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Some Lightlike Submanifolds

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Abstract. We investigate some properties of lightlike submanifolds especially which are coisotropic and some totally umbilical screen distribution.

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§1. Introduction

Some properties of lightlike submanifolds are studied in [1], [2]. A coisotropic submanifold has some similar properties to a lightlike hypersurface since both have $S(TM^{\perp})=\{0\}$. Moreover some properties of lightlike hypersurface hold on a r-lightlike submanifold with $r < \min\{m, n\}$ and a coisotropic submanifold. The geometry of lightlike submanifold which is not hypersurface is rather complicated than that of lightlike hypersurface. There are four types of lightlike submanifold which is not a curve or a hypersurface.

In this paper, we studied the difference and similarity between the properties of lightlike hypersurface and lightlike submanifold. And also we study the conditions of lightlike submanifold which admits the same result on lightlike hypersurface.

After preliminaries of section 2, in section 3, we consider some lightlike submanifolds. In section 4, we consider the totally umbilical screen distribution of lightlike submanifolds.

§2. Preliminaries

We recall notations and fundamental equations for lightlike submanifolds. They are due to the book [1] by Duggal and Bejancu.

Let $(\bar{M},\bar{g})$ be an $(m+n)$-dimensional semi-Riemannian manifold with the semi-Riemannian metric $\bar{g}$ of constant index $q \in \{1,\cdots,m+n-1\}$ and M be a submanifold of $\bar{M}$ of codimensional n. In this paper, we assume that both m and n are larger than 1.

At a point $u \in M$, we define the orthogonal compliment $T_uM^{\perp}$ of the tangent space T_uM by

$$T_uM^{\perp} = \{V_u \in T_u\bar{M} : \bar{g}(V_u, W_u) = 0 \quad \text{for any } W_u \in T_uM\}.$$

We put $RadT_uM = RadT_uM^{\perp} = T_uM \cap T_uM^{\perp}$. The submanifold M of $\bar{M}$ is said to be an *r-lightlike submanifold* if the mapping

$$RadTM : u \in M \longrightarrow RadT_uM$$

defines a smooth distribution on M of rank $r > 0$. We call $RadTM$ the radical distribution of M. In the sequel, an r-lightlike submanifold will simply be called a lightlike submanifold and g is lightlike metric, unless we need to specify r.

We consider the following four cases of lightlike submanifolds:
Case 1. $0 < r < \min\{m,n\}$.
Case 2. $1 < r = n < m$.
Case 3. $1 < r = m < n$.
Case 4. $1 < r = m = n$.

Case 1. $(0 < r < \min\{m,n\})$. Let $S(TM)$ be a complementary distribution of $RadTM$ in TM. $S(TM)$ is orthogonal to $RadTM$ and non-degenerate with respect to $\bar{g}$. Then TM has the orthogonal direct sum

$$TM = Rad(TM) \perp S(TM).$$

Let $S(TM^{\perp})$ be a complementary distribution of $RadTM$ in $TM^{\perp}$. Let $S(TM^{\perp})$ be a complementary distribution of $RadTM$ in $TM^{\perp}$. Then $TM^{\perp}$ has the following orthogonal direct decomposition

$$TM^{\perp} = Rad(TM) \perp S(TM^{\perp}).$$

Since $S(TM^{\perp})$ is a vector subbundle of the orthogonal complement $S(TM)^{\perp}$ of $S(TM)$ in $T\bar{M}$, and $S(TM)^{\perp}$ and $S(TM^{\perp})$ are non-degenerate, we have the following orthogonal direct decomposition

$$S(TM)^{\perp} = S(TM^{\perp}) \perp S(TM^{\perp})^{\perp}.$$

where $S(TM^{\perp})^{\perp}$ is an orthogonal complement of $S(TM^{\perp})$ in $S(TM)^{\perp}$.

