_{i+1}), k_{t'-1} \le i < k_{t'}\} \\ \qquad (\text{if } \{i \mid w_{l_{t'}} \in K(v_i, v_{i+1}), k_{t'-1} \le i < k_{t'}\} \ne \emptyset) \\ k_{t'} \quad (\text{if } \{i \mid w_{l_{t'}} \in K(v_i, v_{i+1}), k_{t'-1} \le i < k_{t'}\} = \emptyset) \end{cases}$$

and

$$q = \begin{cases} \max\{i \mid w_{l_{t'}+1} \in K(v_i, v_{i-1}), k_{t'-1} < i \le k_{t'}\} \\ \qquad (\text{if } \{i \mid w_{l_{t'}+1} \in K(v_i, v_{i-1}), k_{t'-1} < i \le k_{t'}\} \ne \emptyset) \\ k_{t'-1} \quad (\text{if } \{i \mid w_{l_{t'}+1} \in K(v_i, v_{i-1}), k_{t'-1} < i \le k_{t'}\} = \emptyset) \end{cases}$$

Then by the definition of p and q,

$$w_{l_{t'}} \notin K(v_i, v_{i+1}) \text{ for all } k_{t'-1} \le i \le p-1, \tag{3.6}$$

$$w_{l_{t'}+1} \notin K(v_i, v_{i+1}) \text{ for all } q \le i \le k_{t'} - 1, \tag{3.7}$$

$$w_{l_{t'}} \in K(v_p, v_{p+1}) \text{ unless } p = k_{t'} \tag{3.8}$$

and

$$w_{l_{t'}+1} \in K(v_{q-1}, v_q) \text{ unless } q = k_{t'-1}. \tag{3.9}$$

By Lemma 3.10(I)(i), it follows from (3.5) and (3.8) that

(3.10) $$w_{l_{t'}} \in K(v_i, v_{i+1}) \text{ for all } p \le i \le k_{t'} - 1.$$

Also by Lemma 3.10(I)(i), it follows from (3.5) and (3.9) that

(3.11) $$w_{l_{t'}+1} \in K(v_i, v_{i+1}) \text{ for all } k_{t'-1} \le i \le q - 1.$$

If $p > q$, then $w_{l_{t'}+1}, w_{l_{t'}} \notin K(v_{p-1}, v_p)$ by (3.6) and (3.7), which contradicts (3.5). Thus we get

(3.12) $$p \le q.$$

Conbining (3.4), (3.6), (3.7), (3.10), (3.11) and (3.12), we get the disired conclusion. ∎

Note that the assertion (II) of the lemma is an immediate consequence of Claim 3.4.

Claim 3.5.

(i) $E(\langle A \rangle) - E(C) = X_1$.

(ii) $X_2 \subseteq E(\langle B \rangle) - E(C) \subseteq X_2 \cup X_3$.

Proof. To prove (ii), we prove the following two subclaims.

Subclaim 3.1. $X_2 \subseteq E(\langle B \rangle) - E(C)$.

Proof. By Claim 3.4(I) and Lemma 3.3, we have

(3.13) $$\bigcup_{t=1}^{r} \{w_j w_{j-2} \mid l_t \ge j \ge l_{t+1} + 2\} \subseteq E(\langle B \rangle) - E(C) \text{ if (11a) holds,}$$

(3.14) $$\bigcup_{t=1}^{r} \{w_j w_{j-2} \mid l_t \ge j \ge l_{t+1} + 2\} - \{w_{h-2}w_h, w_{h-1}w_{h+1}\} \subseteq E(\langle B \rangle) - E(C) \quad \text{if (11b-1) holds,}$$

and

(3.15) $$\bigcup_{t=1}^{r} \{w_j w_{j-2} \mid l_t \ge j \ge l_{t+1} + 2\} - \{w_{l_{t'}} w_{l_{t'}+2}\} \subseteq E(\langle B \rangle) - E(C) \quad \text{if (11b-2) holds.}$$

Thus if either (11a) holds, or (11b-2) holds and $p \neq k_{t'}$ and $q \neq k_{t'-1}$, we immediately get $X_2 \subseteq E(\langle B\rangle) - E(C)$. Now assume first that (11b-1) holds. Then by (3.2) and (11b-1), we can take t with $1 \leq t \leq r$ such that

$$l_t > h \text{ and } h-1 > l_{t+1}. \tag{3.16}$$

Since $w_{h-1}w_h$ is contractible, $\{w_{h-1}, w_h, v_{k_t}\}$ is not a cutset, and hence

$$\begin{aligned} E(Q &\cup \{v_i \mid 0 \leq i \leq k_t - 1\} \cup \{w_j \mid h+1 \leq j \leq s\}, \\ P &\cup \{v_i \mid k_t + 1 \leq i \leq m\} \cup \{w_j \mid 0 \leq j \leq h-2\}) \neq \emptyset. \end{aligned} \tag{3.17}$$

On the other hand, $v_{k_t} \in K(w_{h-2}, w_{h-1})$ and $v_{k_t} \in K(w_h, w_{h+1})$ by (3.16) and Claim 3.4(I), and hence applying Lemma 3.1(i) to $\{w_{h-2}, w_{h-1}, v_{k_t}\}$ and $\{w_h, w_{h+1}, v_{k_t}\}$, we get

$$\begin{aligned} E(Q &\cup \{v_i \mid 0 \leq i \leq k_t - 1\} \cup \{w_j \mid h+1 \leq j \leq s\}, \\ P &\cup \{v_i \mid k_t + 1 \leq i \leq m\} \cup \{w_j \mid 0 \leq j \leq h-3\}) = \emptyset, \end{aligned} \tag{3.18}$$

and

$$\begin{aligned} E(Q &\cup \{v_i \mid 0 \leq i \leq k_t - 1\} \cup \{w_j \mid h+2 \leq j \leq s\}, \\ P &\cup \{v_i \mid k_t + 1 \leq i \leq m\} \cup \{w_j \mid 0 \leq j \leq h-2\}) = \emptyset. \end{aligned} \tag{3.19}$$

By (3.17), (3.18) and (3.19), we obtain $w_{h-2}w_{h+1} \in E(G)$, and this together with (3.14) inplies that $X_2 \subseteq E(\langle B\rangle) - E(C)$.

Next asuume that (11b-2) holds and $p = k_{t'}$ (so $q = k_{t'}$). Then since $w_{l_{t'}} \notin K(v_{k_{t'}-1}, v_{k_{t'}})$ $(= K(v_{p-1}, v_p))$ by Claim 3.4(II)(ii)(b), $\{v_{k_{t'}-1}, v_{k_{t'}}, w_{l_{t'}}\}$ is not a cutset, and hence we get

$$\begin{aligned} E(Q &\cup \{v_i \mid 0 \leq i \leq k_{t'} - 2\} \cup \{w_j \mid l_{t'}+1 \leq j \leq s\}, \\ P &\cup \{v_i \mid k_{t'} + 1 \leq i \leq m\} \cup \{w_j \mid 0 \leq j \leq l_{t'}-1\}) \neq \emptyset. \end{aligned} \tag{3.20}$$

On the other hand, $w_{l_{t'}+1} \in K(v_{k_{t'}-1}, v_{k_{t'}})$ $(= K(v_{p-1}, v_p))$ by Claim 3.4(II)(ii)(b) and $v_{k_{t'}} \in K(w_{l_{t'}-1}, w_{l_{t'}})$ by Claim 3.4(I), and hence applying Lemma 3.1(i) to $\{v_{k_{t'}-1}, v_{k_{t'}}, w_{l_{t'}+1}\}$ and $\{w_{l_{t'}-1}, w_{l_{t'}}, v_{k_{t'}}\}$ we get

$$\begin{aligned} E(Q &\cup \{v_i \mid 0 \leq i \leq k_{t'} - 2\} \cup \{w_j \mid l_{t'}+2 \leq j \leq s\}, \\ P &\cup \{v_i \mid k_{t'} + 1 \leq i \leq m\} \cup \{w_j \mid 0 \leq j \leq l_{t'}-1\}) = \emptyset, \end{aligned} \tag{3.21}$$

and

$$\begin{aligned} E(Q &\cup \{v_i \mid 0 \leq i \leq k_{t'} - 2\} \cup \{w_j \mid l_{t'}+1 \leq j \leq s\}, \\ P &\cup \{v_i \mid k_{t'} + 1 \leq i \leq m\} \cup \{w_j \mid 0 \leq j \leq l_{t'}-2\}) = \emptyset. \end{aligned} \tag{3.22}$$

By (3.20), (3.21) and (3.22), we obtain $w_{l_{t'}-1}w_{l_{t'}+1} \in E(G)$, and this together with (3.15) implies that $X_2 \subseteq E(\langle B\rangle) - E(C)$.

Finally if (11b-2) holds and $q = k_{t'-1}$(so $p = k_{t'-1}$), then by virtue of the symmetry of the roles of p and q, we get $w_{l_{t'}}w_{l_{t'}+2} \in E(G)$ by arguing exactly as in the preceding paragraph, and hence $X_2 \subseteq E(\langle B\rangle) - E(C)$. ∎

Subclaim 3.2. $E(\langle B\rangle) - E(C) \subseteq X_2 \cup X_3$.

Proof. Take $w_{j_1}w_{j_2} \in E(\langle B\rangle) - E(C)$ with $j_2 \geq j_1 + 2$.

Asuume first that $j_2 = j_1 + 2$, and choose t with $2 \leq t \leq r + 1$ so that $l_{t-1} \geq j_2 \geq l_t + 1$. Assume for the moment that $j_2 = l_t + 1$. Then $w_{l_t} \notin K(v_{k_t-1}, v_{k_t})$ by Lemma 3.1(i). In view of Claim 3.4(II), this means that (11b-2) holds and $t = t'$ and $p = k_{t'}$ (note that we have $2 \leq t \leq r$ because $l_t = j_2 - 1 > j_1 \geq 0$). Hence we get $w_{j_1}w_{j_2} \in X_2$ by the definition of X_2. Thus we may assume

(3.23) $$l_{t-1} \geq j_2 \geq l_t + 2.$$

Moreover, since $w_{j_2-1} \notin K(v_{k_{t-1}}, v_{k_{t-1}+1})$ by Lemma 3.1(i), it follows from Claim 3.4(II)(ii)(b) that

(3.24) if (11b-2) holds and $t = t'$ and $q \neq k_{t'-1}$, then $j_2 \neq l_{t'} + 2$.

By (3.23) and (3.24), we get $w_{j_1}w_{j_2} \in X_2 \cup X_3$, as desired.

Next assume that $j_2 > j_1 + 2$. Then in view of Lemma 3.9(i), (11a) cannot hold. Thus (11b) holds, and $j_1 = h - 2$ and $j_2 = h + 1$ by Lemma 3.9(ii). Now if (11b-2) holds, then $w_{l_{t'}+1}(= w_h) \in K(v_{k_{t'}-1}, v_{k_{t'}})$ or $w_{l_{t'}}(= w_{h-1}) \in K(v_{k_{t'}-1}, v_{k_{t'}})$ by Claim 3.4(II)(ii)(b), and we therefore get a contradiction by applying Lemma 3.1(i) to $\{v_{k_{t'}-1}, v_{k_{t'}}, w_{l_{t'}+1}\}$ or $\{v_{k_{t'}-1}, v_{k_{t'}}, w_{l_{t'}}\}$. Thus (11b-1) holds, and hence we get $w_{j_1}w_{j_2} \in X_2 \cup X_3$ in this case as well. Consequently $E(\langle B\rangle) - E(C) \subseteq X_2 \cup X_3$. ∎

Now (ii) of Claim 3.5 follows from Subclaims 3.1 and 3.2, and (i) can be verified in a similar way. This completes the proof of Claim 3.5. ∎

Claim 3.6.

$$\begin{aligned} & Y_1 \cup \bar{Y}_1 \cup Y_2 \cup \bar{Y}_2 \cup Z_1 \cup \bar{Z}_2 \\ \subseteq\ & E(A, B) \\ \subseteq\ & Y_1 \cup \bar{Y}_1 \cup Y_2 \cup \bar{Y}_2 \cup Z_1 \cup \bar{Z}_2 \cup Y_3' \cup Y_4' \cup Z_1' \cup \bar{Z}_1' \cup Z_2' \cup \bar{Z}_2'. \end{aligned}$$

Proof. We first prove the following subclaim.

Subclaim 3.3. *If* (11b-2) *holds, then* $v_{k_{t'-1}+1}w_{l_{t'}+2} \in E(G)$ *and* $v_{k_{t'}-1}w_{l_{t'}-1} \in E(G)$.

Proof. Since $w_{l_{t'}}w_{l_{t'}+1}$ is contractible, $\{w_{l_{t'}}, w_{l_{t'}+1}, v_{k_{t'-1}}\}$ is not a cutset, and hence

$$\begin{aligned}(3.25)\quad E(Q &\cup \{v_i \mid 0 \le i \le k_{t'-1} - 1\} \cup \{w_j \mid l_{t'} + 2 \le j \le s\},\\ P &\cup \{v_i \mid k_{t'-1} + 1 \le i \le m\} \cup \{w_j \mid 0 \le j \le l_{t'} - 1\}) \ne \emptyset.\end{aligned}$$

On the other hand, $v_{k_{t'}-1} \in K(w_{l_{t'}+1}, w_{l_{t'}+2})$ by Claim 3.4(I), and hence applying Lemma 3.2(i) to $\{w_{l_{t'}+1}, w_{l_{t'}+2}, v_{k_{t'-1}}\}$, we get

$$\begin{aligned}(3.26)\quad E(Q &\cup \{v_i \mid 0 \le i \le k_{t'-1} - 1\} \cup \{w_j \mid l_{t'} + 3 \le j \le s\},\\ P &\cup \{v_i \mid k_{t'-1} + 1 \le i \le m\} \cup \{w_j \mid 0 \le j \le l_{t'} - 1\}) = \emptyset.\end{aligned}$$

Also either $w_{l_{t'}} \in K(v_{k_{t'-1}}, v_{k_{t'-1}+1})$ or $w_{l_{t'}+1} \in K(v_{k_{t'-1}}, v_{k_{t'-1}+1})$ by Claim 3.4(II)(ii)(b), and hence applying Lemma 3.2(i) to $\{v_{k_{t'-1}}, v_{k_{t'-1}+1}, w_{l_{t'}}\}$ or $\{v_{k_{t'-1}}, v_{k_{t'-1}+1}, w_{l_{t'}+1}\}$, we get

$$\begin{aligned}(3.27)\quad E(Q &\cup \{v_i \mid 0 \le i \le k_{t'-1} - 1\} \cup \{w_j \mid l_{t'} + 2 \le j \le s\},\\ P &\cup \{v_i \mid k_{t'-1} + 2 \le i \le m\} \cup \{w_j \mid 0 \le j \le l_{t'} - 1\}) = \emptyset.\end{aligned}$$

By (3.25), (3.26) and (3.27), we obtain $v_{k_{t'-1}+1}w_{l_{t'}+2} \in E(G)$. By the symmetry of the roles of $k_{t'-1}$, $l_{t'} + 1$ and $k_{t'}$, $l_{t'}$ we similarly obtain $v_{k_{t'}-1}w_{l_{t'}-1} \in E(G)$. ∎

In view of Subclaim 3.3, we can prove the following two subclaims by arguing exactly as in the first paragraph of the proof of Claim 5.8 of [4]:

Subclaim 3.4. $Y_1 \cup \bar{Y_1} \subseteq E(A, B)$.

