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Local analytic solutions to the DNLS equation in higher dimension

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Abstract. We prove the local well-posedness of the evolution system

$$\begin{cases} q_t + (q(|q|^2 + 2A))_x - (|q|^2 + 2A)_y + iq_{xx} = 0 \\ A_x + \frac{1}{2}(q + \overline{q})_y = 0 \\ q(0, X) = q_o(X) \end{cases}$$

in the analytic Sobolev spaces $X^m(r(T))$, $m \geq 7$, with the initial compatibility conditions $(q_o + \overline{q_o})_y = -2A_{o\,x}$ and $|q_o|^2 + 2A_o = \psi_{o\,x}$, where $A_o, \psi_o \in X^m(r(T))$. This model, due to E. Mjolhus and J. Wyller ([11]), appears in the physical context of Alfvén waves propagating in a parallel (or quasi-parallel) direction to the ambient magnetic field and is a generalisation of the DNLS equation.

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§1. Introduction

When studying weakly nonlinear waves in fluids, the reductive perturbation method ([15],[2]) appears to be a useful unifying technique. In fact, it leads to well known partial differential equations such as the Burgers equation, the Korteweg de Vries (KdV) equation and the Nonlinear Schrödinger Equation. In particular, for weakly nonlinear, weakly dispersive, MHD waves, propagating perpendicular or obliquely to the ambient magnetic field, this reductive perturbation method allows to derive the KdV equation for magnetosonic waves ([8],[9]):

$$u_t + u_{xxx} + uu_x = 0. \tag{1}$$

Then, by taking into account weak dependence on the transverse direction, this KdV equation was generalized in [7] by Kadomtsev and Petviashvili:

$$(2) \qquad \begin{cases} u_t + u_{xxx} + v_y + uu_x = 0 \\ v_x = u_y. \end{cases}$$

Similarly, when dealing with the singular case of parallel propagation, the above mentioned reduction technique results in the Derivative Non Linear Schrödinger equation (DNLS) (see for example [3]):

$$(3) \qquad q_t + iq_{xx} + (|q|^2 q)_x = 0.$$

The DNLS equation has been studied by many authors (see for instance [14], [16],[6], [10]).

Then, by allowing weak dependence on the transverse directions, Mjolhus and Wyller derived a 2-dimensional model for parallel propagating MHD waves, with the assumptions of weak nonlinearity, weak dispersion, weak diffraction and slow evolution (see [11], [13] and [17]), namely

$$(4) \quad \begin{cases} \frac{\partial \overline{B}}{\partial \tau} + \alpha(\frac{\partial}{\partial \xi}((|\overline{B}|^2 + 2B_oB_x)\overline{B}) - B_o\frac{\partial}{\partial \eta}(|\overline{B}|^2 + 2B_oB_x)) + i\beta\frac{\partial^2 \overline{B}}{\partial \xi^2} = 0 \\ \frac{\partial B_x}{\partial \xi} + \frac{\partial B_y}{\partial \eta} = 0, \end{cases}$$

where B_o is the ambient magnetic field, and the magnetic field B is such that

$$B = (B_o + \epsilon B_x, \epsilon^{\frac{1}{2}} B_y, \epsilon^{\frac{1}{2}} B_z),$$

(weak nonlinearity)

$$\overline{B} = B_y + iB_z,$$

$$\alpha = \frac{v_A}{4B_o^2} \text{ and } \beta = \frac{\Omega_i}{2v_A},$$

with $v_A = \frac{B_o^2}{4\pi\rho_o}$ the Alfven velocity and Ω_i the ions gyrofrequency, and finally

$$(5) \qquad \begin{cases} \xi = \epsilon(x - v_A t) \text{ (weak dispersion)} \\ \eta = \epsilon^{\frac{3}{2}} y \text{ (weak diffraction)} \\ \tau = \epsilon^2 t \text{ (slow evolution).} \end{cases}$$

Note that the small parameter ϵ does not appear in (4) since this equation is invariant by the transformation

$$(\overline{B'}, \overline{B'_x}, \xi', \eta', \tau') \rightarrow (\epsilon^{\frac{1}{2}}\overline{B}, \epsilon\overline{B_x}, \epsilon^{-1}\xi, \epsilon^{\frac{-3}{2}}\eta, \epsilon^{-2}\tau).$$

Finally by normalisation of the physical constants, (4) can easily be brought to the canonical form:

$$\begin{cases} q_t + (q(|q|^2 + 2A))_x - (|q|^2 + 2A)_y + iq_{xx} = 0 \\ A_x + \frac{1}{2}(q + \overline{q})_y = 0, \end{cases} \tag{6}$$

which is clearly a 2-dimensional extension of the DNLS equation, in a similar way that the KP equation is a 2-dimensional extension of the KdV equation.

We will be concerned here with the Cauchy problem associated to (6).
Results in the classical Sobolev spaces $H^s(\mathbb{R}^2)$ do not seem easy to get, essentially for two reasons:

First, when considering the linearized problem

$$\begin{cases} q_t + iq_{xx} - 2A_y = 0 \\ A_x + \frac{1}{2}(q + \overline{q})_y = 0, \end{cases} \tag{7}$$

the linear operator which generates the solutions of the Cauchy problem associated to (7) has the form, when $D_x = \frac{1}{i}\partial_x$ and $D_y = \frac{1}{i}\partial_y$,

$$L(D_x, D_y) = e^{itS(D_x, D_y)},$$

where $D_x = \frac{1}{i}\partial_x$, $D_y = \frac{1}{i}\partial_y$, and

$$S(\xi_1, \xi_2) = -\frac{\xi_2^2}{\xi_1} + \sqrt{\xi_1^4 + \frac{\xi_2^4}{\xi_1^2}}.$$

The expression $e^{itS(\xi_1,\xi_2)}$ does not oscillate for fixed ξ_1 and letting ξ_2 tend to infinity. Actually, for fixed ξ_1,

$$\lim_{\xi_2 \to \infty} \exp itS(\xi_1, \xi_2) = 1.$$

For that reason, one cannot obtain any dispersive estimates with smoothing for the free evolution of (6). In particular, local smoothing estimates or Strichartz type inequalities with smoothing do not seem to be available here.
Secondly, we only know three conservation laws for (6)

$$\frac{d}{dt} J_i(t) = 0,$$

with

$$J_o(t) = \int q\,dxdy, \ \ J_1(t) = \int |q|^2 dxdy$$

and

$$J_2(t) = \int [(|q|^2 + 2A)^2 + 2iq\overline{q_x}]dxdy,$$

which are natural extensions of the three first time-invariants for the DNLS equation:

$$J_o = \int q dx, \ J_1 = \int |q|^2 dx$$

and

$$J_2 = \int (|q|^4 + 2iq\overline{q_x})dx.$$

One may try to use these invariants in order to derive a local existence result for (6) in the classical Sobolev spaces $H^s(\mathbb{R}^2)$, similarly to the case of the KdV equation, the KP equation, and many other dispersive models. This energy approach seems to fail in the case of (6) because the nonlinear term contains the formal anti-derivative $\partial_x^{-1}(q+\overline{q})_y$. For this same reason, we were not able to find any a priori estimates in $H^s(\mathbb{R}^2)$ for possible solutions of (6).

