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A remark on the two level orthogonal array of order 12

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Abstract. A new proof of the uniqueness of $OA(12, 11, 2, 2)$ is proposed. It is shown that any $OA(12, 11, 2, 2)$ can be presented as a juxtaposition of two unique $OA(12, 5, 2, 2)$ and $OA(12, 6, 2, 2)$.

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1. Introduction

An orthogonal array of strength t, index λ, k rows, $k \geq 2$, and s symbols is a $k \times \lambda s^t$ array such that in any $t \times \lambda s^t$ subarray every t-tuple based on s symbols $0, 1, \ldots, s-1$ occurs λ times. Henceforth it will be denoted $OA(\lambda s^t, k, s, t)$. Plackett and Burman (1946) constructed a cyclic orthogonal array with $s = t = 2, \lambda = 3, k = 11$. This array received the attention of several authors in recent years and some of them asserted its uniqueness without giving a supporting reference. Hedayat et al. (1999, §7.5, p. 155) asserted the uniqueness of $OA(12, 11, 2, 2)$ referring to Todd (1933) with no indication of the way it had been shown. Todd proved that the BIB design with parameters $(11, 11, 5, 5, 2)$ is unique. The proof of the uniqueness of $OA(12, 11, 2, 2)$ presented here is direct, using only the basic properties of orthogonal arrays. Three, four and five rowed subarrays of this array were discussed in the statistical literature because they were interesting on their own or proved useful for construction of other arrays.

2. Construction of an orthogonal array with $s = t = 2,\ \lambda = 3,\ k \geq 2$

The main tools of construction of an array are necessary equations $(*)$ for the existence of an orthogonal array as formulated in Hedayat et al. (1997),

$$(*) \qquad \sum_{i=j}^{k} \binom{i}{j} n_i(c) = \binom{k}{j} \lambda s^{t-j}, \quad j = 0, \dots, t,$$

where $n_i(c)$ denotes the number of columns which have i coincidences with any chosen column "c" which belongs to the array. Thus $n_k(c) \geq 1$ and is equal to 1 if the column is not repeated in the array. Subsequently we shall choose "c" to be the first all zero column of the array and the letter "c" will be omitted.

Presently we shall describe the construction of $OA(12, k, 2, 2)$ for $k = 3, 4$ and 5.

Case 1. $k = 3$. The solutions of equations $(*)$ are:

1. $n_3 = 3 \quad n_2 = 0 \quad n_1 = 9 \quad n_0 = 0$
2. $n_3 = 2 \quad n_2 = 3 \quad n_1 = 6 \quad n_0 = 1$
3. $n_3 = 1 \quad n_2 = 6 \quad n_1 = 3 \quad n_0 = 2$

It was shown in Seiden and Zemach (1966) (Proposition 2.7) that if λ is odd and every two columns differ in an even number of elements, then the array cannot be extended beyond $t + 1$ rows. Thus the array satisfying solution 1 cannot be extended. Solutions 2 and 3 are obtainable from each other by permuting the labels. Hence the three row array $OA(12, 3, 2, 2)$ that can be extended beyond three rows is unique.

Case 2. $k = 4$. The solutions of equations $(*)$ are:

1. $n_4 = 2 \quad n_3 = 0 \quad n_2 = 6 \quad n_1 = 4 \quad n_0 = 0$
2. $n_4 = 1 \quad n_3 = 3 \quad n_2 = 3 \quad n_1 = 5 \quad n_0 = 0$
3. $n_4 = 1 \quad n_3 = 2 \quad n_2 = 6 \quad n_1 = 2 \quad n_0 = 1$

The arrays representing these solutions are:

$$\begin{array}{cccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0
\end{array}$$

$$\begin{array}{cccccccccccc}
0&0&0&0&0&0&1&1&1&1&1&1\\
0&0&0&1&1&1&0&0&0&1&1&1\\
0&0&1&0&1&1&0&1&1&0&0&1\\
0&1&0&1&0&1&0&1&1&0&1&0
\end{array}$$

$$\begin{array}{cccccccccccc}
0&0&0&0&0&0&1&1&1&1&1&1\\
0&0&0&1&1&1&0&0&0&1&1&1\\
0&0&1&0&1&1&0&1&1&0&0&1\\
0&1&1&0&0&1&1&0&0&0&1&1
\end{array}$$

It was already pointed out in Cheng (1995) that every $OA(12,4,2,2)$ has at least one column repeated. Note that each of the three arrays has exactly one column repeated. Hence they are isomorphic. The $OA(12,4,2,2)$ is again unique.