Subclaim 3.5. $Y_2 \cup \bar{Y_2} \subseteq E(A, B)$.

Subclaim 3.6. $Z_1 \cup Z_2 \subseteq E(A, B)$.

Proof. We first prove $Z_1 \subseteq E(A, B)$. We may assume that (11b-2) holds and $k_{t'-1} < p < k_{t'}$. Then by Claim 3.4(II)(ii)(b),

$$(3.28)\qquad w_{l_{t'}} \notin K(v_{p-1}, v_p),$$

$$(3.29)\qquad w_{l_{t'}+1} \in K(v_{p-1}, v_p),$$

and

$$(3.30)\qquad w_{l_{t'}} \in K(v_p, v_{p+1}).$$

Since $\{v_{p-1}, v_p, w_{l_{t'}}\}$ is not a cutset by (3.28), we have

$$\begin{aligned}(3.31)\quad E(Q &\cup \{v_i \mid 0 \le i \le p - 2\} \cup \{w_j \mid l_{t'} + 1 \le j \le s\},\\ P &\cup \{v_i \mid p + 1 \le i \le m\} \cup \{w_j \mid 0 \le j \le l_{t'} - 1\} \ne \emptyset.\end{aligned}$$

By (3.29), we can apply Lemma 3.1(i) to $\{v_{p-1}, v_p, w_{l_{t'}+1}\}$ to get

$$(3.32) \quad \begin{aligned} E(Q &\cup \{v_i \mid 0 \le i \le p-2\} \cup \{w_j \mid l_{t'}+2 \le j \le s\}, \\ P &\cup \{v_i \mid p+1 \le i \le m\} \cup \{w_j \mid 0 \le j \le l_{t'}-1\}) = \emptyset. \end{aligned}$$

By (3.30), we can apply Lemma 3.1(i) to $\{v_p, v_{p+1}, w_{l_{t'}}\}$ to get

$$(3.33) \quad \begin{aligned} E(Q &\cup \{v_i \mid 0 \le i \le p-2\} \cup \{w_j \mid l_{t'}+1 \le j \le s\}, \\ P &\cup \{v_i \mid p+2 \le i \le m\} \cup \{w_j \mid 0 \le j \le l_{t'}-1\}) = \emptyset. \end{aligned}$$

By (3.31), (3.32) and (3.33), we obtain $v_{p+1}w_{l_{t'}+1} \in E(G)$. Thus $Z_1 \subseteq E(A,B)$. By the symmetry of the roles of p and q, we similarly obtain $Z_2 \subseteq E(A,B)$. ∎

Subclaim 3.7. *Take $v_iw_j \in E(A,B)$.*

(I) *Let $1 \le t \le r$, and suppose that $i < k_t$ and $j < l_t$. Then $v_iw_j \in Y_2 \cup \bar{Y}_2$.*

(II) *Let $1 \le t \le r$. Suppose that $i > k_t$ and $j > l_{t+1}$, and in the case where* (11b-2) *holds and $t = t'-1$, suppose further that $j > l_{t'}+1$. Then $v_iw_j \in Y_1 \cup \bar{Y}_1$.*

(III) *Suppose that* (11b-2) *holds. Then the following hold.*

(i) *Suppose that $k_{t'-1} \le p < k_{t'}$, and let $j = l_{t'}+1$. Then $i \le p+1$.*

(ii) *Suppose that $k_{t'-1} < q \le k_{t'}$, and let $j = l_{t'}$. Then $i \ge q-1$.*

Proof. We can prove (I) and (II) by arguing as is Claim 5.7 of [4]. To prove (III)(i), suppose that (11b-2) holds and $k_{t'-1} \le p < k_{t'}$ and $j = l_{t'}+1$. If $i > p+1$, then by Lemma 3.1(i), $\{v_p, v_{p+1}, w_{l_{t'}}\}$ is not a cutset, which contradicts Claim3.4 (II)(ii)(b). Thus $i \le p+1$, which proves (III)(i). By the symmetry of the roles of p and q, we can similarly prove (III)(ii). ∎

Subclaim 3.8.

$$E(A,B) \subseteq Y_1 \cup \bar{Y}_1 \cup Y_2 \cup \bar{Y}_2 \cup Z_1 \cup Z_2 \cup Y_3' \cup Y_4' \cup Z_1' \cup \bar{Z}_1' \cup Z_2' \cup \bar{Z}_2'.$$

Proof. Take $v_iw_j \in E(A,B)$ and suppose that $v_iw_j \notin Y_3' \cup Y_4' \cup Z_1' \cup Z_2'$.

For the moment assume that $j \notin \{l_t \mid 1 \le t \le r+1\}$, and in the case where (11b-2) holds, assume further that $j \ne l_{t'}+1$. Take t with $l_t > j > l_{t+1}$. Then since $v_iw_j \notin Y_3'$, we have $i \ne k_t$, and hence it follows from (II) or (I) of Subclaim 3.7 that $v_iw_j \in Y_1 \cup \bar{Y}_1$ or $Y_2 \cup \bar{Y}_2$, according as $i > k_t$ or $i < k_t$.

Throughout the rest of the proof, we assume that $j \in \{l_t \mid 1 \le t \le r+1\}$ in the case where (11a) or (11b-1) holds, and assume that $j \in \{l_t \mid 1 \le t \le r+1\} \cup \{l_{t'}+1\}$ in the case where (11b-2) holds. Let t denote the index such that $j = l_t$ (in the case where (11b-2) holds and $j = l_{t'} + 1$, we let $t = t'$). If $i > k_t$, then we get $v_i w_j \in Y_1 \cup \bar{Y}_1$ by Subclaim 3.7(II); if $i < k_{t-1}$, then applying Subclaim 3.7(I) with t replaced by $t-1$, we get $v_i w_j \in Y_2 \cup \bar{Y}_2$. Thus we may assume

$$k_{t-1} \leq i \leq k_t. \tag{3.34}$$

By the assumption that $v_i w_j \notin Y_4'$, this implies that (11b-2) holds and $t = t'$.

We first consider the case where $j = l_{t'} + 1$. From (3.34) and the assumption that $v_i w_j \notin Z_1'$, we get $p + 1 \leq i \leq k_{t'}$. This in particular implies $p < k_{t'}$, and hence $i \leq p + 1$ by Subclaim 3.7(III)(i). Consequently, $i = p + 1$, and hence $v_i w_j \in Z_1$ or $\bar{Z}_1'$ accrding as $k_{t'-1} < p < k_{t'}$ or $p = k_{t'-1}$. This completes the discussion for the case $j = l_{t'} + 1$.

If $j = l_{t'}$, then by the symmetry of the roles of p and q, we can argue as in the case $j = l_{t'} + 1$, using Subclaim 3.7(III)(ii) in place of Subclaim 3.7(III)(i), to see that $v_i w_j \in Z_2 \cup \bar{Z}_2'$. ∎

By Subclaims 3.4, 3.5, 3.6 and 3.8, we get the desired conclusion, and this completes the proof of Claim 3.6. ∎

Now the assertion (III) of the lemma follows from Claims 3.5 and 3.6, and this completes the proof of Lemma 3.11. ∎

§4. Initial Reduction

Throughout the rest of this paper, we let G and C be as in Theorem 1, and write $C = a_0 a_1 \cdots a_{n_0} b_0 b_1 \cdots b_{n_1} c_0 c_1 \cdots c_{n_2} d_0 d_1 \cdots d_{n_3} e_0 e_1 \cdots e_{n_4} a_0$, where $a_{n_0} b_0$, $b_{n_1} c_0$, $c_{n_2} d_0$, $d_{n_3} e_0$ and $e_{n_4} a_0$ are the five contractible edges contained in C. Note that C is a hamiltonian cycle by the result of Ellingham, Hemminger and Johnson [3] mentioned in Section 1; thus $|V(G)| = n_0 + n_1 + n_2 + n_3 + n_4 + 5$. Let $C_0 = \{a_0, a_1, \ldots, a_{n_0}\}$, $C_1 = \{b_0, b_1, \ldots, b_{n_1}\}$, $C_2 = \{c_0, c_1, \ldots, c_{n_2}\}$, $C_3 = \{d_0, d_1, \ldots, d_{n_3}\}$ and $C_4 = \{e_0, e_1, \ldots, e_{n_4}\}$.

In this section, we derive some basic properties of (G, C). Lemmas 4.1 and 4.5 are proved in Section 4 of [4], and we can prove Lemmas 4.2 through 4.4, 4.6 and 4.7 by arguing exactly as in the corresponding lemmas in Section 4 of [4].

Lemma 4.1. *Suppose that $n_1 = 2$. Then one of the following holds:*

(i) $K(b_0, b_1) = \{c_0\}$ *and* $K(b_1, b_2) = \{a_{n_0}\}$; *or*

(ii) $K(b_0, b_1) \neq \{c_0\}$ *and* $K(b_1, b_2) \neq \{a_{n_0}\}$.

Lemma 4.2. *Suppose that $n_1 \geq 1$.*

(i) *If* $n_1 \neq 2$, *then* $K(b_0, b_1) \subseteq C_3 \cup C_4 \cup \{c_{n_2}, a_0\}$.

(ii) *If* $n_1 = 2$, *then* $K(b_0, b_1) \subseteq C_3 \cup C_4 \cup \{c_0, c_{n_2}, a_0\}$.

Lemma 4.3. *One of the following holds:*

(i) $n_1 = 0$;

(ii) $n_1 = 2$ *and* $K(b_0, b_1) = \{c_0\}$ *and* $K(b_1, b_2) = \{a_{n_0}\}$; *or*

(iii) $n_1 \geq 1$ *and* $K(b_i, b_{i+1}) \cap (C_3 \cup C_4) \neq \emptyset$ *for all* $0 \leq i \leq n_1 - 1$.

With Lemma 4.3 in mind, we difine the terms *degenerate* and *nondegenerate* as follows: for each $0 \leq l \leq 4$, C_l is said to be nondegenerate if $n_l \geq 1$ and $K(u, v) \cap (C_{l+2} \cap C_{l+3}) \neq \emptyset$ for all $uv \in E(\langle C_l \rangle_C)$ (indices of the letter C are to be read modulo 5); otherwise C_l is said to be degenerate. Thus, for example, C_1 is nondegenerate if and only if (iii) of Lemma 4.3 holds, and it is degenerate if and only if (i) or (ii) of Lemma 4.3 holds.

Lemma 4.4. *At most three of the* C_l $(0 \leq l \leq 4)$ *are nondegenerate.*

Lemma 4.5. *Suppose that* C_0 *is degenerate and* $n_0 = 2$. *Then the following hold.*

(i) $E(a_0, V(G)) - E(C) = \{a_0 a_2\}$, *and* $E(a_2, V(G)) - E(C) = \{a_0 a_2\}$.

(ii) $E(\{a_0, a_2\}, V(G)) - E(C) = \{a_0 a_2\}$.

Lemma 4.6. *Suppose that* C_0 *is degenerate, and that* C_4 *is nondegenerate and* $b_0 \in K(e_{n_4 - 1}, e_{n_4})$.

(I) *If* $n_0 = 0$, *then* $E(C_0, V(G)) - E(C) = \{a_0 e_{n_4 - 1}\}$.

(II) *Suppose that* $n_0 = 2$. *Then the following hold.*

(i) $\{a_0 a_2, a_1 e_{n_4 - 1}\} \subseteq E(C_0, V(G)) - E(C) \subseteq \{a_0 a_2, a_1 b_0, a_1 e_{n_4 - 1}, a_1 e_{n_4}\}$.

(ii) *Suppose further that* C_1 *is degenerate, and that either* $n_1 = 2$, *or* $n_1 = 0$ *and* $n_2 \geq 1$ *and* $a_2 \in K(c_0, c_1)$. *Then* $\{a_0 a_2, a_1 e_{n_4 - 1}\} \subseteq E(C_0, V(G)) - E(C) \subseteq \{a_0 a_2, a_1 e_{n_4 - 1}, a_1 e_{n_4}\}$.

Lemma 4.7. *Suppose that* C_4 *is nondegenerate. Then* $e_i e_j \notin E(G)$ *for any* i, j *with* $i + 3 \leq j$.

Lemma 4.8. *Suppose that* C_3 *and* C_4 *are nondegenerate. Then the following hold.*

(i) $K(d_j, d_{j+1}) \cap C_1 \neq \emptyset$ *for all* $0 \leq j \leq n_3 - 1$.

(ii) $E(C_3, C_4) - E(C) \subseteq \{d_{n_3-1}e_0, d_{n_3-1}e_1, d_{n_3}e_1\}$.

Proof.

(i) Let $0 \leq j \leq n_3 - 1$ and $0 \leq x \leq n_4 - 1$. Then by assumption, we can take $y \in K(d_j, d_{j+1}) \cap (C_0 \cup C_1)$ and $z \in K(e_x, e_{x+1}) \cap (C_1 \cup C_2)$. If $y \in C_1$, there is nothing to be proved. Thus we may assume $y \in C_0$. Then by Lemma 3.5, ($y = a_{n_0}$ and) $z = b_0 \in K(d_j, d_{j+1})$. Thus $K(d_j, d_{j+1}) \cap C_1 \neq \emptyset$.