Let $ltr(TM)$ be a complementary vector bundle to $Rad(TM)$ in $S(TM^{\perp})^{\perp}$ on which $\bar{g}$ vanishes. It is called a *lightlike transversal vector bundle* of M

with respect to the pair $((S(TM), S(TM^{\perp}))$. The following range for various indices is used in this paper.

$$i, j, k \cdots \in \{1, \cdots, r\}; a, b, c \cdots \in \{r+1, \cdots, m\}; \alpha, \beta, \gamma \cdots \in \{r+1, \cdots, n\}$$

Let U be a coordinate neighbourhood of M and $\{\xi_i\}_{i=1}^{r}$ be a basis of $\Gamma(RadTM|_U)$. Then there exists a basis $\{N_i\}_{i=1}^{r}$ of $\Gamma(ltr(TM)|_U)$ satisfying the following conditions:

$$g(N_i, \xi_j) = \delta_{ij}$$

Consider the vector bundle

$$tr(TM) = ltr(TM) \perp S(TM^{\perp}),$$

which is called a *transversal vector bundle* of M. Then we obtain

$$\begin{aligned} T\bar{M}|_M &= TM \oplus tr(TM) \\ &= S(TM) \perp S(TM^{\perp}) \perp (RadTM \oplus ltr(TM)) \end{aligned}$$

Hence we have a local quasi-orthonormal field $\{\xi_i, N_i, X_a, W_\alpha\}$ of frames of $T\bar{M}$ along M, where $\{X_a\}$ and $\{W_\alpha\}$ are orthonormal bases of $\Gamma(S(TM)|_U)$ and $\Gamma(S(TM^{\perp})|_U)$, respectively.

Case 2. In this case $Rad(TM)$=$TM^{\perp}$. We call M a *coisotropic submanifold* of $\bar{M}$. Note that $S(TM^{\perp})$=$\{0\}$. Then we have

$$T\bar{M}|_M = TM \oplus ltr(TM) = S(TM) \perp (TM^{\perp} \oplus ltr(TM)).$$

Hence a local quasi-orthonormal field of frames of $T\bar{M}$ along M is given by

$$\{\xi_1, \cdots, \xi_n, N_1, \cdots, N_n, X_{n+1}, \cdots, X_m\}.$$

Case 3. In this case, $RadTM$=TM. M is called to be an *isotropic submanifold* of $\bar{M}$. Note that $S(TM)$=$\{0\}$. Then
we have

$$T\bar{M}|_M = TM \oplus tr(TM) = (TM \oplus ltr(TM)) \perp S(TM^{\perp}).$$

Thus we obtain a local quasi-orthonormal field of frames

$$\{\xi_1, \cdots, \xi_m, N_1, \cdots, N_m, W_{m+1}, \cdots, W_n\}.$$

Case 4. $(1 < r = m = n)$. In this case, $Rad(TM) = TM = TM^{\perp}$, we call M

a *totally lightlike submanifold*. Note that $S(TM) = \{0\} = S(TM^{\perp})$. Hence we have

(2.1) $$T\bar{M}|_M = TM \oplus ltr(TM),$$

and the quasi-orthonormal field of frames of $T\bar{M}$ along M is given by

$$\{\xi_1, \cdots, \xi_m, N_1, \cdots, N_m\}.$$

Suppose $\bar{\nabla}$ is the Levi-Civita connection on $\bar{M}$. We set

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y) \quad \forall X, Y \in \Gamma(TM)$$

and

$$\bar{\nabla}_X V = -A_V X + \nabla^t_X V \quad \forall X \in \Gamma(TM) \quad V \in \Gamma(tr(TM))$$

where $\{\nabla_X Y, A_V X\}$ and $\{h(X,Y), \nabla^t_X V\}$ belong to $\Gamma TM)$ and $\Gamma(tr(TM))$ respectively. Suppose $S(TM^{\perp}) \neq 0$ (either in *Case* 1 or in *Case* 3). We denote by L and S the projection of $tr(TM)$ on $ltr(TM)$ and $S(TM^{\perp})$, respectively. Then we have

$$\bar{\nabla}_X Y = \nabla_X Y + h^{\ell}(X,Y) + h^s(X,Y)$$

and

$$\bar{\nabla}_X V = -A_V X + D^{\ell}_X V + D^s_X V \quad \forall X, Y \in \Gamma(TM) \quad V \in \Gamma(tr(TM))$$