However, an easy formal computation shows that, if we take the auxiliar evolution variable $u = |q|^2 + 2A$, we obtain

(8)

$$\begin{cases} q_t + iq_{xx} - u_y + (uq)_x = 0 \\ u_t - \partial_x^{-1} u_{yy} + \frac{i}{2}(q-\overline{q})_{xy} - (u(q+\overline{q}))_y + i(\overline{q}q_{xx} - q\overline{q_{xx}}) + 2Re(\overline{q}(uq)_x) = 0 \end{cases}$$

and we only get an anti-derivative in the linear part. The price to pay for this operation is the appearance of a big "derivative-loss" in the non linear term. Many authors deal with this kind of problem by introducing the analytic Sobolev spaces, which are essentially the functions $f \in L^2(\mathbb{R}^2)$ that can be extended to holomorphic functions in a band containing the real axis. For instance, those spaces were used by Hayashi (see [4],[5]) to overcome derivative losses in nonlinear Schrödinger equations.

In [1], de Bouard extends this result to nonlinear "non elliptic" Schrödinger equations, by proving the well-posedness in the analytic Sobolev spaces of:

$$\begin{cases} i\mathbf{U}_t + L(\mathbf{U}) + G(\mathbf{U}, \overline{\mathbf{U}}, \nabla\mathbf{U}, \nabla\overline{\mathbf{U}}) = 0 \\ \mathbf{U}(X,0) = \mathbf{U}_o(X) \end{cases} \tag{9}$$

where G is a polynomial such that $G(0,0,0,0) = 0$, the self-adjoint operator L is given by

$$L\hat{\mathbf{U}}(\xi) = \mathbf{P}(\xi)\hat{\mathbf{U}}(\xi),$$

and $\mathbf{P}(\xi)$ is a symmetric matrix symbol with real entries $P_j(\xi) \in L^\infty_{loc}$.

Here, we will generalize this result to the case where the linear part presents a singularity of the kind:

$$L = L_1 + L_2, \tag{10}$$

with L_1 self-adjoint as in [1] and

$$L_2 = Diag(i\epsilon_1 \partial_{x_1}^{-1}\partial^2_{x_{k_1}}, ..., i\epsilon_n \partial_{x_n}^{-1}\partial^2_{x_{k_n}}),\ \epsilon_j \in \{0;1\} \text{ and } k_j \in [1,n],$$

since, as we will see later, (8) can be reduced to this form.
The rest of this paper is organized as follows: In the second section, we introduce the analytic spaces and equip them with a local structure, which is crucial for our proof. At the end of this chapter, we will state our main theorem.

In the third section, we prove the local well-posedness of (9), with L given by (10): we first build a family of approximate solutions $\{u_\epsilon\}_{\epsilon>0}$ by regularizing the anti-derivatives, and then take its limit as $\epsilon \to 0$.

In the last section, we apply these previous results to equation (6). For technical reasons, we will be dealing with a system possessing seven unknowns:

$$(q+\overline{q}, q-\overline{q}, (q+\overline{q})_x, (q-\overline{q})_x, (q+\overline{q})_y, (q-\overline{q})_y, |q|^2+2A).$$

Finally, we will return to the initial system (6).

§2. Sobolev analytic spaces

2.1. Definitions and first properties

We begin by introducing the analytic spaces used in [4],[5].

Definition 2.1. *Let $m \in \mathbb{N}$, $r > 0$ and $n \geq 1$.*
We set(analytic Sobolev space of order m)

$$X^m(r) = \{f \in L^2(\mathbb{R}^n), ||f||^2_{X^m(r)} =< q(r,\xi)\sigma^{2m}(\xi)\hat{f}, \hat{f} >_{L^2(\mathbb{R}^n)} < +\infty\}$$

where $\sigma^m(\xi) = (1+|\xi|^2)^{\frac{m}{2}}$ and

$$q(r,\xi) = q_1(r,\xi) + q_2(r,\xi) := \prod_{j=1}^n \cosh(2r\xi_j) + \sum_{k=1}^n \xi_k \sinh(2r\xi_k) \prod_{j\neq k} \cosh(2r\xi_j).$$

We also introduce the Hardy analytic spaces:

Definition 2.2. *Let $r > 0$ and $n \geq 1$:*
We set $S(r) = \{z \in C^n, \forall 1 \leq j \leq n,\ |Imz_j| < r\}$,

$$L_r = \{f \in L^2(\mathbb{R}^n),\ f \text{ is analytic on the band } S(r),$$

$$\text{and } ||f||^2_{L_r} = Sup_{y \in]-r,r[^n} ||f(.+iy)||^2_{L^2(\mathbb{R}^n)} < +\infty\},$$

$$Y_m(\mathbb{R}^n) = \{f \in L_r, ||f||^2_{Y_m(\mathbb{R}^n)} = \sum_{|\alpha| \leq m} ||\partial_z^\alpha f||^2_{L_r} < +\infty\},$$

and, for $m \geq 1$,

$$Y_m^*(\mathbb{R}^n) = \{f / \nabla f \in L_r, ||f||^2_{Y_m^*(\mathbb{R}^n)} = \sum_{1 \leq |\alpha| \leq m} ||\partial_z^\alpha f||^2_{L_r} < +\infty\}.$$

We now state two results proved by Hayashi in [4]:

Lemma 2.3. *Let $f \in L_r$.*
Then

$$\int q_1(r,\xi)|\hat{f}^*(\xi)|^2 d\xi \leq 2^n ||f||^2_{L_r},$$

where f^* is the trace of f on the real axis. Conversely, if $\int q_1(r,\xi)|\hat{f}(\xi)|^2 d\xi$ is finite, then f can be extended to an analytic function over $S(r)$, and

$$||f||^2_{L_r} \leq 2^n \int q_1(r,\xi)|\hat{f}(\xi)|^2 d\xi.$$

Lemma 2.4. *We set*

$$L_{r,2} = \{f \in L^2(\mathbb{R}^n),\ f \text{ is analytic in the band } S(r),$$

$$\text{and } ||f||^2_{L_{r,2}} = \sum_{k=1}^n Sup_{y/y_k \in]-r,r[^{n-1}} \int_{-r}^r ||\partial_{z_k} f(.+iy)||^2_{L^2(\mathbb{R}^n)} dy_k < +\infty\}.$$