(ii) Applying (i) to C_4 as well as to C_3, we see that $K(d_j, d_{j+1}) \cap C_1 \neq \emptyset$ for all $0 \leq j \leq n_3 - 1$ and $K(e_x, e_{x+1}) \cap C_1 \neq \emptyset$ for all $0 \leq x \leq n_4 - 1$, and hence the desired conclusion follows from Lemma 3.9. ∎

Lemma 4.9. *Suppose that* C_3 *is nondegenerate, and* C_1 *is degenerate and* $n_1 = 2$. *Then* $b_1 \notin K(d_j, d_{j+1})$ *for all* $0 \leq j \leq n_3 - 1$.

Proof. If $b_1 \in K(d_j, d_{j+1})$, then since $b_0b_2 \in E(G)$ by Lemma 4.5, we get a contradiction by Lemma 3.1(i). ∎

§5. Proof of Theorem 1

We continue with the notation of the preceding section, and complete the proof of Theorem 1.

Since $|V(G)| \geq 16$ by assumption, at least one of the C_l ($0 \leq l \leq 4$) has four or more vertices, and hence is nondegenerate. Thus by Lemma 4.4 and by symmetry, it suffices to consider the following five cases:

- C_0, $C_1$and C_4 are nondegenerate,and C_2 and C_3 aredegenerate;
- C_1, C_3 and C_4 are nondegenerate, and C_0 and C_2 are degenerate;
- C_3 and C_4 are nondegenerate, and C_0, C_1 and C_2 are degenerate;
- C_1 and C_4 are nondegenerate, and C_0, C_2 and C_3 are degenerate; or
- C_4 is nondegenerate and $n_4 \geq 3$, and C_0, C_1, C_2 and C_3 are degenerate.

We consider these five cases separately in five propositions, Propositions 1 through 5. Note that in the proof of Propositions 1 through 5, we do not make use of the assumption that $|V(G)| \geq 16$, and thus these five propositions hold for graphs of order less than 16 as well.

Proposition 1. *Suppose that C_0, C_1 and C_4 are nondegenerate, and C_2 and C_3 are degenerate. Then (G,C) is of Type 1.*

Proof. Let $0 \le i \le n_1 - 1$ and $0 \le x \le n_4 - 1$. Applying Lemma 4.8(i) to C_1 and C_0, we get $K(b_i, b_{i+1}) \cap C_3 \neq \emptyset$. Applying Lemma 4.8(i) to C_4 and C_0, we get $K(e_x, e_{x+1}) \cap C_2 \neq \emptyset$. In view of Lemma 3.5, these imply that $K(b_i, b_{i+1}) \cap C_3 = \{d_0\}$ and $K(e_x, e_{x+1}) \cap C_2 = \{c_{n_2}\}$. Thus

$$(5.1) \qquad d_0 \in K(b_i, b_{i+1}) \text{ for all } 0 \le i \le n_1 - 1,$$

$$(5.2) \qquad c_{n_2} \in K(e_x, e_{x+1}) \text{ for all } 0 \le x \le n_4 - 1.$$

Let now $0 \le h \le n_0 - 1$. Applying Lemma 4.8(i) to C_0 and C_1, we get $K(a_h, a_{h+1}) \cap C_3 \neq \emptyset$. By Lemma 3.5, this together with (5.2) implies $K(a_h, a_{h+1}) \cap C_3 = \{d_0\}$. Similarly $K(a_h, a_{h+1}) \cap C_2 = \{c_{n_2}\}$. Thus

$$(5.3) \qquad \{d_0, c_{n_2}\} \subseteq K(a_h, a_{h+1}) \text{ for all } 0 \le h \le n_0 - 1.$$

In view of (5.1) through (5.3), we can now argue as in Proposition 2 of [4], to see that G is of Type 1.

Proposition 2. *Suppose that C_1, C_3 and C_4 are nondegenerate, and C_0 and C_2 are degenerate. Then (G,C) is of Type 2.*

Proof. By Lemma 4.8(i), we get $K(d_j, d_{j-1}) \cap C_1 \neq \emptyset$ for all $1 \le j \le n_3$. By the symmetry of the roles of C_3 and C_4, we also have $K(e_x, e_{x-1}) \cap C_1 \neq \emptyset$ for all $1 \le x \le n_4$. Further by the assumption that C_1 is nondegenerate, $K(b_i, b_{i+1}) \cap (C_3 \cup C_4) \neq \emptyset$ for all $0 \le i \le n_1 - 1$. Consequently we can apply Case (11b) of Lemma 3.11 with $A = C_1$, $B = C_3 \cup C_4$, $m = n_1$, $s = n_3 + n_4 + 1$ and $h = n_3 + 1$, and argue as in Claims 5.9 and 5.10 in the proof of Proposition 1 of [4], to see that (G,C) is of Type 2.

Proposition 3. *Suppose that C_3 and C_4 are nondegenerate, and C_0, C_1 and C_2 are degenerate. Then (G,C) is of Type 3, 4, 5, 6 or 7.*

Proof. Let $0 \le j \le n_3 - 1$ and $0 \le x \le n_4 - 1$. By Lemma 4.8(i), $K(d_j, d_{j+1}) \cap C_1 \neq \emptyset$. Moreover if $n_1 = 2$, then $b_1 \notin K(d_j, d_{j+1})$ by Lemma 4.9. Thus, we get

$$(5.4) \qquad \text{if } n_1 = 0, \text{then } b_0 \in K(d_j, d_{j+1}) \text{ for all } 0 \le j \le n_3 - 1;$$

and

$$(5.5) \qquad \text{if } n_1 = 2, \text{then } K(d_j, d_{j+1}) \cap \{b_0, b_2\} \neq \emptyset \text{ for all } 0 \le j \le n_3 - 1.$$

By symmetry, we also see that

$$(5.6) \qquad \text{if } n_1 = 0, \text{then } b_0 \in K(e_x, e_{x+1}) \text{ for all } 0 \le x \le n_4 - 1;$$

(5.7) if $n_1 = 2$, then $K(e_x, e_{x+1}) \cap \{b_0, b_2\} \neq \emptyset$ for all $0 \leq x \leq n_4 - 1$.

By Lemma 4.8(ii),

$$E(C_3, C_4) - E(C) \subseteq \{d_{n_3-1}e_0, d_{n_3-1}e_1, d_{n_3}e_1\}. \tag{5.8}$$

If $n_1 = 0$, then combining the proof of Proposition 3 of [4] for the case $n_1 = 0$, and the argument used in the proof of (5.6) and Claim 5.11 in Proposition 2 of [4], we see that (G, C) is of Type 3. Thus we henceforth assume that $n_1 = 2$. Applying Lemma 4.5 to C_1, we get

$$E(\{b_0, b_2\}, V(G)) - E(C) = \{b_0b_2\}. \tag{5.9}$$

Claim 5.1. *One of the following holds:*

(i) $K(e_x, e_{x+1}) \cap \{b_0, b_2\} = \{b_2\}$ *for all* $0 \leq x \leq n_4 - 1$ *and* $K(d_j, d_{j+1}) \cap \{b_0, b_2\} = \{b_2\}$ *for all* $0 \leq j \leq n_3 - 1$;

(ii) *there exists* p *with* $1 \leq p \leq n_4 - 1$ *such that* $b_0 \in K(e_x, e_{x+1})$ *for all* $p \leq x \leq n_4 - 1$ *and* $b_2 \in K(e_x, e_{x+1})$ *for all* $0 \leq x \leq p - 1$, *and* $K(d_j, d_{j+1}) \cap \{b_0, b_2\} = \{b_2\}$ *for all* $0 \leq j \leq n_3 - 1$;

(iii) $n_4 = 1$ *and* $b_0, b_2 \in K(e_0, e_1)$, *and* $K(d_j, d_{j+1}) \cap \{b_0, b_2\} = \{b_2\}$ *for all* $0 \leq j \leq n_3 - 1$;

(iv) $K(e_x, e_{x+1}) \cap \{b_0, b_2\} = \{b_0\}$ *for all* $0 \leq x \leq n_4 - 1$ *and* $K(d_j, d_{j+1}) \cap \{b_0, b_2\} = \{b_2\}$ *for all* $0 \leq j \leq n_3 - 1$;

(v) $K(e_x, e_{x+1}) \cap \{b_0, b_2\} = \{b_0\}$ *for all* $0 \leq x \leq n_4 - 1$, *and there exists* p *with* $1 \leq p \leq n_3 - 1$ *such that* $b_0 \in K(d_j, d_{j+1})$ *for all* $p \leq j \leq n_3 - 1$ *and* $b_2 \in K(d_j, d_{j+1})$ *for all* $0 \leq j \leq p - 1$;

(vi) $K(e_x, e_{x+1}) \cap \{b_0, b_2\} = \{b_0\}$ *for all* $0 \leq x \leq n_4 - 1$, *and* $n_3 = 1$ *and* $b_0, b_2 \in K(d_0, d_1)$; *or*

(vii) $K(e_x, e_{x+1}) \cap \{b_0, b_2\} = \{b_0\}$ *for all* $0 \leq x \leq n_4 - 1$, *and* $K(d_j, d_{j+1}) \cap \{b_0, b_2\} = \{b_0\}$ *for all* $0 \leq j \leq n_3 - 1$.

Proof. Assume first that

$$b_2 \in K(e_0, e_1). \tag{5.10}$$

Then by Lemma 3.5, it follows from (5.5) that

$$K(d_j, d_{j+1}) \cap \{b_0, b_2\} = \{b_2\} \text{ for all } 0 \leq j \leq n_3 - 1. \tag{5.11}$$

If $K(e_{n_4-1}, e_{n_4}) \cap \{b_0, b_2\} = \{b_2\}$, then by Lemma 3.10(I)(ii) and (5.7), $K(e_x, e_{x+1}) \cap \{b_0, b_2\} = \{b_2\}$ for all $0 \leq x \leq n_4 - 1$, and hence it follows from (5.11) that (i) holds. Thus by (5.7), we may assume

$$b_0 \in K(e_{n_4-1}, e_{n_4}). \tag{5.12}$$

Now if $n_4 = 1$, then it follows from (5.12), (5.10) and (5.11) that (iii) holds; and if $n_4 > 1$, then in veiw of (5.12), (5.10) and (5.11), arguing as in Claim 5.16 of [4], we see that (ii) holds. Thus we may assume $b_2 \notin K(e_0, e_1)$, and hence by Lemma 3.10(I)(ii) and (5.7),

$$K(e_x, e_{x+1}) \cap \{b_0, b_2\} = \{b_0\} \text{ for all } 0 \leq x \leq n_4 - 1. \tag{5.13}$$

By symmetry, we may also assume $b_0 \notin K(d_{n_3-1}, d_{n_3})$, and hence

$$K(d_j, d_{j+1}) \cap \{b_0, b_2\} = \{b_2\} \text{ for all } 0 \leq j \leq n_3 - 1,$$

and we now see that (iv) holds. ∎

If (i), (ii) or (iii) of Claim 5.1 holds, then combining the proof of Proposition 3 of [4] (in the case where Claim 5.1(iii) holds, we apply the argument in the proof of Claim 5.17 of [4] with $Y = \{b_1e_0, b_1e_1\}$), and the proof of (5.6) and Claim5.11 of [4], we see that (G, C) is of Type 4, 5 or 6. Thus by symmetry, we may assume Claim 5.1(iv) holds. Applying Lemmas 3.3 and 4.7 to C_3 and C_4, we have

$$E(\langle C_3 \rangle) - E(C) = \{d_jd_{j+2} \mid 0 \leq j \leq n_3 - 2\} \tag{5.14}$$

and

$$E(\langle C_4 \rangle) - E(C) = \{e_xe_{x+2} \mid 0 \leq x \leq n_4 - 2\}. \tag{5.15}$$

Applying (I) and (II)(ii) of Lemma 4.6 to C_0 and C_2, we get the following two claims:

Claim 5.2.

(i) *If* $n_0 = 0$, *then* $E(C_0, V(G)) - E(C) = \{a_0e_{n_4-1}\}$.

(ii) *If* $n_0 = 2$, *then* $\{a_0a_2, a_1e_{n_4-1}\} \subseteq E(C_0, V(G)) - E(C) \subseteq \{a_0a_2, a_1e_{n_4-1}a_1e_{n_4}\}$.

Claim 5.3.

(i) *If* $n_2 = 0$, *then* $E(C_2, V(G)) - E(C) = \{c_0d_1\}$.

(ii) *If* $n_2 = 2$, *then* $\{c_0c_2, c_1d_1\} \subseteq E(C_2, V(G)) - E(C) \subseteq \{c_0c_2, c_1d_1, c_1d_0\}$.

Further applying Lemma 3.1(i) to $\{d_{n_3-1}, d_{n_3}, b_2\}$ and $\{e_0, e_1, b_0\}$, we get

$$E(b_1, V(G)) - E(C) \subseteq \{b_1 d_{n_3-1}, b_1 d_{n_3}, b_1 e_0, b_1 e_1\}. \tag{5.16}$$

Claim 5.4.

(i) $\{b_1 d_{n_3-1}, b_1 d_{n_3}\} \cap E(G) \neq \emptyset$.

(ii) $\{b_1 e_1, b_1 e_0\} \cap E(G) \neq \emptyset$.

Proof. By the assumption of this case, $\{e_0, e_1, b_2\}$ is not a cutset, and hence $E(C_0 \cup (C_1 - \{b_2\}) \cup (C_4 - \{e_0, e_1\}), C_2 \cup C_3) \neq \emptyset$. Since $E(C_0 \cup (C_1 - \{b_2\}) \cup (C_4 - \{e_0, e_1\}), C_2) = \emptyset$ by Claim 5.3, and since $E(C_0 \cup \{b_0\} \cup (C_4 - \{e_0, e_1\}), C_3) = \emptyset$ by Claim 5.2, (5.9) and (5.8), this means $E(b_1, C_3) \neq \emptyset$. Hence it follows from (5.16) that $E(b_1, V(G))$ must contain $b_1 d_{n_3-1}$ or $b_1 d_{n_3}$. Thus (i) is proved, and by symmetry, we see that (ii) holds. ∎

Claim 5.5.

(i) $\{d_{n_3-1} e_1, d_{n_3-1} e_0\} \cap E(G) \neq \emptyset$.

(ii) $\{e_1 d_{n_3-1}, e_1 d_{n_3}\} \cap E(G) \neq \emptyset$.

