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Invariance of closed convex sets under semigroups of nonlinear operators in complex Hilbert spaces

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Abstract. From a viewpoint of the general theory for convex functions it is shown that invariance of closed convex subsets under semigroups of nonlinear operators is characterized in complex Hilbert spaces. Some examples, which generalize positivity of semigroups, are given for semigroups generated by nonlinear elliptic operators.

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§1. Introduction

Let H be a complex Hilbert space and A a nonlinear quasi-m-accretive operator with domain $D(A)$ dense in H, that is, $A + \alpha$ is m-accretive in H for some $\alpha \geq 0$ (cf. Kato [**8**, Section V.3.10]). Then $-A$ generates a nonlinear C_0-semigroup $\{S(t)\}_{t\geq 0}$ of type α on H. Let K be a closed convex subset of H and denote by I_K the indicator function of K: $I_K(v) = 0$ if $v \in K$, and $I_K(v) = \infty$ if $v \notin K$. This paper is concerned with the characterizations of the invariance of K under $\{S(t)\}_{t\geq 0}$:

$$S(t)K \subset K \ \ \forall\, t \geq 0. \tag{1.1}$$

In terms of I_K, (1.1) is characterized by

$$I_K(S(t)v) \leq I_K(v) \ \ \forall\, v \in H \ \ \forall\, t \geq 0. \tag{1.2}$$

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A typical example of such a pair of K and $\{S(t)\}_{t\geq 0}$ is the positive cone $L^2_+ := \{u \in L^2(\Omega;\mathbb{R});\ u \geq 0\}$ and positive semigroups (for example, generated by the usual Laplacian Δ with suitable boundary condition) on $H := L^2(\Omega;\mathbb{R})$. It is well-known that positivity of semigroups generated by "linear" elliptic operators is characterized by Kato's inequality (see Arendt [**1**], [**2**] and [**9**]) and by the Beurling-Deny criterion (see e.g. Davies [**7**, Section 1.3], Ouhabaz [**13**] and [**14**]), respectively. Moreover, positivity of semigroups generated by "nonlinear" elliptic operators is characterized by Barthelemy [**3**].

The purpose of this paper is to reveal that there exist simple examples of such closed convex subsets and associated semigroups in the "complex" Hilbert space $L^2(\Omega;\mathbb{C})$. Now let φ be a proper lower semi-continuous convex function on H. Then (1.2) is a particular case of the following condition:

$$\exists\, \beta \geq \alpha\,;\ \ \varphi(S(t)v) \leq e^{2\beta t}\varphi(v)\ \ \forall\, v \in H\ \ \forall\, t \geq 0. \tag{1.3}$$

In particular, if H is a "real" Hilbert space and A is "m-accretive" in H ($\alpha = 0$), then the criteria for (1.3) with $\beta = 0$ have been intensively studied by Brezis [**4**, Section IV.4] (see also Brezis-Pazy [**5**]).

In the first part of this paper we shall give a practical criterion for (1.1) by generalizing [**4**, Theoreme 4.4 and Proposition 4.5] to the case where H is a "complex" Hilbert space and A is "quasi-m-accretive" in H ($\alpha \neq 0$) (see condition (ii) in Theorem 2.4 below).

In the second part we construct two examples of invariant sets under nonlinear semigroups. Let $\{S(t)\}_{t\geq 0}$ be a nonlinear semigroup on $H := L^2(\Omega;\mathbb{C})$ generated by the p-Laplacian with monotone perturbation (see Section 3). Here we consider two examples of closed convex subsets:

Example 1. $K(a_1,a_2,b_1,b_2) := \{u \in H;\ (\operatorname{Re} u, \operatorname{Im} u) \in [-a_1,a_2]\times[-b_1,b_2]\}$, where $a_j, b_j \geq 0$ ($j = 1,2$) (see Figure 1 below).

Example 2. $K(\theta) := \{u \in H;\ |\arg u| \leq \theta\}$, where $0 \leq \theta \leq \pi/2$ (see Figure 2 below).

We shall show that $K(a_1,a_2,b_1,b_2)$ and $K(\theta)$ are invariant under $\{S(t)\}_{t\geq 0}$. Note that $K(0,a_2,0,0)$ tends to the positive cone L^2_+ if $a_2 \to \infty$. On the other hand, it is obvious that $K(0) = L^2_+$. In this sense the subsets in Examples 1 and 2 may be regarded as complex generalizations of L^2_+. Accordingly, the invariance of these subsets complexifies the notion of positivity of semigroups. In fact, we are supposed to show that condition (ii) in Theorem 2.4 is satisfied in order to prove that $K(a_1,a_2,b_1,b_2)$ and $K(\theta)$ are invariant under $\{S(t)\}_{t\geq 0}$. Here we would like to emphasize that the maximum principle for parabolic differential equations does not work in the complex space though it is very strong in the real space.