Then

$$\int q_2(r,\xi)|\hat{f}(\xi)|^2 d\xi \leq 2^{n-1} ||f||^2_{L_{r,2}}.$$

Conversely, if $\int q_2(r,\xi)|\hat{f}(\xi)^2 d\xi$ is finite, then f admits an analytic extension to the band $S(r)$ and

$$||f||^2_{L_{r,2}} \leq 2^{n-1} \int q_2(r,\xi)|\hat{f}(\xi)|^2 d\xi.$$

This leads to the following corollary:

Corollary 2.5. *Let $m \in \mathbb{N}$ and $r > 0$. Then*

$$Y_{m+1}(r) \subset X^m(r) \subset Y_m(r)$$

Finally, we give a last result:

Lemma 2.6. *Assume that $m \geq [n/2] + 1$. Then, for $w_1, ..., w_k \in Y_m(r)$,*

$$||\prod_{j=1}^{k} w_j||_{Y_m(r)} \leq C \sum_{l=1}^{k} (\prod_{j \neq l} ||w_j||_{Y_{m-1}(r)})||w_l||_{Y_m(r)}.$$

Proof: see [4].
We can state now our main result:

Theorem 2.7. *Let $(X, t) \in \mathbb{R}^2 \times \mathbb{R}$ and $m \geq 7$.*
Let $q_o \in X^m(r_o)$, $r_o > 0$, such that

$$(q_o + \overline{q_o})_y = -2A_{ox} \text{ where } A_o \in X^m(r_o) \tag{11}$$

and

$$|q_o|^2 + 2A_o = \psi_x \text{ with } \psi \in X^m(r_o). \tag{12}$$

Then there exists $M > 0$, $T > 0$, depending only on q_o and $\exists r : [-T, T] \to [r_o, r(T)]$ strictly decreasing on $[0, T]$, even, with $r(0) = r_o$ such that the system

$$\begin{cases} q_t + (q(|q|^2 + 2A))_x - (|q|^2 + 2A)_y + iq_{xx} = 0 \\ A_x + \frac{1}{2}(q + \overline{q})_y = 0 \\ q(0, x, y) = q_o(x, y) \end{cases} \tag{13}$$

has an analytic solution

$$q \in L^\infty(-T, T, X^m(r(T))) \cap L^2(-T, T, Y^*_{m+1}(r(T))),$$

with

$$q' \in L^\infty(-T, T, X^{m-2}(r(T))) \cap L^2(-T, T, Y^*_{m-1}(r(T))).$$

Moreover,

$$q \in C_w([-T, T]; X^m(r(T)))$$

and is unique in the class

$$B^w_m(T) = \{w \in C_w([-T, T]; X^m(r(T))), \\ \sup_{|t| \leq T} ||w||^2_{X^m(r(|t|))} + M \int_{-T}^{T} ||w||^2_{Y^*_{m+1}}(r(t))dt < \infty\}.$$

Here C_w denotes the weak continuity.

Remark 2.8.

Condition (12) is natural, since by differentiating the equation with respect to y and taking the real part, we get:

$$(|q|^2 + 2A)_{yy} = \partial_x[-2A_t + iq_{xy} + (uq)_y].$$

2.2. Local Spaces

In this section, we define the local spaces $X_B^m(r)$ where B is a open ball of $\mathbb{R}^2$.

Lemma 2.9. *Let $r > 0$ and $m \in \mathbb{N}$.*
There exists a family $\{C_\alpha(r)\}_{\alpha\in\mathbb{N}^n}$ of $\mathbb{R}_+^*$, with $\sum_\alpha C_\alpha < +\infty$ such that

$$f \in X^m(r) \text{ if and only if } \sum_{|\alpha|\geq 0} C_\alpha(r)||\partial_X^\alpha f||_{H^m(\mathbb{R}^n)}^2 < +\infty.$$

Moreover,

$$||f||_{X^m(r)}^2 = \sum_{|\alpha|\geq 0} C_\alpha(r)||\partial_X^\alpha f||_{H^m(\mathbb{R}^n)}^2.$$

Proof:
One can write

$$q(r,\xi) = \prod_{j=1}^n(\sum_{k\geq 0}\frac{(2r)^{2k}}{(2k)!}\xi_j^{2k}) + \sum_{j_0=1}^n\prod_{j\neq j_o}(\sum_{k\geq 0}\frac{(2r)^{2k}}{(2k)!}\xi_j^{2k})(\sum_{k\geq 0}\frac{(2r)^{2k+1}}{(2k+1)!}\xi_{j_0}^{2k+2})$$

and, by Fubini-Tonnelli,

$$q(r,\xi) = \sum_{|\alpha|\geq 0} C_\alpha(r)\xi^{2\alpha}.$$

Thus, for $f \in X^m(r)$,

$$\begin{aligned} ||f||_{X_m(r)}^2 &= < q(r,\xi)\sigma^{2m}(\xi)\hat{f},\hat{f} >_{L^2(\mathbb{R}^n)} \\ &= \int q(r,\xi)|\hat{\Lambda^m f}(\xi)|^2 d\xi \\ &= \int (\sum_{|\alpha|\geq 0} C_\alpha(r)\xi^{2\alpha})|\hat{\Lambda^m f}(\xi)|^2 d\xi \end{aligned}$$

and again by Fubini-Tonnelli,

$$||f||_{X_m(r)}^2 = \sum_{|\alpha|\geq 0} C_\alpha(r)||\partial_X^\alpha f||_{H^m(\mathbb{R}^n)}^2.$$

Note that we have obtained another formulation for the scalar product of $X^m(r)$:

$$\begin{aligned}< f,g >_{X_m(r)} &=< q(r,\xi)\sigma^{2m}(\xi)\hat{f},\hat{g} >_{L^2(\mathbb{R}^n)} \\ &= \sum_{|\alpha|\geq 0} C_\alpha(r) < \partial_X^\alpha f, \partial_X^\alpha g >_{H^m(\mathbb{R}^n)} .\end{aligned}$$

Definition 2.10. *For all ball $B \subset \mathbb{R}^n$, we set*

$$X_B^m(r) = \{f \in L^2(B)/||f||^2_{X_B^m(r)} = \sum_{|\alpha|\geq 0} C_\alpha(r)||\partial_X^\alpha f||^2_{H^m(B)} < +\infty\}.$$

We now shortly prove the following elementary property:

Lemma 2.11. *Let $m \in \mathbb{N}$ and $\delta > 0$. Then, for all ball $B \subset \mathbb{R}^n$, the embedding*

$$X_B^{m+\delta}(r) \hookrightarrow X_B^m(r)$$

is compact.