Proof. Since C_1 is degenerate by the assumption of Proposition 3, $\{b_1, b_2, d_{n_3}\}$ is not a cutset, and hence $E(C_0 \cup \{b_0\} \cup C_4, C_2 \cup (C_3 - \{d_{n_3}\})) \neq \emptyset$. Since $E(C_0 \cup \{b_0\}, C_3 - \{d_{n_3}\}) = \emptyset$ by Claim 5.2 and (5.9), and since $E(C_0 \cup \{b_0\} \cup C_4, C_2) = \emptyset$ by Claim 5.3, this means $E(C_4, C_3 - \{d_{n_3}\}) \neq \emptyset$. Hence it follows from (5.8) that $E(C_3, C_4)$ must contain $d_{n_3-1} e_1$ or $d_{n_3-1} e_0$. Thus (i) is proved, and by symmetry, we see that (ii) holds. ∎

Claim 5.6.

(i) $\{d_{n_3-1} e_1, d_{n_3-1} b_1\} \cap E(G) \neq \emptyset$.

(ii) $\{e_1 d_{n_3-1}, e_1 b_1\} \cap E(G) \neq \emptyset$.

Proof. Since $d_{n_3} e_0$ is contractible, $\{d_{n_3}, e_0, b_2\}$ is not a cutset, and hence

$$E(C_0 \cup (C_1 - \{b_2\}) \cup (C_4 - \{e_0\}), C_2 \cup (C_3 - \{d_{n_3}\})) \neq \emptyset.$$

In view of Claim 5.3, (5.9) and Claim 5.2, we have

$$E(C_0 \cup (C_1 - \{b_2\}) \cup (C_4 - \{e_0\}), C_2) = \emptyset,$$

$$E(b_0, C_3 - \{d_{n_3}\}) = \emptyset,$$

and

$$E(C_0, C_3 - \{d_{n_3}\}) = \emptyset.$$

Consequently

$$E(\{b_1\} \cup (C_4 - \{e_0\}), C_3 - \{d_{n_3}\}) \neq \emptyset.$$

Hence it follows from (5.16) and (5.8) that (i) holds. By symmetry, we also see that (ii) holds. ∎

Claim 5.7.

(i) *If* $n_2 = 0$ *and* $n_3 = 1$, *then* $\{d_1 b_1, d_1 e_1\} \cap E(G) \neq \emptyset$.

(ii) *If* $n_0 = 0$ *and* $n_4 = 1$, *then* $\{e_0 b_1, e_0 d_{n_3 - 1}\} \cap E(G) \neq \emptyset$.

Proof. Suppose that $n_2 = 0$ and $n_3 = 1$. Since $c_0 d_0$ is contractible, $\{c_0, d_0, e_0\}$ is not a cutset, and hence

$$E(d_1, C_0 \cup C_1 \cup (C_4 - \{e_0\})) \neq \emptyset.$$

In view of (5.9) and Claim 5.2, we have

$$E(d_1, \{b_0, b_2\}) = \emptyset$$

and

$$E(d_1, C_0) = \emptyset.$$

Consequently

$$E(d_1, \{b_1\} \cup (C_4 - \{e_0\})) \neq \emptyset.$$

Hence it follows from (5.16) and (5.8) that (i) holds. By symmetry, we also see that (ii) holds. ∎

Now combining (5.8), (5.9), (5.14), (5.15), (5.16), and Claims 5.2 through 5.7, we see that (G, C) is of Type 7.

Proposition 4. *Suppose that* C_1 *and* C_4 *are nondegenerate, and* C_0, C_2 *and* C_3 *are degenerate. Then* (G, C) *is of Type* 8 *or Type* 9.

Proof. Let

$$i' = \begin{cases} \min\{i \mid 0 \le i \le n_1 - 1, d_0 \in K(b_i, b_{i+1})\} \\ \qquad\qquad (\text{if } \{i \mid 0 \le i \le n_1 - 1, d_0 \in K(b_i, b_{i+1})\} \neq \emptyset) \\ n_1 \qquad (\text{if } \{i \mid 0 \le i \le n_1 - 1, d_0 \in K(b_i, b_{i+1})\} = \emptyset), \end{cases}$$

and

$$x' = \begin{cases} \max\{x \mid 1 \le x \le n_4, c_{n_2} \in K(e_x, e_{x-1})\} \\ \qquad\qquad (\text{if } \{x \mid 1 \le x \le n_4, c_{n_2} \in K(e_x, e_{x-1})\} \neq \emptyset) \\ 0 \qquad (\text{if } \{x \mid 1 \le x \le n_4, c_{n_2} \in K(e_x, e_{x-1})\} = \emptyset). \end{cases}$$

Then by Lemma 3.10(I)(i),

$$(5.17) \qquad d_0 \in K(b_i, b_{i+1}) \text{ for all } i' \le i \le n_1 - 1,$$

and

$$(5.18) \qquad c_{n_2} \in K(e_x, e_{x-1}) \text{ for all } 1 \le x \le x'.$$

We divide the proof into two cases.

Case 1. $i' > 0$ or $x' < n_4$.

By the symmetry of the roles of $\{b_i \mid 0 \le i \le i'\}$ and $\{e_x \mid n_4 \ge x \ge x'\}$, we may assume $x' < n_4$. We show that

$$(5.19) \qquad K(b_i, b_{i+1}) \cap \{e_x \mid n_4 \ge x \ge x'\} \ne \emptyset \text{ for all } 0 \le i \le i' - 1.$$

If $i' = 0$, then (5.19) trivially holds. Thus assume $i' > 0$, and take i with $0 \le i \le i' - 1$. We first show that $K(b_i, b_{i+1}) \cap C_4 \ne \emptyset$. Since C_1 is nondegenerate, we can take $v \in K(b_i, b_{i+1}) \cap (C_3 \cup C_4)$. By the definition of i', $v \ne d_0$. If $v \in C_4$, there is noting to be proved. Thus we may assume $n_3 = 2$ and $v \in C_3 - \{d_0\}$. Then $v \ne d_1$ by Lemma 4.9, and hence $v = d_2$. Consequently, we get $e_0 \in K(b_i, b_{i+1})$ by applying Lemma 3.5 to $\{b_i, b_{i+1}, d_2\}$ and $\{d_0, d_1, e_0\}$. Thus $K(b_i, b_{i+1}) \cap C_4 \ne \emptyset$, as desired. Further if there exists x with $x < x'$ such that $e_x \in K(b_i, b_{i+1}) \cap C_4$, then since $c_{n_2} \in K(e_{x+1}, e_x)$ by (5.18), we get a contradiction by applying Lemma 3.4 to $\{b_i, b_{i+1}, e_x\}$ and $\{e_x, e_{x+1}, c_{n_2}\}$. Thus (5.19) is proved. By symmetry, we also see that

$$(5.20) \qquad K(e_x, e_{x-1}) \cap \{b_i \mid 0 \le i \le i'\} \ne \emptyset \text{ for all } n_4 \ge x \ge x' + 1.$$

By (5.19) and (5.20), we can apply Case (11a) of Lemma 3.11 with $A = \{b_i \mid 0 \le i \le i'\}$, $B = \{e_x \mid x' \le x \le n_4\}$, $m = i'$ and $s = n_4 - x'$, to get the following claim:

Claim 5.8. *There exists an integer r with $1 \le r \le \min\{i' + 1, n_4 - x'\}$, and there exist integers $k_0, k_1, \dots, k_r, k_{r+1}$ and $l_1, l_2, \dots, l_r, l_{r+1}$ with*

$$(5.21) \qquad 0 = k_0 \le k_1 < \cdots < k_r \le k_{r+1} = i'$$

and

$$(5.22) \qquad n_4 = l_1 > l_2 \cdots > l_r > l_{r+1} = x'$$

such that the following hold.

(I) (i) (a) *If $k_1 = 0$, then $b_{k_1} = b_0 \in K(e_{l_1}, e_{l_1 - 1}) = K(e_{n_4}, e_{n_4 - 1})$.*

(b) *If $k_1 > 0$, then $i' > 0$ and $e_{l_1} = e_{n_4} \in K(b_{k_0}, b_{k_0+1}) = K(b_0, b_1)$.*

(ii) (a) *If $k_r = i'$, then $b_{i'} = b_{k_r} \in K(e_{l_{r+1}+1}, e_{l_{r+1}}) = K(e_{x'+1}, e_{x'})$.*

(b) *If* $k_r < i'$, *then* $i' > 0$ *and* $e_{l_{r+1}} = e_{x'} \in K(b_{k_{r+1}-1}, b_{k_{r+1}}) = K(b_{i'-1}, b_{i'})$.

(II) *Set*

$$X_1 = \bigcup_{t=1}^{r+1} \{b_i b_{i+2} \mid k_{t-1} \le i \le k_t - 2\},$$

$$X_2 = \bigcup_{t=1}^{r} \{e_x e_{x-2} \mid l_t \ge x \ge l_{t+1} + 2\},$$

$$Y_1 = \{b_{k_t+1} e_{l_{t+1}+1} \mid 1 \le t \le r-1\},$$

$$\bar{Y_1} = \begin{cases} \{b_{k_r+1} e_{l_{r+1}+1}\} & (\textit{if } k_r < i') \\ \emptyset & (\textit{if } k_r = i'), \end{cases}$$

$$Y_2 = \{b_{k_t-1} e_{l_t-1} \mid 2 \le t \le r\},$$

$$\bar{Y_2} = \begin{cases} \{b_{k_1-1} e_{l_1-1}\} & (\textit{if } k_1 > 0) \\ \emptyset & (\textit{if } k_1 = 0), \end{cases}$$

$$Y_3' = \bigcup_{t=1}^{r} \{b_{k_t} e_x \mid l_t \ge x \ge l_{t+1}\},$$

and

$$Y_4' = \bigcup_{t=1}^{r+1} \{b_i e_{l_t} \mid k_{t-1} \le i \le k_t\}.$$

Then we have

$$E(\langle\{b_i \mid 0 \le i \le i'\}\rangle) - E(C) = X_1,$$

$$E(\langle\{e_x \mid x' \le x \le n_4\}\rangle) - E(C) = X_2,$$

and

$$(5.23)\quad Y_1 \cup \bar{Y_1} \cup Y_2 \cup \bar{Y_2} \subseteq E(\{b_i \mid 0 \le i \le i'\}, \{e_x \mid x' \le x \le n_4\}) \subseteq Y_1 \cup \bar{Y_1} \cup Y_2 \cup \bar{Y_2} \cup Y_3' \cup Y_4'.$$

In Claim 5.8(I)(ii), if $i' > 0$, then without loss of generality, we may assume that

$$(5.24)\qquad i' = k_r (= k_{r+1})$$

and

$$(5.25)\qquad b_{k_r} \in K(e_{l_{r+1}+1}, e_{l_{r+1}})$$

by the symmetry of the roles of $\{b_i \mid 0 \le i \le i'\}$ and $\{e_x \mid n_4 \ge x \ge x'\}$. Note that if $i' = 0$, then (5.24) and (5.25) clealy hold because the condition forces $r = 1$ and $k_1 = 0 (= i')$. Thus we henceforth assume that (5.24) and (5.25) hold. Further by (5.22),

$$x' = l_{r+1}. \tag{5.26}$$

By (5.25) and Lemma 3.1(i), we have

$$\begin{aligned} E(\{b_i \mid k_r + 1 \le i \le n_1\} \cup C_2 \cup C_3 \cup \{e_x \mid 0 \le x \le l_{r+1} - 1\}, \\ C_0 \cup \{b_i \mid 0 \le i \le k_r - 1\} \cup \{e_x \mid l_{r+1} + 2 \le x \le n_4\}) \quad = \emptyset. \end{aligned} \tag{5.27}$$

Claim 5.9.

(I) *Suppose that* $k_r = n_1$ *and* $n_2 = 2$. *Then* $E(\{c_0, c_2\}, V(G)) - E(C) = \{c_0c_2\}$.

(II) *Suppose that* $l_{r+1} = 0$ *and* $n_3 = 2$. *Then* $E(\{d_0, d_2\}, V(G)) - E(C) = \{d_0d_2\}$.

(III) *Suppose that* $k_r < n_1$. *Then the following hold.*

- (i) *If* $n_2 = 0$, *then* $E(C_2, V(G)) - E(C) = \{c_0b_{n_1-1}\}$.
- (ii) *If* $n_2 = 2$, $n_3 = 0$ *and* $l_{r+1} = 0$, *then* $\{c_0c_2, c_1b_{n_1-1}\} \subseteq E(C_2, V(G)) - E(C) \subseteq \{c_0c_2, c_1b_{n_1-1}, c_1b_{n_1}, c_1d_0\}$.
- (iii) *If* $n_2 = 2$, *and either* $n_3 = 2$ *or* $l_{r+1} > 0$, *then* $\{c_0c_2, c_1b_{n_1-1}\} \subseteq E(C_2, V(G)) - E(C) \subseteq \{c_0c_2, c_1b_{n_1-1}.c_1b_{n_1}\}$.

(IV) *Suppose that* $l_{r+1} > 0$. *Then the following hold.*

- (i) *If* $n_3 = 0$, *then* $E(C_3, V(G)) - E(C) = \{d_0e_1\}$.
- (ii) *If* $n_3 = 2$, $n_2 = 0$ *and* $k_r = n_1$, *then* $\{d_0d_2. d_1e_1\} \subseteq E(C_3, V(G)) - E(C) \subseteq \{d_0d_2, d_1c_0, d_1e_0, d_1e_1\}$.
- (iii) *If* $n_3 = 2$, *and either* $n_2 = 2$ *or* $k_r < n_1$, *then* $\{d_0d_2. d_1e_1\} \subseteq E(C_3, V(G)) - E(C) \subseteq \{d_0d_2, d_1e_0, d_1e_1\}$.

(V) *Supppose that* $k_1 = 0$. *Then the following hold.*

- (i) *If* $n_0 = 0$, *then* $E(C_0, V(G)) - E(C) = \{a_0e_{n_4-1}\}$.
- (ii) *If* $n_0 = 2$, *then* $\{a_0a_2, a_1e_{n_4-1}\} \subseteq E(C_0. V(G)) - E(C) \subseteq \{a_0a_2, a_1e_{n_4-1}, a_1e_{n_4}, a_1b_0\}$.

(VI) *Supppose that* $k_1 > 0$. *Then the following hold.*

- (i) *If* $n_0 = 0$, *then* $E(C_0, V(G)) - E(C) = \{a_0b_1\}$.