This paper is organized as follows. In Section 2 we characterize invariance of closed convex subsets under semigroups of nonlinear operators in complex

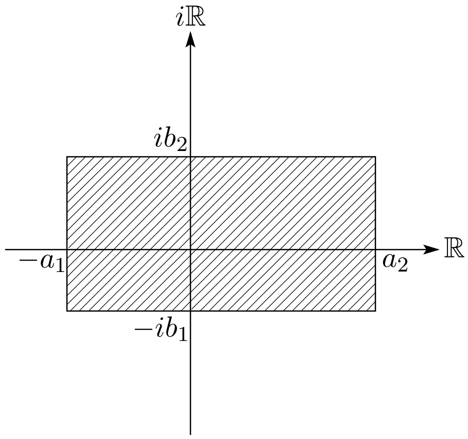

Figure 1: $K(a_1, a_2, b_1, b_2)$

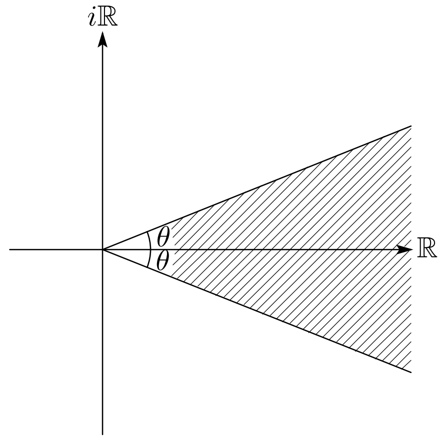

Figure 2: $K(\theta)$

Hilbert spaces by using a general result concerning convex functions. Applying the abstract result prepared in Section 2 to semigroups generated by nonlinear elliptic operators, we construct some examples of closed convex subsets in Section 3. In particular, we shall show that $K(\theta)$ is invariant under nonlinear C_0-semigroups of "type α".

§2. Abstract Results

Let H be a complex Hilbert space and A a nonlinear operator in H such that $A + \alpha$ is m-accretive in H for some $\alpha \geq 0$:

$$\begin{cases} \mathrm{Re}(Au_1 - Au_2, u_1 - u_2) \geq -\alpha \|u_1 - u_2\|^2 \text{ for } u_1, u_2 \in D(A), \\ R(1 + \lambda A) = H \text{ for } \lambda > 0 \text{ with } \lambda\alpha < 1. \end{cases} \tag{2.1}$$

We assume for simplicity that A is single-valued; however, we need not assume that $D(A)$ is dense in H. It is well-known that $-A$ generates a nonlinear C_0-semigroup $\{S(t)\}_{t \geq 0}$ of type α on $\overline{D(A)}$ (the closure of $D(A)$ in H):

$$\begin{cases} S(0) = 1, \ S(t+s) = S(t)S(s) \text{ for } t, s \geq 0, \\ S(t)v \to v \ (t \downarrow 0) \text{ for } v \in \overline{D(A)}, \\ \|S(t)v_1 - S(t)v_2\| \leq e^{\alpha t} \|v_1 - v_2\| \text{ for } v_1, v_2 \in \overline{D(A)} \text{ and } t \geq 0. \end{cases}$$

In this section, given a closed convex subset K of H, we shall present several criteria to guarantee that

$$S(t)(\overline{D(A)} \cap K) \subset K \ \ \forall \, t \geq 0.$$

We modify the arguments in [**4**] and [**5**] in which H is a real Hilbert space and $\alpha = 0$.

First we prepare two lemmas. Let $\{A_\lambda; \lambda > 0\ (\lambda\alpha < 1)\}$ be the Yosida approximation of A:

$$A_\lambda := \lambda^{-1}(1 - (1 + \lambda A)^{-1}) \ \text{ for } \ \lambda > 0 \ \text{ with } \ \lambda\alpha < 1. \tag{2.2}$$

The next lemma shows the accretivity of $A_\lambda + \alpha(1 - \lambda\alpha)^{-1}$. The proof is the same as that of Okazawa [**11**, Lemma 2.2].

Lemma 2.1. *Let A be a nonlinear operator in H such that $A + \alpha$ is m-accretive in H for some $\alpha \geq 0$. Let $\lambda > 0$ with $\lambda\alpha < 1$. Then $A_\lambda + \alpha_\lambda$ is accretive in H:*

$$\operatorname{Re}(A_\lambda v_1 - A_\lambda v_2, v_1 - v_2) \geq -\alpha_\lambda \|v_1 - v_2\|^2, \ \ v_1, v_2 \in H, \tag{2.3}$$

where $\alpha_\lambda := \alpha(1 - \lambda\alpha)^{-1}$.

The following is derived from the maximality of $A + \alpha$.