Proof:
Let B_1 be the unit ball of $X_B^{m+\delta}(r)$ and $\{f_n\}_{n\geq 0}$ a sequence in B_1.

For all $n \in \mathbb{N}$,

$$\sum_{|\alpha|\geq 0} C_\alpha(r)||\partial_X^\alpha f_n||^2_{H^{m+\delta}(B)} \leq 1.$$

Thus, for every fixed α, the sequence $||\partial_X^\alpha f_n||_{H^{m+\delta}(B)}$ is bounded. The injection

$$H_B^{m+\delta}(r) \hookrightarrow H_B^m(r)$$

being compact, there exists a subsequence $\{\partial_X^\alpha f_{\phi_\alpha(n)}\}_{n\geq 0}$ which converges to F_α in $H^m(B)$.
Plainly,

$$F_\alpha = \partial_X^\alpha F_o,$$

and using a classical diagonal process, it is clear that there exists a subsequence $f_{\phi(n)}$, still denoted f_n, such that for all $\alpha \in \mathbb{N}^n$,

$$\lim_{n\to\infty} ||\partial_X^\alpha f_n - \partial_X^\alpha F_o||_{H_B^m(r)} = 0,$$

and, for example,

$$||\partial_X^\alpha f_n - \partial_X^\alpha F_o||_{H_B^m(r)} \leq 1, \text{ for all n.}$$

Hence, since $C_\alpha||\partial_X^\alpha f_n - \partial_X^\alpha F_o||_{H_B^m(r)} \leq C_\alpha$ (with $\sum_\alpha C_\alpha < +\infty$), we have by dominated convergence that:

$$\lim_{n\to\infty} ||f_n - F_o||_{X_B^m(r)} = 0,$$

and yet the proposition is proved.

We end this section by the following remark:

Lemma 2.12. *Let $m > 2$ and $n = 2$.*
Then

$$\forall (f,g) \in X^m(r), fg \in X^m(r) \text{ and } ||fg||_{X^m(r)} \leq C||f||_{X^m(r)}||g||_{X^m(r)}.$$

Proof:

From Lemma 2.6, one gets, for $f, g \in X^m(\mathbb{R}^2)$,

$$||fg||_{Y_m(r)} \in C||f||_{Y_m(r)}||g||_{Y_m(r)}.$$

Moreover, note that for $f \in Y_m(\mathbb{R}^n)$,

$$||\partial_z^\alpha f(. + iy)||_{L^2(\mathbb{R}^2)} = ||\partial_X^\alpha f(. + iy)||_{L^2(\mathbb{R}^2)}.$$

Thus, for $\partial_X^\alpha f, \partial_X^\alpha g \in L_{r,2}$ and for all $|\alpha| \leq m$,

$$\begin{aligned}
&\sum_{|\alpha\leq m} ||\partial_X^\alpha (fg)||^2_{L_{r,2}} \\
&= \sum_{|\alpha\leq m} \sum_{k=1}^{n} Sup_{y/y_k\in]-r,r[^{n-1}} \int_{-r}^{r} ||\partial_{z_k}\partial_X^\alpha (fg)(. + iy)||^2_{L^2(\mathbb{R}^n)} dy_k \\
&\leq C \sum_{k=1}^{n} Sup_{y/y_k\in]-r,r[^{n-1}} \int_{-r}^{r} ||\partial_{z_k}(fg)(. + iy)||^2_{H^m(\mathbb{R}^n)} dy_k \\
&\leq C \sum_{k=1}^{n} Sup_{y/y_k\in]-r,r[^{n-1}} \int_{-r}^{r} ||f(. + iy)\partial_{z_k}(g)(. + iy)||^2_{H^m(\mathbb{R}^n)} dy_k \\
&\quad + \int_{-r}^{r} ||g(. + iy)\partial_{z_k}(f)(. + iy)||^2_{H^m(\mathbb{R}^n)} dy_k \\
&\leq C \sum_{k=1}^{n} Sup_{y/y_k\in]-r,r[^{n-1}} \int_{-r}^{r} (||f(. + iy)||^2_{H^m(\mathbb{R}^n)} ||\partial_{z_k}(g)(. + iy)||^2_{H^m(\mathbb{R}^n)} \\
&\quad + ||g(. + iy)||_{H^m(\mathbb{R}^n)} ||\partial_{z_k}(f)(. + iy)||^2_{H^m(\mathbb{R}^n)}) dy_k \\
&\leq C(||f||^2_{Y^m(r)} \sum_{|\alpha|\leq m} ||\partial_X^\alpha g||_{L^2_{r,2}} + ||g||^2_{Y^m(r)} \sum_{|\alpha|\leq m} ||\partial_X^\alpha f||^2_{L_{r,2}})
\end{aligned}$$

and the lemma is proved by Lemmas 2.3 and 2.4.

§3. Approximate solutions for the System (9)

In this section we treat the scalar case, the generalisation for systems being straightforward.

For all $\epsilon > 0$, we consider the approximate system S_ϵ:

$$\left\{ \begin{array}{l} iu_t + L_\epsilon u + G(u, \overline{u}, \nabla u, \nabla \overline{u}) = 0 \\ u(X,0) = u_o(X) \end{array} \right. \tag{14}$$

where $(X,t) \in \mathbb{R}^n \times \mathbb{R}$, G is a polynomial such that $G(0,0,0,0) = 0$ and

$$\hat{L_\epsilon(u)}(\xi) = \xi_{j_o}^2 p_\epsilon(\xi) \text{ with } p_\epsilon(\xi) = \frac{\xi_{j_1}}{(\epsilon + \xi_{j_1}^2)}, \ (j_o, j_1) \in \{1; n\}.$$

The system S_ϵ fulfils the conditions of theorem 1 of ([1]), hence:

Lemma 3.1. *Let $m = 2[n/2] + 5$ and $u_o \in X^m(r_o)$ for some $r_o > 0$.*
Then, there exists $M_\epsilon > 0$ and $T_\epsilon > 0$ such that the system S_ϵ has an analytic solution

$$u_\epsilon(t) \in C([-T_\epsilon, T_\epsilon]; X^m(r(T))) \cap L^2(-T_\epsilon, T_\epsilon; Y^*_{m+1}(r(T)))$$

with, for all $t \in [-T_\epsilon, T_\epsilon]$, $u(t) \in X^m(r(t))$, $r(t) = r_o e^{\frac{-|t| M_\epsilon T_\epsilon}{r_o}}$.
Moreover, this solution is unique in the class:

$$B_m(T) = \{ w \in C([-T_\epsilon, T_\epsilon]; X^m(r(T_\epsilon))),$$

$$||w||^2_{B_m(T)} = \sup_{|t| \le T} ||w||^2_{X^m(r(t))} + M_\epsilon \int_{-T_\epsilon}^{T_\epsilon} ||w||^2_{Y^*_{m+1}}(r(t)) dt < \infty \}.$$

3.1. Uniform estimates for u_ϵ

Lemma 3.2. *In the Lemma 3.1, one can choose $T = T_\epsilon$ and $M = M_\epsilon$ independently of ϵ.*
Moreover, there exists $C = C(u_o) > 0$ such that

$$||u_\epsilon(t)||_{L^\infty(-T,T;X^m(r(T)))} \le C.$$

Proof: The proof of this lemma relies essentially on the proof of Theorem 1 of [1]. For clarity, we will shortly present this proof. For details, one can refer to that paper.

We denote B_ρ the ball of radius $\rho = 2||u_o||_{X^m(r_o)}$ in $B_m(T_\epsilon)$, and for every $v \in B_\rho$, we consider the equation

$$(15) \qquad \begin{cases} iu_{\epsilon t} + L_\epsilon u_\epsilon = -G(v, \overline{v}, \nabla v, \nabla \overline{v}) \\ u(X,0) = u_o(X). \end{cases}$$

By setting $w = w_\epsilon = \partial_X^\beta u_\epsilon$, $|\beta| \leq m$, and taking the Fourier transform of (13), we get

$$(16) \qquad i\hat{w}_t + L_\epsilon \hat{w} = -\partial_X^\beta \hat{F}(v).$$

Multiplying this last expression by $q(r(t),\xi)\overline{w(\xi)}$ and integrating the imaginary part,

$$\frac{\partial}{\partial t}||w(t)||^2_{X_m(r(t))} - 2r'(t) < |\xi^2| q_1(r(t),\xi)\hat{w}(\xi), \hat{w}(\xi) >_{L^2(\mathbb{R}^n)}$$
$$\leq -2Im < \partial_X^\beta \hat{F}(v), q(r(t),\xi)\hat{w}(\xi) >_{L^2(\mathbb{R}^n)},$$

since

$$\partial_t q(r(t),\xi) = \sum_{k=1}^n 2r'(t)\xi_k^2 q_1(r(t),\xi) + h(\xi,t),$$

where $h \leq 0$.

Thus, by Cauchy-Schwarz inequality, and integrating between $-T_\epsilon$ and T_ϵ:

$$||u_\epsilon||^2_{B_m(T)} \leq ||u_o||^2_{X^m(r_o)} + 2\int_{-T_\epsilon}^{T_\epsilon} ||\partial_X^\beta F(v(s))||_{L_{r(t)}} ||w(s)||_{Y_1(r(s))} ds$$

Finally, using the fact that

$$(17) \qquad ||F(v)||_{Y_m(r)} \leq Q(||v||_{Y_m(r)})||v||_{Y_{m+1}(r)} \text{ if } m \geq [n/2]+1$$

where Q is a polynomial with positive coefficients, we get(see [1] for details)

$$(18) \qquad ||u_\epsilon||_{B_m(T)} \leq \frac{\rho}{\sqrt{2}} + 2\rho Q(\rho)(2T_\epsilon + \frac{1}{M_\epsilon}).$$

In order to get a contraction in B_ρ and therefore obtain a fixed point u_ϵ solution of S_ϵ, it is enough to choose M_ϵ and T_ϵ such that

$$2Q(\rho)(2T_\epsilon + \frac{1}{M_\epsilon}) + \frac{1}{\sqrt{2}} < 1, \tag{19}$$

which can be done obviously independently of ϵ. More, for all $t \in [-T, T]$,

$$||u||^2_{B_m(T)} \leq C(u_o)$$

and we have proved Lemmas 3.1 and 3.2.

Remark 3.3. *Uniqueness*

In [1], the uniqueness of the solutions of (14) is given in the class

$$B_m(T) = \{w \in C([-T,T]; X^m(r(T))),$$
$$\sup_{|t|\leq T} ||w||^2_{X^m(r(|t|))} + M\int_{-T}^{T} ||w||^2_{Y^*_{m+1}}(r(t))dt < \infty\}.$$

More precisely, it is shown in [1] that if $u_1, u_2 \in B_\rho$ are two fixed points solutions of (14) corresponding to the same initial data $\phi \in X^m(r_o)$,

$$||u_1 - u_2||_{B_m(T)} \leq (2Q_1(\rho,\rho) + 2\sqrt{\rho}Q_2(\rho,\rho))(2T + \frac{1}{M}||u_1 - u_2||_{B_m(T)})$$

where Q_1 and Q_2 are two polynomials and once again $\rho = 2||\phi||_{X^m(r_o)}$. Therefore, by eventually choosing new values for T and M, we get $u_1 = u_2$.

3.2. Uniform estimates for u'_ϵ

Lemma 3.4. *Assume that* $u_o \in \partial^{-1}_{x_{j_1}} X^m(r_o)$, *i.e.* $u_o = \partial_{x_{j_1}}\phi$, *with* $\phi \in X^m(r_o)$.
Then there exists $C = C(u_o) > 0$ *such that*

$$||u'_\epsilon||_{L^\infty(-T,T;X^{m-2}r(T))} \leq C.$$

Proof:
We begin by proving the existence of u''_ϵ.
We fix $t \in]-T, T[$.
For h small enough (in order to have $t + h \in]-T, T[$), we set

$$f^{(h)}_\epsilon(t) = \frac{1}{h}(u_\epsilon(t+h) - u_\epsilon(t)).$$

It can be seen from the system S_ϵ that

$$u'_\epsilon \in C([-T,T];X^{m-2}(r(T))). \tag{20}$$

Therefore, a standard argument shows that

$$f_\epsilon^{(h)}(t) \to u'_\epsilon \text{ as } h \to 0 \text{ in } X^{m-2}(r(T)) \text{ weak.}$$

An elementary computation shows that

$$\partial_t f_\epsilon^{(h)}(t) + L_\epsilon f_\epsilon^{(h)}(t) + R(f_\epsilon^{(h)}(t), u_\epsilon(t)) = 0,$$

where

$$\begin{aligned} &R(f_\epsilon^{(h)}(t), u_\epsilon(t)) \\ &\quad = f_\epsilon^{(h)}(t)R_1(u_\epsilon) + \overline{f^{(h)}}_\epsilon(t)R_2(u_\epsilon) + \nabla f_\epsilon^{(h)}(t)R_3(u_\epsilon) + \nabla\overline{f^{(h)}}_\epsilon(t)R_4(u_\epsilon) \end{aligned}$$

and

$$R_j(u_\epsilon) = R_j(u_\epsilon(t), \overline{u}_\epsilon(t+h), \nabla u_\epsilon(t), \nabla\overline{u}_\epsilon(t+h))$$

are polynomials.