(ii) *If* $n_0 = 2$, *then* $\{a_0a_2, a_1b_1\} \subseteq E(C_0, V(G)) - E(C)$
$\subseteq \{a_0a_2, a_1e_{n_4}, a_1b_0, a_1b_1\}$.

Proof. Statements (I) and (II) are proved by applying Lemma 4.5 to C_2 and C_3. Note that if $k_r < n_1$, then $d_0 \in K(b_{n_1-1}, b_{n_1})$ by (5.17) and (5.24), and that if $l_{r+1} > 0$, then $c_{n_2} \in K(e_0, e_1)$ by (5.18) and (5.26). Thus we can prove (III) and (IV) by applying Lemma 4.6 to C_2 and C_3. Also note that if $k_1 = 0$, then $b_0 \in K(e_{n_4}, e_{n_4-1})$ by Claim 5.8 (I)(i)(a), and that if $k_1 > 0$, then $e_{n_4} \in K(b_0, b_1)$ by Claim 5.8(I)(i)(b). Thus we can prove (V) and (VI) by applying (I) and (II)(i) of Lemma 4.6 to C_0. ∎

Claim 5.10.

(i) $E(\{b_i \mid k_r + 1 \le i \le n_1\}, \{b_i \mid k_r \le i \le n_1\}) - E(C) = \{b_ib_{i+2} \mid k_r \le i \le n_1 - 2\}$.

(ii) $\{e_xe_{x-2} \mid l_{r+1} \ge x \ge 2\} \subseteq E(\{e_x \mid l_{r+1} - 1 \ge x \ge 0\}, \{e_x \mid l_{r+1} + 1 \ge x \ge 0\}) - E(C) \subseteq \{e_xe_{x-2} \mid l_{r+1} + 1 \ge x \ge 2\}$.

Proof. $\{e_xe_{x-2} \mid l_{r+1} \ge x \ge 2\} \subseteq E(\{e_x \mid l_{r+1} - 1 \ge x \ge 0\}, \{e_x \mid l_{r+1} + 1 \ge x \ge 0\} - E(C)$ by (5.18) and Lemma 3.3, and
$E(\{e_x \mid l_{r+1} - 1 \ge x \ge 0\}, \{e_x \mid l_{r+1} + 1 \ge x \ge 0\}) - E(C) \subseteq \{e_xe_{x-2} \mid l_{r+1} + 1 \ge x \ge 2\}$ by applying Lemma 4.7 to C_4. Thus (ii) is proved, and (i) is verified in a similar way. ∎

Claim 5.11.

(i) $E(\{b_i \mid k_r + 2 \le i \le n_1\}, (C_3 - \{d_0\}) \cup \{e_x \mid l_{r+1} + 1 \ge x \ge 0\}) - E(C) = \emptyset$.

(ii) $E(\{e_x \mid l_{r+1} - 2 \ge x \ge 0\}, \{b_i \mid k_r \le i \le n_1\} \cup (C_2 - \{c_{n_2}\})) - E(C) = \emptyset$.

Proof. We first prove (i). We may assume that $\{b_i \mid k_r + 2 \le i \le n_1\} \ne \emptyset$. Then $d_0 \in K(b_{k_r}, b_{k_r+1})$ by (5.17), and hence (i) follows from Lemma 3.1(i). Thus (i) is proved, and (ii) is verified in a similar way. ∎

Set

$$u = \begin{cases} b_{k_r+1} & (\text{if } k_r < n_1) \\ c_0 & (\text{if } k_r = n_1 \text{ and } n_2 = 0) \\ c_1 & (\text{if } k_r = n_1 \text{ and } n_2 = 2), \end{cases}$$

$$w = \begin{cases} e_{l_{r+1}-1} & (\text{if } l_{r+1} > 0) \\ d_0 & (\text{if } l_{r+1} = 0 \text{ and } n_3 = 0) \\ d_1 & (\text{if } l_{r+1} = 0 \text{ and } n_3 = 2), \end{cases}$$

and set

$$S = \begin{cases} \emptyset & (\text{if } k_r < n_1) \\ \{u\} & (\text{if } k_r = n_1) \end{cases}$$

and

$$T = \begin{cases} \emptyset & (\text{if } l_{r+1} > 0) \\ \{w\} & (\text{if } l_{r+1} = 0). \end{cases}$$

Note that

(5.28) $$\{b_i \mid k_r + 1 \le i \le n_1\} \cup S = \{b_i \mid k_r + 2 \le i \le n_1\} \cup \{u\},$$

(5.29) $$\{e_x \mid l_{r+1} - 1 \ge x \ge 0\} \cup T = \{e_x \mid l_{r+1} - 2 \ge x \ge 0\} \cup \{w\}.$$

Claim 5.12. *Let* $P = \{b_i \mid k_r + 1 \le i \le n_1\} \cup S \cup \{e_x \mid l_{r+1} - 1 \ge x \ge 0\} \cup T$ *and* $Q = P \cup \{b_{k_r}, e_{l_{r+1}}, e_{l_{r+1}+1}\}$.

(I) *Suppose that* $k_r = n_1$.

(i) *Suppose that* $n_2 = 0$. *Then the following hold.*

(a) $E(u,Q) - E(C) \subseteq \{ue_x \mid l_{r+1} - 2 \ge x \ge 0\} \cup \{uw, ue_{l_{r+1}}, ue_{l_{r+1}+1}\}$ *(in the case where* $l_{r+1} = 0$ *and* $n_3 = 0$*, we have* $uw \notin E(u,Q) - E(C)$ *because* $uw \in E(C)$*).*

(b) $E(b_{k_r}, P) - E(C) \subseteq \{b_{k_r}w\}$.

(ii) *Suppose that* $n_2 = 2$. *Then the following hold.*

(a) $E(u,Q) - E(C) \subseteq \{b_{k_r}u, uw, ue_{l_{r+1}}, ue_{l_{r+1}+1}\}$.

(b) $E(b_{k_r}, P) - E(C) \subseteq \{b_{k_r}u, b_{k_r}w\}$.

(II) *Suppose that* $l_{r+1} = 0$.

(i) *Suppose that* $n_3 = 0$. *Then the following hold.*

(a) $E(w,Q) - E(C) \subseteq \{b_iw \mid k_r + 2 \le i \le n_1\} \cup \{b_{k_r}w, uw, we_{l_{r+1}+1}\}$ *(in the case where* $k_r = n_1$ *and* $n_2 = 0$*, we have* $uw \notin E(w,Q) - E(C)$*).*

(b) $E(\{e_{l_{r+1}}, e_{l_{r+1}+1}\}, P) - E(C) \subseteq \{ue_{l_{r+1}}, ue_{l_{r+1}+1}, we_{l_{r+1}+1}\}$.

(ii) *Suppose that* $n_3 = 2$. *Then the following hold.*

(a) $E(w,Q) - E(C) \subseteq \{b_{k_r}w, uw, we_{l_{r+1}}, we_{l_{r+1}+1}\}$.

(b) $E(\{e_{l_{r+1}}, e_{l_{r+1}+1}\}, P) - E(C) \subseteq \{ue_{l_{r+1}}, ue_{l_{r+1}+1}, we_{l_{r+1}}, we_{l_{r+1}+1}\}$.

(III) *Suppose that* $k_r < n_1$ *and* $l_{r+1} > 0$ *(so* $S = T = \emptyset$*). Then* $E(P,Q) - E(C) \subseteq \{b_ib_{i+2} \mid k_r \le i \le n_1 - 2\} \cup \{e_xe_{x-2} \mid l_{r+1} \ge x \ge 2\} \cup \{b_{k_r}w, uw, ue_{l_{r+1}}, ue_{l_{r+1}+1}, we_{l_{r+1}+1}\}$.

Proof. In view of (5.29), (I)(i)(a) trivially holds, and (I)(i)(b) and (I)(ii) follow from Claim 5.11(ii) and (5.28). Thus (I) is proved. Now (II) can be verified in a similar way, and (III) follows from Claims 5.10 and 5.11 and (5.28) and (5.29) (note that $e_{l_{r+1}+1}e_{l_{r+1}-1} = we_{l_{r+1}+1}$ if $l_{r+1} > 0$). ∎

Set

$$D = \begin{cases} \{ue_x \mid l_{r+1} - 2 \geq x \geq 0\} & (\text{if } k_r = n_1, l_{r+1} > 0 \text{ and } n_2 = 0) \\ \emptyset & (\text{otherwise}), \end{cases}$$

$$F = \begin{cases} \{b_i w \mid k_r + 2 \leq i \leq n_1\} & (\text{if } k_r < n_1, l_{r+1} = 0 \text{ and } n_3 = 0) \\ \emptyset & (\text{otherwise}), \end{cases}$$

$$H_1 = \begin{cases} \{d_1 e_0\} & (\text{if } l_{r+1} = 0 \text{ and } n_3 = 2) \\ \emptyset & (\text{otherwise}), \end{cases}$$

$$H_2 = \begin{cases} \{b_{n_1} c_1\} & (\text{if } k_r = n_1 \text{ and } n_2 = 2) \\ \emptyset & (\text{otherwise}), \end{cases}$$

$$I = \begin{cases} \{b_{k_r} w, ue_{l_{r+1}}, ue_{l_{r+1}+1}, we_{l_{r+1}+1}\} \\ \qquad (\text{if } k_r = n_1, l_{r+1} = 0, n_2 = 0 \text{ and } n_3 = 0) \\ \{b_{k_r} w, uw, ue_{l_{r+1}}, ue_{l_{r+1}+1}, we_{l_{r+1}+1}\} \quad (\text{otherwise}), \end{cases}$$

$$W_1 = \{ue_{l_{r+1}+1}, we_{l_{r+1}+1}\},$$

$$W_2 = \begin{cases} \{ud_1, uw, ue_{l_{r+1}}, ue_{l_{r+1}+1}\} \cup D \\ \qquad (\text{if } k_r = n_1, l_{r+1} > 0, n_2 = 0 \text{ and } n_3 = 2) \\ \{uw, ue_{l_{r+1}}, ue_{l_{r+1}+1}\} \cup D \quad (\text{otherwise}), \end{cases}$$

$$W_3 = \begin{cases} \{b_{k_r} w, uw, c_1 w, we_{l_{r+1}+1}\} \cup F \\ \qquad (\text{if } k_r < n_1, l_{r+1} = 0, n_2 = 2 \text{ and } n_3 = 0) \\ \{b_{k_r} w, uw, we_{l_{r+1}+1}\} \cup F \quad (\text{otherwise}), \end{cases}$$

$$Z_2 = \begin{cases} \{ue_0\} & (\text{if } k_r = n_1, l_{r+1} \geq 2, n_2 = 0 \text{ and } n_3 = 0) \\ \{uw\} & (\text{if } l_{r+1} = 1 \text{ and } n_3 = 0) \\ \{uw, ue_{l_{r+1}}\} & (\text{if } l_{r+1} = 0 \text{ and } n_3 = 0) \\ \emptyset & (\text{otherwise}), \end{cases}$$

and

$$Z_3 = \begin{cases} \{b_{n_1}w\} & (\text{if } k_r \le n_1 - 2, l_{r+1} = 0, n_2 = 0 \text{ and } n_3 = 0) \\ \{uw\} & (\text{if } k_r = n_1 - 1 \text{ and } n_2 = 0) \\ \{b_{k_r}w, uw\} & (\text{if } k_r = n_1 \text{ and } n_2 = 0) \\ \emptyset & (\text{otherwise}). \end{cases}$$

Claim 5.13. $\{b_ib_{i+2} \mid k_r \le i \le n_1 - 2\} \cup \{e_xe_{x-2} \mid l_{r+1} \ge x \ge 2\}$
$\subseteq E\Big(\{b_i \mid k_r + 1 \le i \le n_1\} \cup S \cup \{e_x \mid l_{r+1} - 1 \ge x \ge 0\} \cup T,$
$\{b_i \mid k_r+1 \le i \le n_1\} \cup S \cup \{e_x \mid l_{r+1}-1 \ge x \ge 0\} \cup T \cup \{b_{k_r}, e_{l_{r+1}}, e_{l_{r+1}+1}\}\Big) - E(C)$
$\subseteq \{b_ib_{i+2} \mid k_r \le i \le n_1 - 2\} \cup \{e_xe_{x-2} \mid l_{r+1} \ge x \ge 2\} \cup I \cup D \cup F \cup H_1 \cup H_2.$

Proof. If $k_r = n_1$ or $l_{r+1} = 0$, then the assertion follows from Claim 5.10 and (I) or (II) of Claim 5.12. If $k_r < n_1$ and $l_{r+1} > 0$, then the assertion follows from Claim 5.10 and (III) of Claim 5.12 (note that $e_{l_{r+1}+1}e_{l_{r+1}-1} = we_{l_{r+1}+1}$ if $l_{r+1} > 0$). ∎

Claim 5.14. $W_1 \cap E(G) \ne \emptyset$.

Proof. Since $b_{k_r} \in K(e_{l_{r+1}}, e_{l_{r+1}+1})$ by (5.25), we have $E(e_{l_{r+1}+1}, \{b_i \mid k_r + 1 \le i \le n_1\} \cup C_2 \cup C_3 \cup \{e_x \mid l_{r+1} - 1 \ge x \ge 0\}) - E(C) \ne \emptyset$ by Lemma 3.1(ii), and hence we can take $e_{l_{r+1}+1}z \in E(e_{l_{r+1}+1}, \{b_i \mid k_r + 1 \le i \le n_1\} \cup C_2 \cup C_3 \cup \{e_x \mid l_{r+1} - 1 \ge x \ge 0\}) - E(C)$. From (I) and (III) of Claim 5.9, we get $z \notin C_2 - S$, and from (II) and (IV) of Claim 5.9, we get $z \notin C_3 - \{T\}$. Hence $z \in \{b_i \mid k_r + 1 \le i \le n_1\} \cup S \cup \{e_x \mid l_{r+1} - 1 \ge x \ge 0\} \cup T$, and we therefore obtain $z \in \{u, w\}$ by Claim 5.13.

Claim 5.15. $(W_2 - Z_2) \cap E(G) \ne \emptyset$.

Proof. We divide the proof into three cases.
`Case(i)`. $k_r = n_1$, $n_2 = 0$ and $n_3 = 2$.