Lemma 2.2. *Let A be as in* Lemma 2.1*. Let $[v_0, w_0] \in H \times H$. Assume that*

$$\operatorname{Re}(w_0 - Au, v_0 - u) \geq -\alpha\|v_0 - u\|^2 \ \ \forall\, u \in D(A).$$

Then $v_0 \in D(A)$ and $w_0 = Av_0$.

Next we give a general result concerning convex functions. Let φ be a proper lower semi-continuous convex function on H, where proper means that the effective domain $D(\varphi) := \{u \in H; \varphi(u) < \infty\}$ is non-empty. Then the subdifferential $\partial\varphi$ of φ is defined as

$$\partial\varphi(u) := \{f \in H; \operatorname{Re}(f, v - u) \leq \varphi(v) - \varphi(u) \ \ \forall\, v \in H\}$$

for $u \in D(\partial\varphi) := \{u \in D(\varphi); \partial\varphi(u) \neq \emptyset\}$. It is well-known that $\partial\varphi$ is a (possibly) multi-valued m-accretive operator in H: $\operatorname{Re}(w_1 - w_2, u_1 - u_2) \geq 0$ for $w_j \in \partial\varphi(u_j)$ $(j = 1, 2)$. Since $(1 + \mu\partial\varphi)^{-1}$ is single-valued, the Yosida approximation $\{(\partial\varphi)_\mu; \mu > 0\}$ of $\partial\varphi$ is also defined as (2.2) with A and λ replaced with $\partial\varphi$ and μ, respectively. For $\mu > 0$ we set

$$\varphi_\mu(u) := \min_{v \in H}\Big\{\frac{1}{2\mu}\|v - u\|^2 + \varphi(v)\Big\} = \frac{\mu}{2}\|(\partial\varphi)_\mu(u)\|^2 + \varphi((1 + \mu\partial\varphi)^{-1}u).$$

Then φ_μ is Fréchet differentiable on H and the derivative $\partial\varphi_\mu$ coincides with $(\partial\varphi)_\mu$ (see [**4**, Proposition 2.11]). We denote by $P_{\overline{D(A)}}$ the projection of H on $\overline{D(A)}$.

Theorem 2.3. *Let A and $\{S(t)\}_{t\geq 0}$ be as above. Let $\varphi : H \to (-\infty, \infty]$ be a proper lower semi-continuous convex function such that $\varphi(P_{\overline{D(A)}}v) \leq \varphi(v)$ for $v \in H$. Then for $\beta\,(\geq \alpha)$ the following conditions are equivalent*:

(i) $\varphi((1+\lambda A)^{-1}v) \leq (1-2\lambda\beta)^{-1}\varphi(v)$ *for* $v \in H$ *and* $\lambda > 0$ *with* $2\lambda\beta < 1$.

(ii) $\mathrm{Re}(A_\lambda v, w) \geq -2\beta(1-2\lambda\beta)^{-1}\varphi(v)$ *for* $v \in D(\partial\varphi)$, $w \in \partial\varphi(v)$ *and* $\lambda > 0$ *with* $2\lambda\beta < 1$.

(iii) $\mathrm{Re}(A_\lambda v, \partial\varphi_\mu(v)) \geq -2\beta(1-2\lambda\beta)^{-1}\varphi_\mu(v)$ *for* $v \in H$ *and* $\lambda, \mu > 0$ *with* $2\lambda\beta < 1$.

(iv) $\mathrm{Re}(Au, \partial\varphi_\mu(u)) \geq -2\beta\varphi_\mu(u)$ *for* $u \in D(A)$ *and* $\mu > 0$.

(v) $\varphi_\mu((1+\lambda A)^{-1}v) \leq (1-2\lambda\beta)^{-1}\varphi_\mu(v)$ *for* $v \in H$ *and* $\lambda, \mu > 0$ *with* $2\lambda\beta < 1$.

(vi) $\varphi_\mu(S(t)v) \leq e^{2\beta t}\varphi_\mu(v)$ *for* $v \in \overline{D(A)}$, $\mu > 0$ *and* $t \geq 0$.

(vii) $\varphi(S(t)v) \leq e^{2\beta t}\varphi(v)$ *for* $v \in \overline{D(A)}$ *and* $t \geq 0$.

Proof. (i) $\Rightarrow$ (ii). Let $v \in D(\partial\varphi)$ and $w \in \partial\varphi(v)$. Then by definition we see that for $\lambda > 0$ with $2\lambda\beta < 1$,

$$\begin{aligned}\mathrm{Re}(A_\lambda v, w) &= -\lambda^{-1}\mathrm{Re}((1+\lambda A)^{-1}v - v, w)\\ &\geq -\lambda^{-1}(\varphi((1+\lambda A)^{-1}v) - \varphi(v))\\ &\geq -\lambda^{-1}((1-2\lambda\beta)^{-1} - 1)\varphi(v) = -2\beta(1-2\lambda\beta)^{-1}\varphi(v).\end{aligned}$$