More, for all t,

$$\begin{cases} u_\epsilon(t+h) \to u_\epsilon(t) \\ \nabla u_\epsilon(t+h) \to \nabla u_\epsilon(t) \text{ in } X^{m-1}(r(T)) \text{ strong.} \end{cases}$$

Hence, $u_{\epsilon tt}(t)$ exists in $X^{m-4}(r(T))$, and, setting $v = v_\epsilon = u_{\epsilon t}$,

$$\begin{cases} v_t + L_\epsilon(v) + R(v(t), u_\epsilon(t)) = 0 \\ v(0) = -L_\epsilon(u_{\epsilon o}) + G(u_{\epsilon o}). \end{cases} \tag{21}$$

We now consider equation (21):
We have $v_o \in X_{m-2}(T)$, $m-2 = 2[n/2]+3$ and

$$||R(v(t), u_\epsilon(t))||_{Y_{m-2}(r)} \le C(||v||_{Y_{m-2}(r)}|| + ||\nabla v||_{Y_{m-2}(r)}).$$

Therefore, the estimate (17) holds for the nonlinear term $R(v, u_\epsilon)$ and we can apply the fixed-point technique described above to equation (21): this equation possesses a unique solution $v_s \in B_{m-2}(T)$. Moreover,

$$||v_s(t)||_{X^{m-2}(r(T))} \le C||v_o||_{X^{m-2}(r_o)}.$$

Also, $u'_\epsilon \in B_m(T)$ satisfies (21). Therefore $u'_\epsilon = v_s$ and

$$||u'_\epsilon(t)||_{X^{m-2}(r(T))} \le C||u'_o(t)||_{X^{m-2}(r(T))}$$

where C is a positive constant independent of ϵ.

Finally,

$$||u_o'||_{X^{m-2}(r_o)} \leq ||L_\epsilon u_o||_{X^{m-2}(r_o)} + ||G(u_o)||_{X^{m-2}(r_o)}$$

and

$$\begin{aligned} ||L_\epsilon u_o||^2_{X^{m-2}(r_o)} &= \int q(r(0),\xi)\sigma^{2(m-2)}(\xi)\frac{\xi_{j_1}^2}{(\epsilon+\xi_{j_1}^2)^2}|\hat{u_o}|^2 d\xi \\ &= \int q(r(0),\xi)\sigma^{2(m-2)}(\xi)\frac{\xi_{j_o}^4\xi_{j_1}^4}{(\epsilon+\xi_{j_1}^2)^2}|\hat{\phi}_o|^2 d\xi \\ &\leq ||\phi_o||^2_{X^m(r(T))}. \end{aligned}$$

§4. Limit of the approximate solutions

Lemma 4.1. *Let $m \geq 2[n/2]+5$ and $u_o \in X^m(r_o)$ such that $\partial_{x_{j_1}}\phi = u_o$ where $\phi \in X^m(r_o)$.*
Then there exists a sequence $\epsilon_n \to 0$ such that

$$u_{\epsilon_n} \to u \text{ in } L^\infty(-T,T;X^m(r(T))) \text{ weak-*}$$

and

$$u'_{\epsilon_n} \to u' \text{ in } L^\infty(-T,T;X^{m-2}(r(T))) \text{ weak-*}.$$

Moreover, for $\alpha \in]0,2[$,

$$u_{\epsilon_n} \to u \text{ in } C([-T,T],X^{m-\alpha}_{loc}(r(T))) \text{ strong.}$$

Proof:
The first two assertions are consequences of Lemmas 3.2 and 3.3.
Since the embedding

$$X_B^{m-2}(r) \hookrightarrow X_B^{m-\alpha}(r)$$

is compact, the standard Aubin's compactness lemma yields, up to a subsequence,

$$u_{\epsilon_n} \to u \text{ in } L^2(-T,T;X^{m-\alpha}_{loc}(r(T))) \text{ strong.}$$

Moreover, one can write, for all $t_o, t_1 \in [-T;T]$:

$$\begin{aligned} & ||u_{\epsilon_n}(t_1) - u_{\epsilon_n}(t_o)||^2_{X^{m-\alpha}} \\ \leq & \int q(r(T),\xi)\sigma^{2(m-\alpha)}||\hat{u_{\epsilon_n}}(t_1) - \hat{u_{\epsilon_n}}(t_1)|^2 d\xi \\ \leq & ||u_{\epsilon_n}(t_1) - u_{\epsilon_n}(t_o)||^{\frac{\alpha}{2}}_{X^{m-2}(r(T))}||u_{\epsilon_n}(t_1) - u_{\epsilon_n}(t_o)||^{1-\frac{\alpha}{2}}_{X^m(r(T))} \\ \leq & C\int_{t_o}^{t_1} ||u'_{\epsilon_n}(\tau)||^{\frac{\alpha}{2}}_{X^{m-2}(r(T))} \\ \leq & C|t_1 - t_o| \end{aligned}$$

and u_{ϵ_n} is equicontinuous in $C([-T;T];X^{m-\alpha}(r(T)))$, $0 < \alpha < 2$.
By the same calculations,

$$||u(t_1) - u(t_o)||^2_{X_{m-\alpha}} \leq C|t_1 - t_o|.$$

Now, by setting $f_n(t) = ||u(t) - u_{\epsilon_n}(t)||^2_{X^{m-\alpha}_{loc}}$,

$$\int_{-T}^{T} f_n(t)dt \to 0 \text{ as } n \to +\infty$$

and f_n is equicontinuous, hence

$$\sup_{[-T,T]} f_n(t) \to 0 \text{ as } n \to +\infty.$$

Lemma 4.2.

$$i\partial_{x_{j_1}} L_{\epsilon_n} u_{\epsilon_n} \to -\partial^2_{x_{j_o}} u \textit{ weakly-* in } L^\infty(-T,T;X^{m-2}(r(T))).$$