Since G is 3-connected, we have $\deg(c_0) \ge 3$, and hence $E(c_0, V(G) - \{b_{n_1}, c_0, d_0\}) \ne \emptyset$. Consequently, we get $(W_2 - Z_2) \cap E(G) \ne \emptyset$ by (5.27), Claim 5.13, and (II) and (IV)(ii) of Claim 5.9.
`Case(ii)`. $n_3 = 0$.

Since d_0e_0 is contractible, $\{d_0, e_0, b_{k_r}\}$ is not a cutset, and hence $E(C_0 \cup \{b_i \mid 0 \le i \le k_r - 1\} \cup (C_4 - \{e_0\}), \{b_i \mid k_r + 1 \le i \le n_1\} \cup C_2) - E(C) \ne \emptyset$, which implies $E(\{e_x \mid l_{r+1}+1 \ge x \ge 1\}, \{b_i \mid k_r+1 \le i \le n_1\} \cup S) - E(C) \ne \emptyset$ by (5.27) and (I) and (III) of Claim 5.9. Consequently, we get $(W_2 - Z_2) \cap E(G) \ne \emptyset$ by Claim 5.13.
`Case(iii)`. $n_3 = 2$, and either $k_r < n_1$ or $n_2 = 2$.

Note that c_{n_2} and b_{k_r} are not consecutive. Since $c_{n_2}d_0$ is contractible, $\{c_{n_2}, d_0, b_{k_r}\}$ is not a cutset, and hence $E(C_0 \cup \{b_i \mid 0 \le i \le k_r - 1\} \cup (C_3 -$

$\{d_0\}) \cup C_4, \{b_i \mid k_r + 1 \le i \le n_1\} \cup (C_2 - \{c_{n_2}\})) - E(C) \ne \emptyset$, which implies $E(\{e_x \mid l_{r+1} + 1 \ge x \ge 0\} \cup T, \{b_i \mid k_r + 1 \le i \le n_1\} \cup S\}) - E(C) \ne \emptyset$ by (5.27) and (I), (II), (III)(iii) and (IV)(iii) of Claim 5.9. Consequently, we get $(W_2 - Z_2) \cap E(G) \ne \emptyset$ by Claim 5.13. ∎

Claim 5.16. $(W_3 - Z_3) \cap E(G) \ne \emptyset$.

Proof. We divide the proof into three cases.

Case(i). $l_{r+1} = 0$, $n_2 = 2$ and $n_3 = 0$.

Since G is 3-connected, we have $\deg(d_0) \ge 3$, and hence $E(d_0, V(G) - \{c_2, d_0, e_0\}) \ne \emptyset$. Consequently, we get $(W_3 - Z_3) \cap E(G) \ne \emptyset$ by (5.27), Claim 5.13, and (I) and (III)(ii) of Claim 5.9.

Case(ii). $n_2 = 0$.

Since $b_{n_1}c_0$ is contractible, $\{b_{n_1}, c_0, e_{l_{r+1}}\}$ is not a cutset, and hence $E(C_0 \cup (C_1 - \{b_{n_1}\}) \cup \{e_x \mid n_4 \ge x \ge l_{r+1} + 1\}, C_3 \cup \{e_x \mid l_{r+1} - 1 \ge x \ge 0\}) - E(C) \ne \emptyset$. Consequently, we get $(W_3 - Z_3) \cap E(G) \ne \emptyset$ by (5.27), Claim 5.13 and (II) and (IV) of Claim 5.9.

Case(iii). $n_2 = 2$, and either $l_{r+1} > 0$ or $n_3 = 2$.

Note that d_0 and $e_{l_{r+1}}$ are not consecutive. Since $c_{n_2}d_0$ is contractible, $\{c_{n_2}, d_0, e_{l_{r+1}}\}$ is not a cutset, and hence $E(C_0 \cup C_1 \cup (C_2 - \{c_{n_2}\}) \cup \{e_x \mid n_4 \ge x \ge l_{r+1} + 1\}, (C_3 - \{d_0\}) \cup \{e_x \mid l_{r+1} - 1 \ge x \ge 0\}) - E(C) \ne \emptyset$. Consequently, we get $(W_3 - Z_3) \cap E(G) \ne \emptyset$ by (5.27), Claim 5.13, and (I), (II), (III)(iii) and (IV)(iii) of Claim 5.9. ∎

Set

$$W_4 = \{b_{k_r}e_x \mid l_r \ge x \ge l_{r+1}\} \cup \{b_{k_r}w\},$$

$$Z_4 = \begin{cases} \{b_{k_r}u\} & (\text{if } n_2 = 2) \\ \emptyset & (\text{if } n_2 = 0), \end{cases}$$

and

$$Z_4' = \begin{cases} \{a_1b_0\} & (\text{if } n_0 = 2) \\ \emptyset & (\text{if } n_0 = 0). \end{cases}$$

Claim 5.17. *Suppose that $n_1 = 1$ and $r = 1$.*

(i) *If $k_1 = 1$ and $n_0 = 0$, then $(W_4 \cup Z_4) \cap E(G) \ne \emptyset$.*

(ii) *If $k_1 = 0$ and $n_2 = 0$, then $(W_4 \cup Z_4') \cap E(G) \ne \emptyset$.*

Proof. First assume that $n_0 = 0$, $n_1 = 1$, $r = 1$ and $k_r = 1$. Then since a_0b_0 is contractible, $\{a_0, b_0, c_0\}$ is not a cutset, and hence $E(b_{k_1}(= b_1), V(G) - \{a_0, b_0, b_1, c_0\}) \ne \emptyset$, which implies that $E(b_{k_1}, C_4 \cup S \cup T) - E(C) \ne \emptyset$ by

(I),(II) and (IV) of Claim 5.9. Consequently, we obtain $(W_4 \cup Z_4) \cap E(G) \neq \emptyset$ by Claim 5.13. Next assume that $n_1 = 1$, $n_2 = 0$, $r = 1$ and $k_r = 0$. Then since $b_1 c_0$ is contractible, $\{b_1, c_0, a_{n_0}\}$ is not a cutset, and hence $E(b_{k_1} (= b_0), V(G) - \{a_{n_0}, b_0, b_1, c_0\}) \neq \emptyset$, which implies that $E(b_{k_1}, (C_0 - \{a_{n_0}\}) \cup C_4 \cup T) - E(C) \neq \emptyset$ by (II), (IV)(i) and (IV)(iii) of Claim 5.9. Consequently, we obtain $(W_4 \cup Z_4') \cap E(G) \neq \emptyset$ by (V)(ii) of Claim 5.9 and Claim 5.13. ∎

Set

$$W_5 = \begin{cases} \{b_{k_r}w, b_{k_r}e_{l_{r+1}}, uw, ue_{l_{r+1}}\} & (\text{if } l_{r+1} > 0) \\ \{b_{k_r}e_{l_{r+1}}, ue_{l_{r+1}}\} & (\text{if } l_{r+1} = 0 \text{ and } n_3 = 0) \\ \{b_{k_r}e_{l_{r+1}}, ue_{l_{r+1}}, we_{l_{r+1}}\} & (\text{if } l_{r+1} = 0 \text{ and } n_3 = 2). \end{cases}$$

Claim 5.18. *Suppose that $n_0 = 0$, $r = 1$, $k_r = 0$ and $l_{r+1} = n_4 - 1$. Then $W_5 \cap E(G) \neq \emptyset$.*

Proof. Since $e_{n_4}a_0$ is contractible, $\{e_{n_4}, a_0, d_0\}$ and $\{e_{n_4}, a_0, d_{n_3}\}$ are not cutsets, and hence we get

$$E(C_1 \cup C_2, (C_3 - \{d_0\})) \cup (C_4 - \{e_{n_4}\}) \neq \emptyset \tag{5.30}$$

and

$$E(C_1 \cup C_2 \cup (C_3 - \{d_{n_3}\}), (C_4 - \{e_{n_4}\}) \neq \emptyset. \tag{5.31}$$

If $l_{r+1} > 0$, i.e., $n_4 > 1$, then (5.30) together with (III)(i), (III)(iii) and (IV)(iii) of Claim 5.9 implies $E(C_1, C_4 - \{e_{n_4}\}) - E(C) \neq \emptyset$ (note that we have $S = T = \emptyset$ by the assumption that $k_r = 0 < n_1$ and $l_{r+1} > 0$), and hence we get $W_5 \cap E(G) \neq \emptyset$ by Claim 5.13; if $l_{r+1} = 0$, then (5.31) together with (II) and (III) of Claim 5.9 implies $E(C_1 \cup T, C_4 - \{e_{n_4}\}) - E(C) \neq \emptyset$ (note that we have $S = \emptyset$), and hence we get $W_5 \cap E(G) \neq \emptyset$ by Claim 5.13. ∎

Combining (5.24), (5.26), Claim 5.8, (5.27), Claim 5.9 and Claims 5.13 through 5.18, we now see that (G, C) is of type 8.

Case 2. $i' = 0$ and $x' = n_4$.

By (5.17) and (5.18),

$$d_0 \in K(b_i, b_{i+1}) \text{ for all } 0 \leq i \leq n_1 - 1, \tag{5.32}$$

and

$$c_{n_2} \in K(e_x, e_{x-1}) \text{ for all } 1 \leq x \leq n_4. \tag{5.33}$$

Claim 5.19.

(I) *If* $n_0 = 2$, *then* $E(\{a_0, a_2\}, V(G)) - E(C) = \{a_0a_2\}$.

(II) (i) *If* $n_2 = 0$, *then* $E(C_2, V(G)) - E(C) = \{c_0b_{n_1-1}\}$.

(ii) *If* $n_2 = 2$, *then* $\{c_0c_2, c_1b_{n_1-1}\} \subseteq E(C_2, V(G)) - E(C)$
$\subseteq \{c_0c_2, c_1b_{n_1-1}, c_1b_{n_1}\}$.

(III) (i) *If* $n_3 = 0$, *then* $E(C_3, V(G)) - E(C) = \{d_0e_1\}$.

(ii) *If* $n_3 = 2$, *then* $\{d_0d_2, d_1e_1\} \subseteq E(C_3, V(G)) - E(C)$
$\subseteq \{d_0d_2, d_1e_0, d_1e_1\}$.

Proof. Statement (I) follows immediately from Lemma 4.5(ii). By (5.32) and (5.33), we have $d_0 \in K(b_{n_1-1}, b_{n_1})$ and $c_{n_2} \in K(e_0, e_1)$, and hence we can prove (II) and (III) by applying (I) and (II)(ii) of Lemma 4.6 to C_2 and C_3. ∎

Claim 5.20.

(i) $E(\langle C_1 \rangle) - E(C) = \{b_ib_{i+2} \mid 0 \leq i \leq n_1 - 2\}$.

(ii) $E(\langle C_4 \rangle) - E(C) = \{e_xe_{x-2} \mid n_4 \geq x \geq 2\}$.

Proof. By (5.32) and (5.33), (i) and (ii) follow immediately from Lemmas 3.3 and 4.7. ∎

Claim 5.21.

(I) (i) *If* $n_0 = 0$, $E(C_1, C_4 \cup \{a_0\}) - E(C)$
$\subseteq \{b_0e_{n_4}, b_0e_{n_4-1}, a_0b_1, b_1e_{n_4}, b_1e_{n_4-1}\}$.

(ii) *If* $n_0 = 2$, $E(C_1, C_4 \cup \{a_1\}) - E(C)$
$\subseteq \{a_1b_0, b_0e_{n_4}, b_0e_{n_4-1}, a_1b_1, b_1e_{n_4}, b_1e_{n_4-1}\}$.

(II) (i) *If* $n_0 = 0$, $E(C_4, C_1 \cup \{a_0\}) - E(C)$
$\subseteq \{b_0e_{n_4}, b_1e_{n_4}, a_0e_{n_4-1}, b_0e_{n_4-1}, b_1e_{n_4-1}\}$.

(ii) *If* $n_0 = 2$, $E(C_4, C_1 \cup \{a_1\}) - E(C)$
$\subseteq \{a_1e_{n_4}, b_0e_{n_4}, b_1e_{n_4}, a_1e_{n_4-1}, b_0e_{n_4-1}, b_1e_{n_4-1}\}$.

Proof. Since $d_0 \in K(b_0, b_1)$ by (5.32), it follows from Lemma 3.1(i) that

$$E(C_1 - \{b_0, b_1\}, C_4 \cup C_0) = \emptyset. \tag{5.34}$$

Also since $c_{n_2} \in K(e_{n_4}, e_{n_4-1})$ by (5.33), it follows from Lemma 3.1(i) that

$$E(C_1 \cup C_0, C_4 - \{e_{n_4}, e_{n_4-1}\}) = \emptyset. \tag{5.35}$$

Combining (5.34) and (5.35), we get all the desired conclusions. ∎

Set

$$W_1 = \begin{cases} \{b_1 e_{n_4}, b_1 e_{n_4-1}\} & (\text{if } n_0 = 0) \\ \{b_1 a_1, b_1 e_{n_4}, b_1 e_{n_4-1}\} & (\text{if } n_0 = 2), \end{cases}$$

and

$$W_2 = \begin{cases} \{b_0 e_{n_4-1}, b_1 e_{n_4-1}\} & (\text{if } n_0 = 0) \\ \{a_1 e_{n_4-1}, b_0 e_{n_4-1}, b_1 e_{n_4-1}\} & (\text{if } n_0 = 2). \end{cases}$$

Claim 5.22. *$W_1 \cap E(G) \neq \emptyset$ and $W_2 \cap E(G) \neq \emptyset$.*

Proof. Since $a_{n_0}b_0$ is contractible, $\{a_{n_0}, b_0, c_{n_2}\}$ is not a cutset, and hence $E((C_1 - \{b_0\}) \cup (C_2 - \{c_{n_2}\}), (C_0 - \{a_{n_0}\}) \cup C_4 \cup C_3) \neq \emptyset$. Consequently, we obtain $W_1 \cap E(G) \neq \emptyset$ by Claim 5.19 and Claim 5.21(I). In view of the symmetry of the roles of C_1 and C_4, we similarly obtain $W_2 \cap E(G) \neq \emptyset$. ∎

Claim 5.23. *If $n_0 = 2$, $\{a_1 b_1, a_1 e_{n_4-1}\} \cap E(G) \neq \emptyset$; if $n_0 = 0$, $\{a_0 b_1, a_0 e_{n_4-1}\} \cap E(G) \neq \emptyset$.*