where we put

$$h^{\ell}(X,Y) = L(h(X,Y)) \quad h^s(X,Y) = S(h(X,Y))$$

and

$$D^{\ell}_X V = L(\nabla^t_X V) \quad D^s_X V = S(\nabla^t_X V).$$

Then we put

$$\nabla^{\ell}_X(LV) = D^{\ell}_X(LV) \quad \nabla^s_X(SV) = D^s_X(SV)$$

$$D^{\ell}(X, SV) = D^{\ell}_X(SV) \quad D^s(X, LV) = D^s_X(LV)$$

for any $X \in \Gamma(TM)$ and $V \in \Gamma(tr(TM))$.

Next, suppose M is a coisotropic or a totally lightlike submanifold of $\bar{M}$ either in *Case* 2 or in *Case* 4. Then we have

$$\bar{\nabla}_X Y = \nabla_X Y + h^{\ell}(X,Y)$$

and

$$\bar{\nabla}_X N = -A_N X + \nabla^{\ell}_X N$$

for any $X, Y \in \Gamma(TM)$ and $N \in \Gamma(ltr(TM))$.

We denote by P the projection of TM on $S(TM)$. Suppose M is either an r-lightlike submanifold with $r < \min\{m, n\}$ or a coisotropic submanifold. We set

$$\nabla_X PY = \nabla^*_X PY + h^*(X, PY)$$

and

$$\bar{\nabla}_X \xi = -A^*_\xi X + \nabla^{*t}_X V\xi$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$, where $\{\nabla^*_X PY, A^*_\xi X\}$ and $\{h^*(X, PY), \nabla^{*t}_X \xi\}$ belong to $\Gamma(S(TM))$ and $\Gamma(Rad(TM))$, respectively.

§3. Some lightlike submanifolds

Let (M, g) be a coisotropic submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. M is said to be totally umbilical if there exists a global section $N \in \Gamma(trTM)$ such that

$$h(X, Y) = g(X, Y)N \quad \forall X, Y \in \Gamma(TM). \tag{3.1}$$

If M is a lightlike hypersurface, that is, M is a coisotropic submanifold of rank 1, then we have $tr(TM) = ltr(TM)$ and hence $tr(TM)$ is uniquely determined.

Taking into account that $\bar{\nabla}$ is a metric connection, we obtain

$$(\nabla_X g)(Y, Z) = \bar{g}(h^\ell(X, Y), Z) + \bar{g}(h^\ell(X, Z), Y) \tag{3.2}$$

for any $X, Y, Z \in \Gamma(TM)$.

Theorem 3.1. *Let (M, g) be a totally umbilical coisotropic submanifold of a $m(\geq 3)$-dimensional semi-Riemannian manifold $(\bar{M}(c), \bar{g})$ of constant curvature . Then*

$$-\bar{\tau}_k(N)\bar{\tau}_i(N) + \tau_i(\xi_k) = 0 \tag{3.3}$$

where $\tau_i(X) = \bar{g}(\nabla^\ell_X N, \xi_i)$ and $\bar{\tau}_i(N) = \bar{g}(N, \xi_i)$ for any $N \in \Gamma(ltr(TM))$ and curvature tensor of M is given by

$$\begin{aligned} R(X,Y)Z & \\ = \; & c\{g(Y,Z)X - g(X,Z)Y)\} + g(Y,Z)A_N X - g(X,Z)A_N Y \end{aligned} \tag{3.4}$$

for any $X, Y \in \Gamma(TM)$. Moreover

$$\tau_i(PX) = 0. \tag{3.5}$$

Proof. The essential technique of the proof is given by [1]. If M is a coisotropic submanifold, we derive

$$\bar{R}(X,Y)Z = R(X,Y)Z + A_{h^\ell(X,Z)}Y - A_{h^\ell(Y,Z)}X + (\nabla_X h^\ell)(Y,Z) - (\nabla_Y h^\ell)(X,Z) \tag{3.6}$$

where

$$(\nabla_X h^\ell)(Y,Z) = \nabla^\ell_X(h^\ell(Y,Z)) - h^\ell(\nabla_X Y, Z) - h^\ell(Y, \nabla_X Z). \tag{3.7}$$