(ii) $\Rightarrow$ (iii). It follows from (2.3) that for $v \in H$ and $\lambda, \mu > 0$ with $2\lambda\beta < 1$,

$$\begin{aligned}&\mathrm{Re}(A_\lambda v, \partial\varphi_\mu(v))\\ &= \mu^{-1}\mathrm{Re}(A_\lambda v - A_\lambda(1+\mu\partial\varphi)^{-1}v, v - (1+\mu\partial\varphi)^{-1}v)\\ &\quad + \mathrm{Re}(A_\lambda(1+\mu\partial\varphi)^{-1}v, \partial\varphi_\mu(v))\\ &\geq -\alpha(1-\lambda\alpha)^{-1}\mu^{-1}\|v - (1+\mu\partial\varphi)^{-1}v\|^2 - 2\beta(1-2\lambda\beta)^{-1}\varphi((1+\mu\partial\varphi)^{-1}v)\\ &\geq -2\beta(1-2\lambda\beta)^{-1}\big[(\mu/2)\|\partial\varphi_\mu(v)\|^2 + \varphi((1+\mu\partial\varphi)^{-1}v)\big]\\ &= -2\beta(1-2\lambda\beta)^{-1}\varphi_\mu(v).\end{aligned}$$

(iii) $\Rightarrow$ (iv). It suffices to note that $A_\lambda u \to Au$ ($\lambda \downarrow 0$) in H for every $u \in D(A)$.

(iv) $\Rightarrow$ (v). Let $v \in H$ and $\lambda, \mu > 0$ with $2\lambda\beta < 1$. Then we have

$$\begin{aligned}\varphi_\mu(v) - \varphi_\mu((1+\lambda A)^{-1}v) &\geq \mathrm{Re}(\partial\varphi_\mu((1+\lambda A)^{-1}v), v - (1+\lambda A)^{-1}v)\\ &= \lambda\mathrm{Re}(\partial\varphi_\mu((1+\lambda A)^{-1}v), A(1+\lambda A)^{-1}v)\\ &\geq -2\lambda\beta\varphi_\mu((1+\lambda A)^{-1}v).\end{aligned}$$

(v) $\Rightarrow$ (vi). Let $v \in \overline{D(A)}$, $\mu > 0$, $t \geq 0$ and $n \in \mathbb{N}$ with $n > 2\beta t$. Then we have

$$\varphi_\mu\Big(\Big(1 + \frac{t}{n}A\Big)^{-n} v\Big) \leq \Big(1 - \frac{2\beta t}{n}\Big)^{-n}\varphi_\mu(v).$$

Letting $n \to \infty$, we obtain (vi).

(vi) $\Rightarrow$ (vii). Note that $\lim_{\mu\downarrow 0}\varphi_\mu(v) = \varphi(v)$ for every $v \in H$.

(vii) $\Rightarrow$ (i). Let $v \in H$ and $\lambda > 0$ with $2\lambda\beta < 1$. Take sufficiently small $t > 0$ such that $(\lambda/t)(e^{2\beta t} - 1) < 1$ and set

$$T(t) := S(t)P_{\overline{D(A)}}.$$

Then $T(t)$ is Lipschitz continuous on H with constant $e^{\alpha t}$. Hence it follows that $(1 - T(t)) + (e^{\alpha t} - 1)$ is m-accretive in H: for $w_1, w_2 \in H$,

$$\mathrm{Re}((1 - T(t))w_1 - (1 - T(t))w_2, w_1 - w_2) \geq -(e^{\alpha t} - 1)\|w_1 - w_2\|^2, \tag{2.4}$$

and $R(1 + \mu(1 - T(t))) = H$ for $\mu > 0$ with $\mu(e^{\alpha t} - 1) < 1$ (see (2.1)). Noting that $(\lambda/t)(e^{\alpha t} - 1) < 1$, we can define

$$v_t := (1 + (\lambda/t)(1 - T(t)))^{-1}v.$$

Writing as

$$v_t = \frac{t}{t+\lambda}v + \frac{\lambda}{t+\lambda}T(t)v_t,$$

we see from the convexity of φ and condition (vii) that

$$\varphi(v_t) \leq \frac{t}{t+\lambda}\varphi(v) + \frac{\lambda}{t+\lambda}e^{2\beta t}\varphi(P_{\overline{D(A)}}v_t).$$

Since $\varphi(P_{\overline{D(A)}}v_t) \leq \varphi(v_t)$ by assumption, we obtain

$$\varphi(v_t) \leq \Big(1 - \lambda\frac{e^{2\beta t} - 1}{t}\Big)^{-1}\varphi(v).$$