Proof:
It is clear that $\partial_{x_{j_1}} L_{\epsilon_n} u_{\epsilon_n} \to v$ in $L^\infty(-T,T;X^{m-2}(r(T)))$ weak-*.
Let $f \in L^1(-T,T;X^{m-2}(r(T)))$.
On one hand,

$$\int_{-T}^{T} < \partial_{x_{j_1}} L_{\epsilon_n} u_{\epsilon_n}(\tau), f(\tau) >_{X^{m-2}(r(T))} d\tau \\ \to \int_{-T}^{T} < v(\tau), f(\tau) >_{X^{m-2}(r(T))} d\tau.$$

On the other hand,

$$\begin{aligned} & \int_{-T}^{T} < \partial_{x_{j_1}} L_{\epsilon_n} u_{\epsilon_n}(\tau), f(\tau) >_{X^{m-2}(r(T))} d\tau \\ = & -\int_{-T}^{T} < \partial^2_{x_{j_o}} u(\tau), f(\tau) >_{X^{m-2}(r(T))} d\tau \\ & + \int_{-T}^{T} < \sigma^{2(m-2)} q(r(T),\xi)\hat{f}(\tau), \frac{\epsilon_n}{\epsilon_n + \xi^2_{j_1}} \hat{\partial_{x_{j_o}} u}(\tau) >_{L^2(\mathbb{R}^n)} . \end{aligned}$$

By dominated convergence, it is clear that the above integral tends to 0, and the lemma is proved.
We can already state that

$$(iu_t + G(u, \overline{u}, \nabla u, \nabla \overline{u}))_{x_{j_1}} = -\partial_{x_{j_o}} u \text{ in } L^\infty(-T, T; X^{m-1}(r(T))).$$

The nonlinear terms also converge, since

$$u_{\epsilon_n} \to u \text{ in } C([-T, T], X_{loc}^{m-\alpha}(r(T))) \text{ strong.}$$

Finally,

Remark 4.3.

One has

$$u \in C_w([-T, T], X^m(r(T))),$$

i.e., for all $\psi \in X^m(r(T))$, $t \to < \psi, u(t) >_{X^m(r(T))}$ is continuous.

In fact, it suffices to notice that

$$q \in L^\infty(-T, T; X^m(r(T)))$$

and

$$q' \in L^\infty(-T, T; X^{m-2}(r(T))).$$

(See for instance [12], Vol.1, Chap.1).
We finish this section by the following theorem:

Theorem 4.4. *Let $(x, t) \in \mathbb{R}^n \times \mathbb{R}$, $m = 2[n/2] + 5$, and G a polynomial such that $G(0, 0, 0, 0) = 0$.*
We assume that for some $r_o > 0$, $u_o \in X^m(r_o)$ such that

$$u_o = \phi_{x_{j_1}} \textit{ with } \phi := \partial_{x_{j_1}}^{-1} u_o \in X^m(r_o).$$

Then, there exists $M > 0$ and $T > 0$, depending only on u_o and $r : [-T, T] \to [r_o, r(T)]$ strictly decreasing over $[0, T]$, even, $r(0) = r_o$, such that the problem

$$\left\{ \begin{array}{l} iu_t + \partial_{x_{j_o}}^2 v + G(u, \overline{u}, \nabla u, \nabla \overline{u}) = 0 \\ v_{x_{j_1}} = u \\ u(0, X) = u_o(X) \end{array} \right. \tag{22}$$

has an analytic solution such that

$$u \in L^\infty(-T, T, X^m(r(T))) \cap L^2(-T, T, Y_{m+1}^*(r(T)))$$

and

$$u' \in L^{\infty}(-T, T, X^{m-2}(r(T))) \cap L^2(-T, T, Y^*_{m-1}(r(T))).$$

Moreover, $u \in C_w([-T,T], X^m(r(T)))$ *and is unique in the class*

$$B^w_m(T) = \{w \in C_w([-T,T]; X^m(r(T))),$$
$$\sup_{|t|\leq T} ||w||^2_{X^m(r(|t|))} + M\int_{-T}^{T} ||w||^2_{Y^*_{m+1}}(r(t))dt < \infty\}.$$

Proof:
We first check that $u \in B^w_m(T)$:
It is easy to see that

$$Sup_{|t|\leq T}||u(t)||_{X^m(r(t))} < \infty$$

by replacing in the proof above t by T.
More, we have, for all $\epsilon > 0$,

$$\int_{-T}^{T} ||u_\epsilon||_{Y^*_{m+1}(r(t))}dt \leq C(u_o).$$

Also,

$$\int_{-T}^{T} ||u_\epsilon||_{Y^*_{m+1}(r(t))}dt = \int_{-T}^{T} ||\sigma_m(\xi)(q_1(r(t),\xi))^{\frac{1}{2}}\xi\hat{u}_\epsilon(\xi,t)||^2_{L^2_\xi}dt.$$

Therefore,

$$\sigma_m(\xi)q_1(r(t),\xi))^{\frac{1}{2}}\xi\hat{u}_\epsilon(\xi,t) \to \psi \text{ in } L^2(-T,T;L^2(\mathbb{R})) \text{ weak.}$$

Since $\sigma_m(\xi)\xi(q_1(r(t),\xi))^{\frac{1}{2}} \in C^\infty([-T,T]\times\mathbb{R})$, a simple argument of uniqueness of the limit in the distributional sense yields

$$\psi = \sigma_m(\xi)(q_1(r(t),\xi))^{\frac{1}{2}}\xi\hat{u}(\xi,t).$$

Uniqueness of this solution can now be obtained as in Remark 3.3, since the same a priori estimate holds here. Therefore, we only need to show that $u(0,.) = u_o(.)$. In fact, for $\phi \in X^m(r(T))$ fixed and for all n,

$$< u_o, \phi >_{X^m(r(T))} = < u_{\epsilon_n}(0), \phi >_{X^m(r(T))} \to < u(0), \phi >_{X^m(r(T))} .$$

§5. Application to Alfvén waves

5.1. The case of systems

As also noticed in [1], it is straightforward to generalize these results to the system (9):