Proof. Suppose first that $n_0 = 2$. Then since $a_{n_0}b_0$ is contractible, $\{a_{n_0}, b_0, e_{n_4}\}$ is not a cutset, and hence $E(\{a_0, a_1\}, (C_1 - \{b_0\}) \cup C_2 \cup C_3 \cup (C_4 - \{e_{n_4}\})) \neq \emptyset$. Consequently, we obtain $\{a_1 b_1, a_1 e_{n_4-1}\} \cap E(G) \neq \emptyset$ by Claim 5.19 and (I)(ii) and (II)(ii) of Claim 5.21. Suppose now that $n_0 = 0$. Then since G is 3-connected, $E(a_0, (C_1 - \{b_0\}) \cup C_2 \cup C_3 \cup (C_4 - \{e_{n_4}\}) \neq \emptyset$. Since $d_0 \in K(b_0, b_1)$ by (5.32), we get $E(a_0, (C_1 - \{b_0, b_1\}) \cup C_2) = \emptyset$, and since $c_{n_2} \in K(e_{n_4}, e_{n_4-1})$ by (5.33), we also get $E(a_0, (C_4 - \{e_{n_4}, e_{n_4-1}\}) \cup C_3) = \emptyset$. Consequently, we obtain $\{a_0 b_1, a_0 e_{n_4-1}\} \cap E(G) \neq \emptyset$. ∎

Set

$$W_3 = \begin{cases} \{b_0 e_{n_4}, b_0 e_{n_4-1}\} & (\text{if } n_0 = 0) \\ \{a_1 b_0, b_0 e_{n_4}, b_0 e_{n_4-1}\} & (\text{if } n_0 = 2), \end{cases}$$

$$W_4 = \begin{cases} \{a_0 e_{n_4-1}, b_0 e_{n_4-1}\} & (\text{if } n_0 = 0) \\ \{a_1 e_{n_4-1}, b_0 e_{n_4-1}\} & (\text{if } n_0 = 2), \end{cases}$$

$$W_5 = \begin{cases} \{b_0 e_{n_4}, b_1 e_{n_4}\} & (\text{if } n_0 = 0) \\ \{a_1 e_{n_4}, b_0 e_{n_4}, b_1 e_{n_4}\} & (\text{if } n_0 = 2), \end{cases}$$

and

$$W_6 = \begin{cases} \{a_0 b_1, b_1 e_{n_4}\} & (\text{if } n_0 = 0) \\ \{a_1 b_1, b_1 e_{n_4}\} & (\text{if } n_0 = 2). \end{cases}$$

Claim 5.24.

(i) *If* $n_1 = 1$ *and* $n_2 = 0$, *then* $W_3 \cap E(G) \neq \emptyset$ *and* $W_4 \cap E(G) \neq \emptyset$.

(ii) *If* $n_4 = 1$ *and* $n_3 = 0$, *then* $W_5 \cap E(G) \neq \emptyset$ *and* $W_6 \cap E(G) \neq \emptyset$.

Proof. To prove (i), suppose that $n_1 = 1$ and $n_2 = 0$. Then since b_1c_0 is contractible, $\{b_1, c_0, a_{n_0}\}$ and $\{b_1, c_0, e_{n_4}\}$ are not cutsets, and hence we get

$$E(b_0, (C_0 - \{a_{n_0}\}) \cup C_3 \cup C_4) \neq \emptyset \tag{5.36}$$

and

$$E(C_0 \cup \{b_0\}, C_3 \cup (C_4 - \{e_{n_4}\}) \neq \emptyset. \tag{5.37}$$

By (5.36), Claim 5.21(I) and (I),(III) of Claim 5.19, we obtain $W_3 \cap E(G) \neq \emptyset$, and by (5.37), Claim 5.21(II), and (I), (III) of Claim 5.19, we obtain $W_4 \cap E(G) \neq \emptyset$. Thus (i) is proved, and by the symmetry of the roles of C_1, C_2 and C_4, C_3, (ii) can be verified in a similar way. ∎

Combining Claims 5.19 through 5.24, we see that (G, C) is of Type 9.

Proposition 5. *Suppose that* C_4 *is nondegenerate and* $n_4 \geq 3$, *and* C_0, C_1, C_2 *and* C_3 *are degenerate. Then* (G, C) *is of Type* 10, 11, 12, 13, 14, 15, 16, 17, 18, 19 *or* 20.

Proof. By symmetry, we may assume $n_1 \leq n_2$. We divide the proof into three cases, according as $(n_1, n_2) = (0, 0)$, $(n_1, n_2) = (0, 2)$, or $(n_1, n_2) = (2, 2)$.

Case 1. $(n_1, n_2) = (0, 0)$.

Since C_4 is nondegenerate,

$$K(e_x, e_{x+1}) \cap \{b_0, c_0\} \neq \emptyset \text{ for all } 0 \leq x \leq n_4 - 1. \tag{5.38}$$

Arguing as in Claim 5.16 in the proof of Proposition 3 of [4], we obtain:

Claim 5.25. *One of the following holds*:

(i) $K(e_x, e_{x+1}) \cap \{b_0, c_0\} = \{c_0\}$ *for all* $0 \leq x \leq n_4 - 1$;

(ii) *there exists* p *with* $1 \leq p \leq n_4 - 1$ *such that* $c_0 \in K(e_x, e_{x+1})$ *for all* $0 \leq x \leq p - 1$ *and* $b_0 \in K(e_x, e_{x+1})$ *for all* $p \leq x \leq n_4 - 1$; *or*

(iii) $K(e_x, e_{x+1}) \cap \{b_0, c_0\} = \{b_0\}$ *for all* $0 \leq x \leq n_4 - 1$.

By symmetry, we may assume that (i) or (ii) of Claim 5.25 holds. If Claim 5.25(ii) holds, then applying the argument in the proof of (5-16) of [4] to b_0c_0 instead of b_0b_1, and the argument in the proof of Claim 5.14 of [4] to c_0 as well as b_0, we see that G is of Type 10. Thus we may assume Claim 5.25(i) holds. Then applying Lemma 4.5 to C_0 and C_3, we get

$$(5.39)\quad E(c_0, V(G)) - E(C) = \begin{cases} \{c_0e_x \mid 0 \le x \le n_4\} \cup \{c_0a_0\} \\ \quad \text{(if } n_3 = 0 \text{ and } n_0 = 0) \\ \{c_0e_x \mid 0 \le x \le n_4\} \cup \{c_0a_1\} \\ \quad \text{(if } n_3 = 0 \text{ and } n_0 = 2) \\ \{c_0e_x \mid 0 \le x \le n_4\} \cup \{c_0a_0, c_0d_1\} \\ \quad \text{(if } n_3 = 2 \text{ and } n_0 = 0) \\ \{c_0e_x \mid 0 \le x \le n_4\} \cup \{c_0a_1, c_0d_1\} \\ \quad \text{(if } n_3 = 2 \text{ and } n_0 = 2). \end{cases}$$

Claim 5.26. *Suppose that $n_0 = 0$.*

(i) *If $n_3 = 0$, then $\{c_0e_x \mid 0 \le x \le n_4\} \cap E(G) \ne \emptyset$.*

(ii) *If $n_3 = 2$, then $(\{c_0e_x \mid 0 \le x \le n_4\} \cup \{c_0d_1\}) \cap E(G) \ne \emptyset$.*

Proof. Since a_0b_0 is contractible, $\{a_0, b_0, d_0\}$ is not a cutset, and hence $E(c_0, (C_3 - \{d_0\}) \cup C_4) \ne \emptyset$. Consequently we get the desired conclusions by (5.39). ∎

Now arguing as in Case 2 in the proof of Proposition 3 of [4] (we apply the second half of the proof of Claim 5.20 of [4] to b_0c_0), we see that G is of Type 11.

Case 2. $(n_1, n_2) = (0, 2)$.

Applying Lemma 4.9 to C_4 and C_2, we see that $c_1 \notin K(e_x, e_{x+1})$ for each $0 \le x \le n_4 - 1$. Since C_4 is nondegenerate, this implies

$$(5.40)\qquad K(e_x, e_{x+1}) \cap \{b_0, c_0, c_2\} \ne \emptyset \text{ for all } 0 \le x \le n_4 - 1.$$

Claim 5.27. *Let $0 \le x \le n_4 - 1$, and suppose that $c_0 \in K(e_x, e_{x+1})$. Then $b_0 \in K(e_x, e_{x+1})$.*

Proof. Let $0 \le x \le n_4 - 1$. Since C_2 is degenerate and $n_2 = 2$ and $n_1 = 0$, $b_0 \in K(c_1, c_2)$, and hence the desired conclusion follows from Lemma 3.5. ∎

In view of (5.40) and Claim 5.27, we obtain the following claim by arguing as in Claim 5.16 of [4]:

Claim 5.28. *One of the following holds:*

(i) *$K(e_x, e_{x+1}) \cap \{b_0, c_0, c_2\} = \{c_2\}$ for all $0 \le x \le n_4 - 1$;*

(ii) *there exists* p *with* $1 \leq p \leq n_4 - 1$ *such that* $c_2 \in K(e_x, e_{x+1})$ *for all* $0 \leq x \leq p-1$ *and* $b_0 \in K(e_x, e_{x+1})$ *for all* $p \leq x \leq n_4 - 1$; *or*

(iii) $K(e_x, e_{x+1}) \cap \{b_0, c_0, c_2\} \subseteq \{b_0, c_0\}$ *for all* $0 \leq x \leq n_4 - 1$.

Now we divide the proof into three subcases, according to whether (i), (ii) or (iii) of Claim 5.28 holds.

Subcase 1. Claim 5.28(i) holds.

For convenience, let $a = a_1$ if $n_0 = 2$, and let $a = a_0$ if $n_0 = 0$.

Arguing as in the first half of the proof of Claim 5.20 of [4], we obtain:

Claim 5.29. $ac_1 \in E(G)$.

Applying the second half of the proof of Claim 5.20 of [4] to $\{c_0, c_1, e_{n_4}\}$ and $\{c_0, c_1, a_{n_0}\}$, we obtain the following two claims:

Claim 5.30. $\{e_{n_4-1}a, e_{n_4-1}b_0\} \cap E(G) \neq \emptyset$.

Claim 5.31.

(i) *If* $n_0 = 2$, *then* $\{b_0a_1, b_0e_{n_4-1}, b_0e_{n_4}\} \cap E(G) \neq \emptyset$.

(ii) *If* $n_0 = 0$, *then* $\{b_0e_{n_4-1}, b_0e_{n_4}\} \cap E(G) \neq \emptyset$.

We can prove the following two claims by applying the proof of Claim 5.21 of [4] to $\{a_1, a_2, c_0\}$ and $\{a_1, a_2, d_0\}$ or $\{e_{n_4}, a_0, c_0\}$ and $\{e_{n_4}, a_0, d_0\}$, according as $n_0 = 2$ or $n_0 = 0$:

Claim 5.32.

(i) *If* $n_0 = 2$, *then* $\{b_0c_1, b_0e_{n_4-1}, b_0e_{n_4}\} \cap E(G) \neq \emptyset$.

(ii) *If* $n_0 = 0$, *then* $\{b_0c_1, b_0e_{n_4-1}\} \cap E(G) \neq \emptyset$.

Claim 5.33.

(i) *If* $n_0 = 2$, *then* $\{e_{n_4-1}b_0, e_{n_4-1}c_1, e_{n_4}b_0, e_{n_4}c_1\} \cap E(G) \neq \emptyset$.

(ii) *If* $n_0 = 0$, *then* $\{e_{n_4-1}b_0, e_{n_4-1}c_1\} \cap E(G) \neq \emptyset$.

Arguing as in Claim 5.26 of Case 1, we obtain:

Claim 5.34. *If* $n_0 = 0$, *then* $\{c_1e_{n_4-1}, c_1e_{n_4}\} \cap E(G) \neq \emptyset$.

Now combining Claims 5.29 through 5.34, and applying Lemma 4.6 to C_3, and Lemmas 3.3 and 4.7 to C_4, we see that (G, C) is of Type 12.

Subcase 2. Claim 5.28(ii) holds.

Applying Lemma 3.1(i) to $\{e_{p-1}, e_p, c_2\}$, and Lemma 4.5 to C_0 and C_2, we get

$$E(b_0, V(G)) - E(C) \subseteq \begin{cases} \{b_0c_1\} \cup \{b_0e_x \mid p-1 \leq x \leq n_4\} & (\text{if } n_0 = 0) \\ \{b_0a_1, b_0c_1\} \cup \{b_0e_x \mid p-1 \leq x \leq n_4\} & (\text{if } n_0 = 2). \end{cases}$$
(5.41)

Applying the proof of (5.16) of [4] to b_0c_0 as well as to c_0c_1, we obtain the following two claims:

Claim 5.35. $\{e_{p-1}b_0, e_{p-1}e_{p+1}\} \cap E(G) \neq \emptyset$.

Claim 5.36. $\{e_{p+1}c_1, e_{p+1}e_{p-1}\} \cap E(G) \neq \emptyset$.

Claim 5.37.

(i) *If* $n_0 = 2$, *then* $(\{b_0a_1\} \cup \{b_0e_x \mid p-1 \leq x \leq n_4\}) \cap E(G) \neq \emptyset$.

(ii) *If* $n_0 = 0$, *then* $\{b_0e_x \mid p-1 \leq x \leq n_4\} \cap E(G) \neq \emptyset$.

Proof. We prove (i) and (ii) simultaneously. Since C_2 is denenerate by the assumption of Proposition 5, $\{c_0, c_1, a_{n_0}\}$ is not a cutset, and hence $E(b_0, (C_0 - \{a_{n_0}\}) \cup \{c_2\} \cup C_3 \cup C_4) \neq \emptyset$. Consequently, we get the desired conclusions by (5.41). ∎

Applying the proof of Claim 5.14 of [4] to $\{a_1, a_2, c_0\}$ or $\{e_{n_4}, a_0, c_0\}$ (according as $n_0 = 2$ or 0), and $\{d_0, d_1, b_0\}$ or $\{d_0, e_0, b_0\}$ (according as $n_3 = 2$ or 0), we obtain the following two claims:

Claim 5.38.

(i) *If* $n_0 = 2$, *then* $(\{b_0c_1\} \cup \{b_0e_x \mid p-1 \leq x \leq n_4\}) \cap E(G) \neq \emptyset$.

(ii) *If* $n_0 = 0$, *then* $(\{b_0c_1\} \cup \{b_0e_x \mid p-1 \leq x \leq n_4 - 1\}) \cap E(G) \neq \emptyset$.