Since every lower semi-continuous convex function on H is also weakly lower semi-continuous on H, it suffices to show that $v_t \to (1 + \lambda A)^{-1}v$ $(t \downarrow 0)$ weakly in H. To this end let $u \in D(A)$. Noting that $(\lambda/t)(1 - T(t))v_t = (v - u) + (u - v_t)$ and $T(t)u = S(t)u$, we see from (2.4) with w_1 and w_2 replaced with v_t and u that

$$\Big(1 - \lambda\frac{e^{\alpha t} - 1}{t}\Big)\|v_t - u\|^2 \leq \mathrm{Re}\Big(v - u + \lambda\frac{S(t)u - u}{t}, v_t - u\Big).$$

This implies that $\{\|v_t\|\}$ is bounded as $t \downarrow 0$. Hence there exist a sequence $\{v_{t_n}\}$ selected from $\{v_t\}$ and $v_0 \in H$ such that $v_{t_n} \to v_0$ $(n \to \infty)$ weakly in H. So we have

$$(1 - \lambda\alpha)\|v_0 - u\|^2 \leq \mathrm{Re}(v - u - \lambda Au, v_0 - u),$$

and hence

$$\mathrm{Re}(\lambda^{-1}(v - v_0) - Au, v_0 - u) \geq -\alpha\|v_0 - u\|^2.$$

Therefore it follows from Lemma 2.2 that $v_0 \in D(A)$ and $\lambda^{-1}(v - v_0) = Av_0$. This shows that $v_0 = (1 + \lambda A)^{-1}v$. Since we could have started with any sequence selected from $\{v_t\}$ instead of $\{v_t\}$ itself, it follows that $v_t \to (1 + \lambda A)^{-1}v$ $(t \downarrow 0)$ weakly in H and the proof is complete. □

Remark 1. 1) When H is a real Hilbert space and $\alpha = \beta = 0$, Theorem 2.3 is proved in [**5**] (see also [**4**, Theoreme 4.4]).

2) When A is a linear operator and $\partial\varphi$ is a selfadjoint operator, the same results with applications to linear evolution equations of hyperbolic type are established by [**11**] and Okazawa-Unai [**12**].

Now we present the main theorem in this section. Let φ be the indicator function of a closed convex subset K of H. Then φ_μ and $\partial\varphi_\mu$ are given by

$$\varphi_\mu(u) = \frac{1}{2\mu}\|u - P_K u\|^2, \quad \partial\varphi_\mu(u) = \frac{1}{\mu}(u - P_K u)$$

(see [**4**, p. 46]). Therefore Theorem 2.3 yields the following

Theorem 2.4. *Let A and $\{S(t)\}_{t\geq 0}$ be as above. Let K be a closed convex subset of H. Assume that $P_{\overline{D(A)}}K \subset K$. Then for $\beta\,(\geq \alpha)$ the following conditions are equivalent*:

(i) $(1 + \lambda A)^{-1}K \subset K$ *for* $\lambda > 0$ *with* $2\lambda\beta < 1$.

(ii) $\mathrm{Re}(Au, u - P_K u) \geq -\beta\|u - P_K u\|^2$ *for* $u \in D(A)$.

(iii) $\mathrm{dist}((1 + \lambda A)^{-1}v, K) \leq (1 - 2\lambda\beta)^{-1/2}\mathrm{dist}(v, K)$ *for* $v \in H$ *and* $\lambda > 0$ *with* $2\lambda\beta < 1$.

(iv) $\mathrm{dist}(S(t)v, K) \leq e^{\beta t}\mathrm{dist}(v, K)$ *for* $v \in \overline{D(A)}$ *and* $t \geq 0$.

(v) $S(t)(\overline{D(A)} \cap K) \subset K$ *for* $t \geq 0$.

Remark 2. When H is a real Hilbert space and $\alpha = \beta = 0$, Theorem 2.4 is proved in [**5**] (see also [**4**, Proposition 4.5]).

§3. Applications

In this section we shall apply the abstract result prepared in Section 2 to semigroups generated by nonlinear elliptic operators.

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 1)$ with C^1-boundary. Let A be the m-accretive operator in $H := L^2(\Omega; \mathbb{C})$ as defined by

$$\begin{cases} D(A) := \{u \in W_0^{1,p}(\Omega; \mathbb{C}) \cap H;\ \Delta_p u,\ g(|u|^2)u \in H\}, \\ Au := -\Delta_p u + g(|u|^2)u \ \text{ for } \ u \in D(A), \end{cases} \tag{3.1}$$

where $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ with $p > 1$ and $g \in C([0,\infty);\mathbb{R}) \cap C^1((0,\infty);\mathbb{R})$ with

$$(d/ds)[g(s^2)s] = g(s^2) + 2s^2 g'(s^2) \geq 0 \quad \forall\, s > 0.$$

The proof of the m-accretivity of A is summarized as follows. First we introduce two proper lower semi-continuous convex functions on H:

$$\phi(u) := \begin{cases} \dfrac{1}{p}\displaystyle\int_\Omega |\nabla u(x)|^p\,dx & \text{for } u \in W_0^{1,p}(\Omega;\mathbb{C}) \cap H, \\ \infty & \text{otherwise}, \end{cases}$$

$$\psi(u) := \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega G(|u(x)|^2)\,dx & \text{for } u \in H \text{ with } G(|u|^2) \in L^1(\Omega;\mathbb{R}), \\ \infty & \text{otherwise}, \end{cases}$$

where $G(t) := \int_0^t g(s)\,ds$. Then their subdifferentials are given by

$$\partial\phi(u) = -\Delta_p u, \quad \partial\psi(u) = g(|u|^2)u.$$

Next, applying the perturbation theory for nonlinear m-accretive operators (see Brezis-Crandall-Pazy [**6**] and Okazawa [**10**]), we see that $A = \partial\phi + \partial\psi$ is m-accretive in H so that

$$A = \partial(\phi + \psi). \tag{3.2}$$

Hence $-A$ generates a nonlinear contraction semigroup $\{S(t)\}_{t\geq 0}$ on $H = \overline{D(A)}$.

As stated in Section 1, we give two examples of invariant sets under $\{S(t)\}_{t\geq 0}$. The first example is concerned with rectangularly-valued functions. Namely, let $a_j, b_j \geq 0$ $(j = 1, 2)$. Then we consider the rectangle including the origin in $\mathbb{C}$:

$$I := \{\zeta = \xi + i\eta \in \mathbb{C};\ (\xi,\eta) \in [-a_1, a_2]\times[-b_1, b_2]\}.$$

In terms of I we have

$$K(a_1, a_2, b_1, b_2) = \{u \in H;\ u(x) \in I \ \text{ a.a. } x \in \Omega\}$$

which may be regarded as a generalization of the positive cone L^2_+.

Theorem 3.1. *Let A be the operator as defined by* (3.1) *and $\{S(t)\}_{t\geq 0}$ a nonlinear contraction semigroup on H generated by $-A$. Then for $a_j, b_j \geq 0$ $(j = 1, 2)$, $K(a_1, a_2, b_1, b_2)$ is invariant under $\{S(t)\}_{t\geq 0}$:*

$$S(t)K(a_1, a_2, b_1, b_2) \subset K(a_1, a_2, b_1, b_2) \ \ \forall\, t \geq 0.$$

To prove Theorem 3.1 we have only to show that condition (ii) in Theorem 2.4 is satisfied. For the purpose we prepare two lemmas. First the projection of $\mathbb{C}$ on I is expressed in the following form.

Lemma 3.2. *Let I be as above. Then for $\zeta = \xi + i\eta \in \mathbb{C}$,*

$$P_I\zeta = \frac{1}{2}(|\xi + a_1| - |\xi - a_2| - a_1 + a_2) + \frac{i}{2}(|\eta + b_1| - |\eta - b_2| - b_1 + b_2).$$

Since $(P_{K(a_1,a_2,b_1,b_2)}u)(x) = P_I(u(x))$ for a.a. $x \in \Omega$, we can obtain the expression for $P_{K(a_1,a_2,b_1,b_2)}$ by virtue of Lemma 3.2. Next we have

Lemma 3.3. *Let $u \in W_0^{1,p}(\Omega;\mathbb{C})$. Then $P_{K(a_1,a_2,b_1,b_2)}u \in W_0^{1,p}(\Omega;\mathbb{C})$ and*

$$\nabla(P_{K(a_1,a_2,b_1,b_2)}u) = \delta_{a_1,a_2}(\operatorname{Re} u)\nabla(\operatorname{Re} u) + i\delta_{b_1,b_2}(\operatorname{Im} u)\nabla(\operatorname{Im} u),$$

where $\delta_{c_1,c_2}(v)$ is given by

$$\delta_{c_1,c_2}(v) := \begin{cases} 1 & \text{if } -c_1 < v < c_2, \\ 1/2 & \text{if } v = -c_1,\ c_2, \\ 0 & \text{if } v < -c_1,\ c_2 < v. \end{cases}$$

Proof. Let $v \in W_0^{1,p}(\Omega;\mathbb{R})$ and for $c_1, c_2 \geq 0$ set

$$f(s) := \frac{1}{2}(|s + c_1| - |s - c_2| - c_1 + c_2).$$

Then it suffices to show that $f(v) \in W_0^{1,p}(\Omega;\mathbb{R})$ and $\nabla(f(v)) = \delta_{c_1,c_2}(v)\nabla v$. Since $|f(v)| \leq |v|$, it follows that $f(v) \in L^p(\Omega;\mathbb{R})$. For $\varepsilon > 0$ define

$$f_\varepsilon(s) := \frac{1}{2}\big(((s + c_1)^2 + \varepsilon)^{1/2} - ((s - c_2)^2 + \varepsilon)^{1/2} - (c_1^2 + \varepsilon)^{1/2} + (c_2^2 + \varepsilon)^{1/2}\big).$$