Theorem 5.1. *Let* $(x,t) \in \mathbb{R}^n \times \mathbb{R}$ *et* $m \geq 2[n/2]+5$.
Let $\mathbf{U} = (u_1, u_2, ..., u_d)$ *and* $\{G_j\}_{1\leq j\leq d}$ *d polynomials such that* $G_j(\mathbf{0}) = 0$.
We set $\mathbf{B^w_m}(T) = \{B^w_m\}^d$ *and* $L = L_1 + L_2$,
where L_1 *is a self-adjoint operator with a symmetric matrix symbol with real entries* $P_j \in L^\infty_{loc}$ *and*

$$L_2 = Diag(i\epsilon_1 \partial_{x_1}^{-1}\partial^2_{x_{k_1}}, ..., i\epsilon_n \partial_{x_n}^{-1}\partial^2_{x_{k_n}})$$

where $\epsilon_j \in \{0;1\}$, $k_j \in [1,n]$.
Then, if for each $\epsilon_j = 1$, $u_{oj} \in \partial_{x_1}^{-1} X^m(r_o)$,$r_o > 0$, *there exists* $T > 0$ *and* $M > 0$ *depending only on* $\mathbf{U_o}$ *and* $r : [-T,T] \to [r_o, r(T)]$ *strictly decreasing over* $[0,T]$, *even,* $r(0) = r_o$, *such that the system*

$$\left\{ \begin{array}{l} i\mathbf{U}_t + L\mathbf{U} + G\mathbf{U} = 0 \\ \mathbf{U}(x,0) = \mathbf{U_o}(x) \end{array} \right. \tag{23}$$

where

$$G(\mathbf{U}) = \begin{pmatrix} G_1(\mathbf{U}) \\ \dots \\ G_d(\mathbf{U}) \end{pmatrix},$$

has a solution $\mathbf{U}$ *such that*

$$\mathbf{U} \in L^\infty(-T,T;\mathbf{X}^m(r(T))) \cap L^2(-T,T;\mathbf{Y}^*_{m+1}(r(T)))$$

and

$$\mathbf{U}' \in L^\infty(-T,T;\mathbf{X}^{m-2}(r(T))) \cap L^2(-T,T;\mathbf{Y}^*_{m-1}(r(T))).$$

Moreover,

$$\mathbf{U} \in C_w([-T,T];\mathbf{X}^m(r(T)))$$

and is unique in the class $\mathbf{B^w_m}(T)$.

5.2. Proof of Theorem (2.7)

We consider the following system:

$$\left\{ \begin{array}{l} iQ_t + LQ + G(Q) = 0 \\ Q(X,O) = Q_o(X) \end{array} \right. \tag{24}$$

with $Q = (a, b, \alpha, \beta, \gamma, \delta, u)$,

$$\mathbf{L} = \begin{pmatrix} i\partial_x^{-1}\partial_{yy} & -\partial_{xx} & 0 & 0 & 0 & 0 & 0 \\ -\partial_{xx} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\partial_{xx} & 0 & 0 & 0 \\ 0 & 0 & -\partial_{xx} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i\partial_x^{-1}\partial_{yy} & -\partial_{xx} & 0 \\ 0 & 0 & 0 & 0 & -\partial_{xx} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & i\partial_x^{-1}\partial_{yy} \end{pmatrix}$$

and

$$2G = \begin{pmatrix} ua_x + af(a,b) - \frac{1}{2}(a^2 - b^2)_y \\ -iub_x - ibf(a,b) \\ u_x a_x + u\alpha_x + a_x f(a,b) + 2ag(a,\alpha,b,\beta) - (a_x a_y - b_x b_y + a\alpha_y - b\beta_y) + \gamma_y \\ -iu_x b_x - iu\beta_x - ib_x f(a,b) - ibg(a,\alpha,b,\beta) \\ u_y a_x + u\gamma_x + a_y f(a,b) + 2ah(a,\alpha,b,\beta,\gamma) - (a_y^2 - b_y^2 + a\gamma_y - b\delta_y) \\ -iu_y b_x + u\gamma_x - b_y f(a,b) - ibh(a,\alpha,b,\beta,\gamma) \\ 2\alpha_y + (a\alpha_x - b\beta_x) + 2(u_y a) + 2(ua)_y - 2u_x(a^2 - b^2) \end{pmatrix}$$

where

$$f(a,b) = \frac{1}{4}(a^2 - b^2)_x - a_y,$$

$$g(a,\alpha,b,\beta) = \frac{1}{4}(a_x^2 + a\alpha_x - b_x^2 - b\beta_x + 4\alpha_y),$$

and

$$h(a,\alpha,b,\beta,\gamma) = \frac{1}{4}(2\gamma_y + a_x a_y - b_x b_y + a\alpha_y - b\beta_y).$$

As mentionned in the introduction, this system was obtained heuristically by setting

$$(a, b, \alpha, \beta, \gamma, \delta, u) = (q + \overline{q}, q - \overline{q}, (q + \overline{q})_x, (q - \overline{q})_x, (q + \overline{q})_y, (q - \overline{q})_y, |q|^2 + 2A),$$

in order to apply Theorem 5.1.
By choosing $a_o \in X^m(r_o)$ and $b_o \in X^m(r_o)$ such that

$$a_o = \phi_x, \phi \in X^m(r_o) \text{ and } \frac{1}{4}(a_o^2 - b_o^2) - \phi_y = \psi_x, \psi_x \in X^m(r_o)$$

with

$$Q_o = (a_o, b_o, a_{ox}, b_{ox}, a_{oy}, b_{oy}, (a_o^2 - b_o^2) - \phi_y),$$

the system (24) has a solution in the product space $\mathbf{B^w_m}(T) = (B^w_m(T))^2 \times (B^w_{m-1}(T))^4$, as in theorem 5.1.
We now prove that

$$V(t) = (\alpha - a_x, \beta - b_x, \gamma - a_y, \gamma - b_y, u_x - \frac{1}{4}(a^2 - b^2)_x + a_y) = 0 \text{ for all } t \in [-T; T].$$

By construction, by taking $U = (v_1, v_2, v_3, v_4)$, it is easy to see that $U \in B^w_{m-1}(T)$ satisfies a system of the form

$$U_t + L_1(U) + R_1(Q, \nabla Q) R_2(U, \nabla U) = 0 \tag{25}$$

where R_j are polynomials, $R_2(0,0) = 0$, and Q is a fixed function (the solution of (24) with initial data Q_o).
Since $Q \in B^w_{m-1}(T)$, the computation presented in Remark 3.3 remains valid for example in $B^w_{m-1}(T)$ and therefore, for all times, $U(t) = 0$.
We can now prove by the same method that $v_5 = 0$, and Theorem 2.7 holds for

$$q(x,t) = \frac{1}{2}(a(x,t) + b(x,t)) \text{ and } A(x,t) = -\frac{1}{2}(u(x,t) - |q(x,t)|^2)$$

by looking at the first two lines of the system (24).

Conclusion
We were able to prove the well-posedness of system (9) for analytic initial data. It remains an interesting and important open problem to prove the existence of solutions for less regular initial data, say $H^s(\mathbb{R}^2)$, for some $s > 0$.

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