For a semi-Riemannian manifold $\bar{M}(c)$ of constant curvature , the curvature tensor field $\bar{R}$ of $\bar{M}$ is given by

$$\bar{R}(X,Y)Z = c\{\bar{g}(Y,Z)X - \bar{g}(X,Z)Y\} \tag{3.8}$$

for any $X, Y, Z \in \Gamma(T\bar{M})$. The right hand side of the above equation belongs to TM if $X, Y, Z \in \Gamma(TM)$. From (3.6) and (3.8), take the tangential part of TM, then we obtain (3.4).

Then we have

$$\begin{aligned} \bar{g}(\bar{R}(X,Y)Z, \xi_i) &= \bar{g}((\nabla_X h^\ell)(Y,Z) - (\nabla_Y h^\ell)(X,Z), \xi_i) \\ &= \{(\nabla_X g)(Y,Z) - (\nabla_Y g)(X,Z)\}\bar{\tau}_i(N) \\ &\quad + g(Y,Z)\tau_i(X) - g(X,Z)\tau_i(Y) \\ &= c\bar{g}(g(Y,Z)X - g(X,Z)Y, \xi_i) \\ &= 0. \end{aligned} \tag{3.9}$$

Take $X = \xi_k$, and $Z = Y \in \Gamma(S(TM)|_U)$ such that $g(Y,Y) \neq 0$, where U is the domain of ξ_k. Then we obtain (3.3) by using (3.2). (3.5) follows from (3.9) taking account (3.1). In fact, take $X = PX, Y = PY, Z = PZ$ in (3.9), we obtain

$$\tau_i(PX)PY = \tau_i(PY)PX.$$

Since $\dim S(TM) > 1$, we can choose PX and PY as a linearly independent system. Hence we obtain (3.5). ∎

Let x be a point of $\bar{M}$ and U be a null vector of $T_x\bar{M}$. A plane H of $T_x\bar{M}$ is called null plane directed by U if it contains U, $\bar{g}_x(U,W) = 0$ for any $W \in H$ and there exists $W_0 \in H$ such that $\bar{g}_x(W_0, W_0) \neq 0$. Define the *null sectional curvature* of K with respect to U and $\bar{\nabla}$, as a real number

$$\bar{K}_U(H) = \frac{\bar{g}_x(\bar{R}(W,U)U,W)}{\bar{g}_x(W,W)}$$

where W is an arbitrary non-null vector in H.

Consider $u \in M$ and a null plane H of T_uM directed by $\xi_u \in Rad(T_uM)$. Similarly, define *the null sectional curvature* of H with respect to ξ_u, and ∇ as a real number

$$K_{\xi_u}(H) = \frac{g_u(R(W_u, \xi_u)\xi_u, W_u)}{g_u(W_u, W_u)}$$

where W_u is an arbitrary non-null vector in H.

In [1], we have the following theorem.

Theorem 3.2. *Let M be an r-lightlike submanifold with $r < \min\{m, n\}$ or a coisotropic submanifold of $\bar{M}$. Then the following assertions are equivalent:*
(1)$RadTM$ is integrable.
(2)The lightlike second fundamental form of M satisfies

$$h^\ell(PX, \xi) = 0 \quad \text{for any } \xi \in \Gamma(RadTM), X \in \Gamma(TM).$$

Using this theorem, we obtain the followings.

Proposition 3.3. *Let M be a coisotropic submanifold and $RadTM$ is integrable. Then $K_{\xi_u}(H) = \bar{K}_{\xi_u}(H)$ for any non-null plane H of T_uM directed by $\xi_u \in Rad(T_uM)$.*

Proof. By straightfoward calculations, we have

$$\begin{aligned} \bar{R}(X, Y, \xi, PU) &= g(R(X, Y)\xi, PU) \\ &+ \bar{g}(h^*(Y, PU), h^\ell(X, \xi)) - \bar{g}(h^*(X, PU), h^\ell(Y, \xi)) \end{aligned}$$

for any $X, Y, U \in \Gamma(TM)$ and $\xi \in \Gamma(RadTM)$.