Case 3. $k=5$. The solutions of equations $(*)$ are:

1. $n_5=2 \quad n_4=0 \quad n_3=0 \quad n_2=10 \quad n_1=0 \quad n_0=0$
2. $n_5=1 \quad n_4=2 \quad n_3=0 \quad n_2=8 \quad n_1=1 \quad n_0=0$
3. $n_5=1 \quad n_4=1 \quad n_3=3 \quad n_2=5 \quad n_1=2 \quad n_0=0$
4. $n_5=1 \quad n_4=0 \quad n_3=6 \quad n_2=2 \quad n_1=3 \quad n_0=0$
5. $n_5=1 \quad n_4=0 \quad n_3=5 \quad n_2=5 \quad n_1=0 \quad n_0=1$

Solution 2 does not represent an orthogonal array. Since $n_4=2$ it would have to have a three row subarray which satisfies solution $n_3=3 \quad n_2=0 \quad n_1=9 \quad n_0=0$. It was already noticed that such an array cannot be extended. The orthogonal arrays satisfying the remaining four solutions can be constructed easily. The common characteristic of the two arrays satisfying solutions 1 and 4 is having one column repeated. The array satisfying solution 1 can be easily constructed extending in a unique way the first $OA(12,4,2,2)$. The structure of this array can be described independently of the way it was obtained. The two all zero columns are followed by ten distinct columns, each containing two zeros in one of the $\binom{5}{2}$ possible positions in a column. The array satisfying solution 4 can be easily obtained by extending uniquely the second $OA(12,4,2,2)$. It has one repeated column. In fact, interchanging the labels $(0 \leftrightarrow 1)$ in the rows which have a one in the repeated column will yield an array with two all zero columns, which corresponds to solution 1. This proves the uniqueness of the $OA(12,5,2,2)$ with one repeated column. The arrays represented by solutions 3 and 5 are characterized by the property of having no repeated columns. The array satisfying solution 5 can be obtained by extending uniquely the third $OA(12,4,2,2)$. It has two columns with no coincidences, an all zero and all one column and a symmetric distribution of 0's and 1's with respect to these two columns. The array satisfying solution 3 can be obtained by extending any of the three $OA(12,4,2,2)$ in a unique way. It can be verified that each of the resulting arrays has two columns with no coincidences. Thus they are all isomorphic to the array satisfying solution 5. Hence the $OA(12,5,2,2)$ array with no repeated columns is also unique. Thus there are two distinct $OA(12,5,2,2)$, one with a repeated column and the other with no repeated columns.

3. The structure and enumeration of two level orthogonal arrays of order 12

The unique $OA(12,5,2,2)$ with a repeated column and the $OA(12,5,2,2)$ with no repeated columns extended to include the sixth row, are presented hereby to facilitate the subsequent discussion.

$$\begin{array}{cccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0
\end{array}$$

$$\begin{array}{cccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1
\end{array}$$