Claim 5.39.

(i) *If either* $n_3 = 2$ *or* $p \geq 2$, *then* $\{c_1e_{p-1}, c_1e_p, c_1e_{p+1}\} \cap E(G) \neq \emptyset$.

(ii) *If either* $n_3 = 0$ *and* $p = 1$, *then* $\{c_1e_p, c_1e_{p+1}\} \cap E(G) \neq \emptyset$.

Now combining Claims 5.35 through 5.39, and applying Lemma 4.6 to C_0 and C_3, and Lemmas 3.3 and 4.7 to C_4, we see that (G, C) is of Type 13.

Subcase 3. Claim 5.28(iii) holds.

Let $d = d_1$ if $n_3 = 2$, and let $d = d_0$ if $n_3 = 0$. Arguing as in the first half of the proof of Claim 5.20 of [4], we obtain:

Claim 5.40. $\{db_0, dc_1\} \cap E(G) \neq \emptyset$.

Arguing as in Claim 5.21 of [4], we obtain:

Claim 5.41.

(i) *If* $n_3 = 2$, *then* $\{c_1e_0, c_1e_1\} \cap E(G) \neq \emptyset$.

(ii) *If* $n_3 = 0$, *then* $c_1e_1 \in E(G)$.

Applying the second half of the proof of Claim 5.20 of [4] to $\{b_0, c_0, e_0\}$ as well as to $\{c_1, c_2, e_0\}$, we obtain the following two claims:

Claim 5.42. $\{db_0, de_1\} \cap E(G) \neq \emptyset$.

Claim 5.43. *If* $n_3 = 2$, *then* $\{e_1c_1, e_1d_1\} \cap E(G) \neq \emptyset$.

Arguing as in Claim 5.37, we obtain:

Claim 5.44.

(i) *If* $n_0 = 2$, *then* $(\{b_0a_1, b_0d\} \cup \{b_0e_x \,|\, 0 \leq x \leq n_4\}) \cap E(G) \neq \emptyset$.

(ii) *If* $n_0 = 0$, *then* $(\{b_0d\} \cup \{b_0e_x \,|\, 0 \leq x \leq n_4\}) \cap E(G) \neq \emptyset$.

Arguing as in Claim 5.38, we obtain:

Claim 5.45.

(i) *If* $n_0 = 2$, *then* $(\{b_0c_1, b_0d\} \cup \{b_0e_x \,|\, 0 \leq x \leq n_4\}) \cap E(G) \neq \emptyset$.

(ii) *If* $n_0 = 0$, *then* $(\{b_0c_1, b_0d\} \cup \{b_0e_x \,|\, 0 \leq x \leq n_4 - 1\}) \cap E(G) \neq \emptyset$.

Now combining Claims 5.40 through 5.45, and applying Lemma 4.6 to C_0, and Lemmas 3.3 and 4.7 to C_4, we see that (G, C) is of Type 14.

Case 3. $(n_1, n_2) = (2, 2)$.

For each $0 \le x \le n_4 - 1$, we get $c_1 \notin K(e_x, e_{x+1})$ by applying Lemma 4.9 to C_4 and C_2, and by symmetry, we also get $b_1 \notin K(e_x, e_{x+1})$. Since C_4 is nondegenerate, this implies

$$(5.42) \qquad K(e_x, e_{x+1}) \cap \{b_0, b_2, c_0, c_2\} \neq \emptyset \text{ for all } 0 \le x \le n_4 - 1.$$

Arguing as in Claim 5.27, we obtain:

Claim 5.46. *Let $0 \le x \le n_4 - 1$. Then either $b_2, c_0 \in K(e_x, e_{x+1})$ or $b_2, c_0 \notin K(e_x, e_{x+1})$.*

Claim 5.47. *One of the following holds:*

(i) *there exists p and q with $1 \le p < q \le n_4 - 1$ such that $c_2 \in K(e_x, e_{x+1})$ for all $0 \le x \le p - 1$, and $\{b_2, c_0\} \subseteq K(e_x, e_{x+1})$ for all $p \le x \le q - 1$, and $b_0 \in K(e_x, e_{x+1})$ for all $q \le x \le n_4 - 1$;*

(ii) *$b_0 \notin K(e_x, e_{x+1})$ for all $0 \le x \le n_4 - 1$, and there exists p with $1 \le p \le n_4 - 1$ such that $c_2 \in K(e_x, e_{x+1})$ for all $0 \le x \le p - 1$, and $\{b_2, c_0\} \subseteq K(e_x, e_{x+1})$ for all $p \le x \le n_4 - 1$;*

(iii) *$K(e_x, e_{x+1}) \cap \{b_2, c_0\} = \emptyset$ for all $0 \le x \le n_4 - 1$, and there exists p with $1 \le p \le n_4 - 1$ such that $c_2 \in K(e_x, e_{x+1})$ for all $0 \le x \le p - 1$, and $b_0 \in K(e_x, e_{x+1})$ for all $p \le x \le n_4 - 1$;*

(iv) *$c_2 \notin K(e_x, e_{x+1})$ for all $0 \le x \le n_4 - 1$, and there exists p with $1 \le p \le n_4 - 1$ such that $\{b_2, c_0\} \subseteq K(e_x, e_{x+1})$ for all $0 \le x \le p - 1$, and $b_0 \in K(e_x, e_{x+1})$ for all $p \le x \le n_4 - 1$;*

(v) *$K(e_x, e_{x+1}) \cap \{b_0, b_2, c_0, c_2\} = \{c_2\}$ for all $0 \le x \le n_4 - 1$;*

(vi) *$K(e_x, e_{x+1}) \cap \{b_0, b_2, c_0, c_2\} = \{b_2, c_0\}$ for all $0 \le x \le n_4 - 1$;*

(vii) *$K(e_x, e_{x+1}) \cap \{b_0, b_2, c_0, c_2\} = \{b_0\}$ for all $0 \le x \le n_4 - 1$;*

(viii) *$K(e_x, e_{x+1}) \cap \{b_0, b_2, c_0, c_2\} = \{c_2\}$ for all $0 \le x \le n_4 - 2$, and $\{b_0, b_2, c_0\} \subseteq K(e_{n_4 - 1}, e_{n_4})$; or*

(ix) *$K(e_x, e_{x+1}) \cap \{b_0, b_2, c_0, c_2\} = \{b_0\}$ for all $1 \le x \le n_4 - 1$, and $\{b_2, c_0, c_2\} \subseteq K(e_0, e_1)$.*

Proof. If $c_2 \notin K(e_0, e_1)$, then by Lemma 3.10(I)(ii) and (5.42), we have $c_2 \notin K(e_x, e_{x+1})$ and $K(e_x, e_{x+1}) \cap \{b_0, b_2, c_0\} \neq \emptyset$ for all $0 \le x \le n_4 - 1$, and in view of Claim 5.46, we therefore see that (iv), (vi) or (vii) holds by arguing as in Claim 5.16 of [4]. Thus we may assume

$$(5.43) \qquad c_2 \in K(e_0, e_1).$$

Similarly we may assume $b_0 \in K(e_{n_4-1}, e_{n_4})$.

We first consider the case where

$$K(e_x, e_{x+1}) \cap \{b_2, c_0\} = \emptyset \text{ for all } 1 \le x \le n_4 - 2. \tag{5.44}$$

In this case, arguing as in Lemma 5.16 of [4], we see that there exists p with $1 \le p \le n_4 - 1$ such that

$$\begin{aligned} c_2 &\in K(e_x, e_{x+1}) \text{ for all } 0 \le x \le p-1 \text{ and} \\ b_0 &\in K(e_x, e_{x+1}) \text{ for all } p \le x \le n_4 - 1. \end{aligned} \tag{5.45}$$

Subclaim 1.

(i) If $\{b_2, c_0\} \subseteq K(e_0, e_1)$, then $c_2 \notin K(e_x, e_{x+1})$ for all $1 \le x \le n_4 - 1$ (so $p = 1$).

(ii) If $\{b_2, c_0\} \subseteq K(e_{n_4-1}, e_{n_4})$, then $b_0 \notin K(e_x, e_{x+1})$ for all $0 \le x \le n_4 - 2$ (so $p = n_4 - 1$).

Proof. If $\{b_2, c_0\} \subseteq K(e_0, e_1)$ and there exists x with $1 \le x \le n_4 - 1$ such that $c_2 \in K(e_x, e_{x+1})$, then $x = 1$ by Lemma 3.5, and hence $\{b_2, c_0\} \subseteq K(e_1, e_2)$ by Lemma 3.6, which contradics (5.44) (recall that we are assuming $n_4 \ge 3$ in Proposition 5). This proves (i), and (ii) can be verified in a similar way. ∎

Returning to the proof of the claim, assume for the moment that $\{b_2, c_0\} \subseteq K(e_0, e_1)$. Then by Subclaim 1(i), $c_2 \notin K(e_x, e_{x+1})$ for all $1 \le x \le n_4 - 1$ and $p = 1$, and hence $K(e_{n_4-1}, e_{n_4}) \cap \{b_2, c_0\} = \emptyset$ by Subclaim 1(ii). Consequently, it follows from (5.43) and (5.44) that (ix) holds. Thus we may assume $K(e_0, e_1) \cap \{b_2, c_0\} = \emptyset$. Similarly we may assume $K(e_{n_4-1}, e_{n_4}) \cap \{b_2, c_0\} = \emptyset$, and it therefore follows from (5.44) and (5.45) that (iii) holds.

We now consider the case where (5.44) does not hold. In this case, we get $\{1 \le x \le n_4 - 2 \,|\, \{b_2, c_0\} \subseteq K(e_x, e_{x+1})\} \ne \emptyset$ from Claim 5.46. Let $p = \min\{1 \le x \le n_4 - 2 \,|\, \{b_2, c_0\} \subseteq K(e_x, e_{x+1})\}$ and $q = 1 + \max\{1 \le x \le n_4 - 2 \,|\, \{b_2, c_0\} \subseteq K(e_x, e_{x+1})\}$. Then $p < q$, and by Lemma 3.10(I)(iii), $\{b_2, c_0\} \subseteq K(e_x, e_{x+1})$ for all $p \le x \le q - 1$. By Lemma 3.10(II) and the minimality of p, $K(e_x, e_{x+1}) \cap \{b_0, b_2, c_0\} = \emptyset$ for all $1 \le x \le p - 1$, and hence $c_2 \in K(e_x, e_{x+1})$ for all $0 \le x \le p - 1$ by (5.42) and (5.43). Similarly $b_0 \in K(e_x, e_{x+1})$ for all $q \le x \le n_4 - 1$, and we now see that (i) holds. ∎

By symmetry, we may assume that (i), (ii), (iii), (v), (vi) or (viii) of Claim 5.47 holds. If Claim 5.47(i) holds, then arguing as in Case 1 of Proposition 3 of [4], we see that (G, C) is of Type 15. If Claim 5.47(ii) holds, then combining the arguments in Cases 1 and 2 of Proposition 3 of [4], we see that (G, C) is of Type 16. If Claim 5.47(vi) holds, then arguing as in Case 2 of Proposition 3 of [4], we see that (G, C) is of Type 19. If Claim 5.47(v) holds, then arguing as in Case 2 of Proposition 3 of [4] (we apply the first half of the proof of Claim 5.20 of [4] to $\{e_{n_4-1}, e_{n_4}, b_0\}$ and $\{e_{n_4-1}, e_{n_4}, c_0\}$, and the second half to $\{b_1, b_2, d_0\}$ as well as to $\{b_0, b_1, e_{n_4}\}$, $\{c_0, c_1, e_{n_4}\}$ and $\{c_0, c_1, a_{n_0}\}$; we apply the proof of Claim 5.21 of [4] to $\{a_1, a_2, c_0\}$ and $\{a_1, a_2, d_0\}$ or $\{e_{n_4}, a_0, c_0\}$ and $\{e_{n_4}, a_0, d_0\}$, according to whether $n_0 = 2$ or $n_0 = 0$), we see that (G, C) is of Type 18. If Claim 5.47(viii) holds, then combining the proof of (5.16) of [4] and the proof of Claim 5.38 in Case 2, and applying the first half of the proof of Claim 5.20 of [4] to $\{e_{n_4-2}, e_{n_4-1}, c_0\}$, we see that (G, C) is of Type 20. Thus we may assume that Claim 5.47 (iii) holds. Arguing as in the proof of (5-16) of [4], we obtain:

Claim 5.48.

(i) $\{e_{p+1}e_{p-1}, e_{p+1}c_1\} \cap E(G) \neq \emptyset$.

(ii) $\{e_{p-1}e_{p+1}, e_{p-1}b_1\} \cap E(G) \neq \emptyset$.

Applying the first half of the proof of Claim 5.20 of [4] to $\{e_{p-1}, e_p, b_2\}$ and $\{e_p, e_{p+1}, c_0\}$, we obtain:

Claim 5.49.

(i) $\{b_1c_1, b_1e_{p-1}\} \cap E(G) \neq \emptyset$.

(ii) $\{c_1b_1, c_1e_{p+1}\} \cap E(G) \neq \emptyset$.

Arguing as in Claim 5.37 of Case 2, we obtain:

Claim 5.50. $\{b_1e_{p-1}, b_1e_p, b_1e_{p+1}\} \cap E(G) \neq \emptyset$.

Arguing as in Claim 5.37 with C_2 and a_{n_0} replaced by C_1 and d_0, we obtain:

Claim 5.51. $\{c_1e_{p-1}, c_1e_p, c_1e_{p+1}\} \cap E(G) \neq \emptyset$.

Applying the proof of Claim 5.14 of [4] to $\{e_{n_4}, a_0, d_0\}$ and $\{e_0, d_0, a_{n_0}\}$, we obtain:

Claim 5.52.

(i) *If* $n_0 = 0$ *and* $p = n_4 - 1$, *then* $\{b_1e_{n_4-1}, b_1e_{n_4-2}, c_1e_{n_4-1}, c_1e_{n_4-2}\} \cap E(G) \neq \emptyset$.

(ii) *If* $n_3 = 0$ *and* $p = 1$, *then* $\{c_1e_1, c_1e_2, b_1e_1, b_1e_2\} \cap E(G) \neq \emptyset$.

Now combining Claims 5.48 through 5.52, and applying Lemma 4.6 to C_0 and C_3, and Lemmas 3.3 and 4.7 to C_4, we see that (G, C) is of Type 17.

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