Now $RadTM$ is integrable, $h^\ell(X, \xi) = 0$ for any $X \in \Gamma(TM)$. So we obtain

$$\bar{R}(X, Y, \xi, PU) = g(R(X, Y)\xi, PU).$$

Then we obtain $K_{\xi_u}(H) = \bar{K}_{\xi_u}(H)$. ∎

Proposition 3.4. *Let M be an r-lightlike submanifold with $r < \min\{m, n\}$. Then we have*

$$\bar{K}_\xi(H) = K_\xi(H) + \frac{\bar{g}(h^s(\xi, PU), h^s(PU, \xi)) - \bar{g}(h^s(U, U), h^s(\xi, \xi))}{g(U, U)}$$

for any $U \in \Gamma(S(TM))$ and $\xi \in \Gamma(RadTM)$.

Proof. We have

$$\begin{aligned} \bar{R}(X, Y, \xi, PU) &= g(R(X, Y)\xi, PU) + \bar{g}(h^s(Y, PU), h^s(X, \xi)) \\ &\quad - \bar{g}((h^s(X, PU), h^s(Y, \xi)) \end{aligned}$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(RadTM)$. Take $X = PU$ and $Y = \xi$, then we obtain the assertion.@ ∎

Since

$$\bar{g}(h^{\ell}(X, PY), \xi) = \bar{g}(A^*_{\xi}X, PY)$$

holds, we obtain the following.

Proposition 3.5. *Let M be a 1-lightlike submanifold. Suppose that $1 < \min\{m, n\}$. Then $A_N X$ is $S(TM)$-valued for $N \in ltr(TM)$ and $X \in TM$.*

Proof. Since rank of $RadTM = 1$, we have rank of $ltr(TM) = 1$. For any $N \in \Gamma(ltr(TM)) and X \in \Gamma(TM)$, we have

$$\bar{g}(A_N X, N) = -\bar{g}(\bar{\nabla}_X N, N) = -\frac{1}{2} X\bar{g}(N, N) = 0.$$

Hence $A_N X$ belongs to $\Gamma(S(TM))$. ∎

§4. Totally umbilical screen distributions of lightlike submanifolds

In section 3, we gave the curvature tensor of totally umbilical coisotropic submanifold M of constant curvature $\bar{M}(c)$, and the sectional curvature of an r-lightlike submanifold M with $r < \min\{m, n\}$. In this section, if M is a lightlike hypersurface or an r-lightlike submanifold with $r < \min\{m, n\}$ of R^{m+n}_q, adding the conditions that RadTM is of rank 1 and both M and $S(TM)$ are totally umbilical, we see that the curvature tensor of M, the sectional curvature spanned by the two unit orthonormal vector fields $X, Y \in \Gamma$ $(S(TM))$ and the null sectional curvature have very simple forms.

Let M be a 1-lightlike submanifold (i.e. rank of $RadTM = 1$) of a semi-Riemannian manifold $\bar{M}$. For any non-degenerate ξ of $RadTM$ on a coordinate neighbourhood $U \subset M$, we take a unique section N of $ltr(TM)$ on U satisfying $\bar{g}(N, \xi) = 1$. We say

that the screen distribution $S(TM)$ is totally umbilical if on any coordinate neighbourhood $U \subset M$, for any non-zero section ξ of

$RadTM$ on U, there exists a smooth function λ such that

$$\bar{g}(h^*(X, PY), N) = \lambda g(X, PY) \quad \text{for any } X, Y \in \Gamma(TM|_U). \tag{4.1}$$

We note that by Chapter 5, Theorem 2.5 in [1], $S(TM)$ is integrable. In case $\lambda = 0$ on U, we say that $S(TM)$ is totally geodesic.