Then we see from the chain rule that $f_\varepsilon(v) \in W_0^{1,p}(\Omega;\mathbb{R})$ and

$$\nabla(f_\varepsilon(v)) = \frac{1}{2}\left(\frac{v + c_1}{((v + c_1)^2 + \varepsilon)^{1/2}} - \frac{v - c_2}{((v - c_2)^2 + \varepsilon)^{1/2}}\right)\nabla v.$$

Noting that

$$f_\varepsilon(v) \to f(v) \ \text{ and } \ \nabla(f_\varepsilon(v)) \to \delta_{c_1,c_2}(v)\nabla v \ \ (\varepsilon \downarrow 0) \ \text{ in } \ L^p(\Omega;\mathbb{R}),$$

we can conclude that $f(v) \in W_0^{1,p}(\Omega;\mathbb{R})$ and $\nabla(f(v)) = \delta_{c_1,c_2}(v)\nabla v$. □

Now we are in a position to complete

Proof of Theorem 3.1. Put $K := K(a_1, a_2, b_1, b_2)$ and let $u \in D(A) \subset W_0^{1,p}(\Omega;\mathbb{C})$. Since $|\nabla(P_K u)| \le |\nabla u|$ by Lemma 3.3, we have

$$\operatorname{Re}(-\Delta_p u, u - P_K u) = \int_\Omega |\nabla u|^{p-2}(|\nabla u|^2 - \operatorname{Re}(\nabla u \cdot \overline{\nabla(P_K u)}))\,dx \ge 0. \tag{3.3}$$

On the other hand, noting that $g \ge 0$ and $|P_K u| \le |u|$, we obtain

$$\operatorname{Re}(g(|u|^2)u, u - P_K u) = \int_\Omega g(|u|^2)(|u|^2 - \operatorname{Re}(u\overline{P_K u}))\,dx \ge 0. \tag{3.4}$$

Adding (3.3) and (3.4) yields that $\operatorname{Re}(Au, u - P_K u) \ge 0$. Therefore the assertion follows from Theorem 2.4 (ii) $\Rightarrow$ (v). □

The second example is concerned with sectorially-valued functions. Namely, let $0 \le \theta \le \pi/2$. Then we consider the sector in $\mathbb{C}$:

$$\Sigma := \{z \in \mathbb{C};\ |\arg z| \le \theta\}.$$

In terms of Σ we have

$$K(\theta) = \{u \in H;\ u(x) \in \Sigma \ \text{ a.a. } x \in \Omega\}$$

which may be also regarded as a generalization of the positive cone L^2_+.

Theorem 3.4. *Let A be the operator as defined by* (3.1) *and $\{S(t)\}_{t\ge 0}$ a nonlinear contraction semigroup on H generated by $-A$. Then for $0 \le \theta \le \pi/2$, $K(\theta)$ is invariant under $\{S(t)\}_{t\ge0}$:*

$$S(t)K(\theta) \subset K(\theta) \ \ \forall\, t \ge 0.$$

As in the proof of Theorem 3.1, we prove Theorem 3.4 by using Theorem 2.4. To see this we need two lemmas. First we can easily obtain

Lemma 3.5. *Let $0 \le \theta \le \pi/2$. Then for $z \in \mathbb{C}$,*

$$P_\Sigma z = \begin{cases} z & \textit{on } \Sigma = \{z \in \mathbb{C};\ |\arg z| \le \theta\}, \\ (1/2)(z + e^{2i\theta}\overline{z}) & \textit{on } \Sigma_1 := \{z \in \mathbb{C};\ \theta < \arg z < \theta + \pi/2\}, \\ 0 & \textit{on } \Sigma_2 := \{z \in \mathbb{C};\ \theta + \pi/2 \le |\arg z| \le \pi\}, \\ (1/2)(z + e^{-2i\theta}\overline{z}) & \textit{on } \Sigma_3 := \{z \in \mathbb{C};\ -(\theta + \pi/2) < \arg z < -\theta\}. \end{cases}$$

In view of Lemma 3.5 we can obtain the expression for $(P_{K(\theta)}u)(x) = P_\Sigma(u(x))$. It would be difficult to use the approximating argument for $P_{K(\theta)}u$ as in the proof of Lemma 3.3; nevertheless, we can obtain

Lemma 3.6. *Let* $u \in W_0^{1,p}(\Omega;\mathbb{C})$. *Then* $P_{K(\theta)}u \in W_0^{1,p}(\Omega;\mathbb{C})$ *and*

$$\partial_{x_j}(P_{K(\theta)}u) = \begin{cases} \partial_{x_j}u & \text{on } \Omega \setminus \bigcup_{k=1}^{3} \Omega_{j,k}, \\ (1/2)(\partial_{x_j}u + e^{2i\theta}\partial_{x_j}\overline{u}) & \text{on } \Omega_{j,1}, \\ 0 & \text{on } \Omega_{j,2}, \\ (1/2)(\partial_{x_j}u + e^{-2i\theta}\partial_{x_j}\overline{u}) & \text{on } \Omega_{j,3}, \end{cases}$$

where $\Omega_{j,k}$ $(k = 1, 2, 3)$ *are disjoint subsets of* Ω.