An $OA(12,11,2,2)$ with the first all zero column has the remaining eleven columns each containing five 0's and six 1's. If we permute its rows so the five zeros of the second column appear in the first five rows, then the resulting array will be a juxtaposition of two arrays isomorphic to the above. Thus any given $OA(12,11,2,2)$ can be presented as a juxtaposition of these two arrays. We turn now to the problem of construction of $OA(12,11,2,2)$. We may assume that the first five rows of the array to be constructed consist of the above $OA(12,5,2,2)$. The question to be solved is to determine a permutation of the $OA(12,6,2,2)$ which will match it. The first, the last and the second column of the $OA(12,6,2,2)$ fit the first, second and the third column of the $OA(12,5,2,2)$. The first two rows of $OA(12,5,2,2)$ determine the distribution of 0's and 1's in each of the six squares composed either of rows 6, 7, 8 or 9, 10, 11 and columns 4, 5, 6, columns 7, 8, 9, and columns 10, 11, 12. In particular the square belonging to rows 6, 7, 8 and columns 4, 5, 6 has to have one 0 and two 1's in each of its rows while the square belonging to rows 9, 10, 11 and columns 4, 5, 6 has to have one 1 and two 0's in each of its rows. Thus we may assume that the first square has 0 in its left diagonal and 1 elsewhere, while the second square has 1 on its left diagonal and 0's elsewhere. This leaves the problem to determine the entries of the remaining six rows and columns. To solve it we use the uniqueness of $OA(12,3,2,2)$ formed by rows 4, 5, 6. There are two ways to complete the entries of the sixth row. One possibility is to assign 0 to columns 8, 11, 12 and 1 to columns 7, 9, 10; the other way is to put 0 in columns 9, 10, 12 and 1 in columns 7, 8, 11. Each of these possibilities determines the remaining entries of $OA(12,11,2,2)$. Thus there are at most two distinct $OA(12,11,2,2)$.

Here are the two resulting arrays:
ARRAY 1

0	0	0	0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	0	0	0	1	1	1
0	0	1	0	1	1	0	1	1	0	0	1
0	0	1	1	0	1	1	0	1	0	1	0
0	0	1	1	1	0	1	1	0	1	0	0
0	1	0	0	1	1	1	0	1	1	0	0
0	1	0	1	0	1	1	1	0	0	0	1
0	1	0	1	1	0	0	1	1	0	1	0
0	1	1	1	0	0	0	0	1	1	0	1
0	1	1	0	1	0	1	0	0	0	1	1
0	1	1	0	0	1	0	1	0	1	1	0

ARRAY 2

0	0	0	0	0	0	1	1	1	1	1	1
0	0	0	1	1	1	0	0	0	1	1	1
0	0	1	0	1	1	0	1	1	0	0	1
0	0	1	1	0	1	1	0	1	0	1	0
0	0	1	1	1	0	1	1	0	1	0	0
0	1	0	0	1	1	1	1	0	0	1	0
0	1	0	1	0	1	0	1	1	1	0	0
0	1	0	1	1	0	1	0	1	0	0	1
0	1	1	1	0	0	0	1	0	0	1	1
0	1	1	0	1	0	0	0	1	1	1	0
0	1	1	0	0	1	1	0	0	1	0	1

Following is a description of a procedure for computing permutations which map Array 1 into Array 2. The first step consists of moving consecutively the five rows having 0 in the last column to become the first five rows. This array is isomorphic to the first five row subarray of Array 2. Numbering its columns in accordance with their positions in the first five row subarray of Array 1 gives the desired column permutation. The positions of the remaining six rows are determined in accordance with the first six entries of their counterparts in Array 1. Analogous procedure will determine the permutations which map Array 2 into Array 1.

Permutations which map Array 1 into Array 2.

Row permutation: 4 5 6 8 11 2 3 1 7 9 10.

(This means: row 4 is moved to the first row, row 5 is moved to the second row, etc.).

Column permutation: 1 3 10 11 6 8 12 4 9 5 7 2.

Permutations which map Array 2 into Array 1.

Row permutation: 4 5 6 7 10 3 1 2 11 8 9.

Column permutation: 1 3 10 11 5 9 12 6 7 4 8 2.

Concluding remark

The uniqueness of $OA(12, 11, 2, 2)$ implies that given any two arrays with the above parameters there are permutations which map each of them into the other. However two isomorphic arrays may differ in their structure and consequently may not be equally interesting for statistical analysis. Properties of the Plackett Burman arrays drew attention of several statisticians.

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