Theorem 4.1. *Let M be a 1-lightlike submanifold with $1 < \min\{m, n\}$ of a semi-Riemannian manifold $\bar{M}(c)$ of constant curvature such that $S(TM)$ is totally umbilical. Then $\lambda \neq 0$ on any $U \subset M$, implies M is totally umbilically immersed in $\bar{M}(c)$. In paticular M is totally geodesically immersed in $\bar{M}(c)$, if and only if, λ is a solution of the partial differential equation*

$$\xi(\lambda) - \lambda\tau(\xi) + c = 0.$$

Proof. This is proved in the same way as a lightlike hypersurface (Chapter4, Theorem 5.4 in [1]). We have

$$\begin{aligned} \bar{g}(\bar{R}(X,Y)PZ,N) &= (\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ) \\ &\quad +\tau(Y)C(X,PZ) - \tau(X)C(Y,PZ) \end{aligned}$$

and

$$\bar{R}(X,Y)Z = c\{g(Y,Z)X - g(X,Z)Y\}$$

where $h^*(X,PY) = C(X,PY)\xi$, $\tau(X) = \bar{g}(\nabla^t_X N, \xi)$ and

$$(\nabla_X C)(Y,PZ) = X(C(Y,PZ)) - C(\nabla_X Y, PZ) - C(Y, \nabla^*_X PZ).$$

Then we obtain

$$\begin{aligned} & (c\eta(X) - X(\lambda) - \lambda\tau(X))g(PY,PZ) \\ = \; & (c\eta(Y) - Y(\lambda) - \lambda\tau(Y))g(PX,PZ) + \lambda\{B(X,PZ)\eta(Y) - B(Y,PZ)\eta(X)\} \end{aligned}$$

where $\eta(X) = \bar{g}(X,N)$ and $h^\ell(X,Y) = B(X,Y)N$ for any $X, Y, Z \in \Gamma(TM|_U)$. Consider $X = \xi$, then

$$\{\xi(\lambda) - \lambda\tau(\xi) + c\}g(PY,PZ) = \lambda B(PY,PZ) \tag{4.2}$$

holds, because of $B(X,\xi) = 0$ for any $X \in \Gamma(TM)$ and $\xi \in \Gamma(RadTM)$. In case $\lambda = 0$ at a point $u \in M$, we have $c = 0$ since $S(TM)$ is non-degenerate. ∎

Proposition 4.2. *Let M be a r-lightlike submanifold with $r < \min\{m, n\}$ of a semi-Riemannian manifold $\bar{M}(c)$ of constant curvature. There exists no lightlike submanifold M in $\bar{M}(c), c \neq 0$, with a totally geodesic screen distribution.*

Proof. Suppose there exists a lightlike submanifold M in $\bar{M}(c)$, $c \neq 0$, with a totally geodesic screen distribution. Then, a contradiction arises from (4.2) with $\lambda = 0$.
∎

Theorem 4.3. *Let M be a lightlike hypersurface of R_q^{m+1}. Suppose M is totally umbilical and $S(TM)$ is totally umbilical. Then*

$$R(X,Y)Z = \lambda\{g(Y,Z)PX - g(X,Z)PY\} \tag{4.3}$$

for any $X, Y, Z \in \Gamma(TM)$ where λ is as in (4.1). Moreover, the sectional curvature determined by two unit orthonormal vector fields $X, Y \in \Gamma(S(TM))$ is given by

$$K(X \Lambda Y) = g(R(X,Y)Y, X) = \lambda. \tag{4.4}$$

In paticular if $\{\xi, W\}$ span the null plane of M, then

$$K(\xi \Lambda W) = 0 \quad \forall \xi \in \Gamma(RadTM) \quad W \in \Gamma(S(TM)). \tag{4.5}$$

Proof. By direct calculations, we find

$$g(A_N X, PW) = \bar{g}(N, h^*(X, PW))$$

and

$$C(X, PW) = \bar{g}(N, h^*(X, PW))$$

for any $X, W \in \Gamma(TM)$. So we have $g(A_N X, PW) = C(Y, PW)$. Since $C(X, PW) = \lambda g(X, PW)$, we obtain

$$A_N X = \lambda PX.$$

From (3.6) and (3.8), taking into account of $c = 0$, we obtain (4.3). (4.4) is obtained by the orthogonality of the two vectors X, Y. (4.5) is obtained since $g(\xi, \xi) = 0$ and $g(\xi, W) = 0$. ∎

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