Proof. Let $u \in W_0^{1,p}(\Omega;\mathbb{C})$. It then follows that for $\varphi \in C_0^\infty(\Omega;\mathbb{C})$,

$$\begin{aligned} -\int_\Omega (P_{K(\theta)}u)(x)\,\partial_{x_j}\varphi(x)\,dx &= -\lim_{h\downarrow 0}\int_\Omega P_\Sigma(u(x))\frac{\varphi(x)-\varphi(x-he_j)}{h}\,dx \\ &= \lim_{h\downarrow 0}\int_\Omega \frac{P_\Sigma(u(x+he_j)) - P_\Sigma(u(x))}{h}\varphi(x)\,dx \\ &= \lim_{h\downarrow 0}\int_\Omega [I_1(x,h) + I_2(x,h)]\varphi(x)\,dx, \end{aligned}$$

where Σ is the same as in Lemma 3.5 and

$$\begin{aligned} I_1(x,h) &:= \frac{1}{h}\big[P_\Sigma(u(x+he_j)) - P_\Sigma\big(u(x) + h\partial_{x_j}u(x)\big)\big], \\ I_2(x,h) &:= \frac{1}{h}\big[P_\Sigma\big(u(x) + h\partial_{x_j}u(x)\big) - P_\Sigma(u(x))\big]. \end{aligned}$$

From the dominated convergence theorem it suffices to compute $\lim_{h\downarrow 0} I_1(x,h)$ and $\lim_{h\downarrow 0} I_2(x,h)$ for a.a. $x \in \Omega$. Since the projection is nonexpansive, we have

$$|I_1(x,h)| \le \left|\frac{u(x+he_j) - u(x)}{h} - \partial_{x_j}u(x)\right| \to 0 \ \ (h \downarrow 0).$$

On the other hand, we can compute $\lim_{h\downarrow 0} I_2(x,h)$ as follows:

Case i) $u(x) \in \mathbb{C}\setminus(\partial\Sigma \cup \partial\Sigma_2)$. We see that for sufficiently small $h > 0$,

$$I_2(x,h) = \begin{cases} \partial_{x_j}u & \text{if } u(x) \in \Sigma\setminus\partial\Sigma, \\ (1/2)(\partial_{x_j}u + e^{2i\theta}\partial_{x_j}\overline{u}) & \text{if } u(x) \in \Sigma_1, \\ 0 & \text{if } u(x) \in \Sigma_2\setminus\partial\Sigma_2, \\ (1/2)(\partial_{x_j}u + e^{-2i\theta}\partial_{x_j}\overline{u}) & \text{if } u(x) \in \Sigma_3. \end{cases}$$

Case ii) $u(x) \in \partial\Sigma \cup \partial\Sigma_2$. In this case, $I_2(x,h)$ depends on the argument of the complex number $\partial_{x_j}u(x)$: for sufficiently small $h > 0$,

$$I_2(x,h) = \begin{cases} \partial_{x_j}u & \text{if } u(x) + h\partial_{x_j}u(x) \in \Sigma, \\ (1/2)(\partial_{x_j}u + e^{2i\theta}\partial_{x_j}\overline{u}) & \text{if } u(x) + h\partial_{x_j}u(x) \in \Sigma_1, \\ 0 & \text{if } u(x) + h\partial_{x_j}u(x) \in \Sigma_2, \\ (1/2)(\partial_{x_j}u + e^{-2i\theta}\partial_{x_j}\overline{u}) & \text{if } u(x) + h\partial_{x_j}u(x) \in \Sigma_3. \end{cases}$$

Therefore we can obtain the assertion. □

Now we can complete

Proof of Theorem 3.4. Let $u \in D(A)$. Then it follows from Lemmas 3.5 and 3.6 that $|P_{K(\theta)}u| \le |u|$ and $|\nabla(P_{K(\theta)}u)| \le |\nabla u|$. Therefore in the same way as in the proof of Theorem 3.1 we can obtain

$$\operatorname{Re}(Au, u - P_{K(\theta)}u) \ge 0. \tag{3.5}$$

Thus condition (ii) in Theorem 2.4 is satisfied and hence the proof is complete. □

Finally, we shall show that $K(\theta)$ is invariant under semigroups of type α which are not necessarily contractions. Let $\alpha \ge 0$. Since $A = (A - \alpha) + \alpha$ is m-accretive in H, it follows that $-(A-\alpha)$ generates a nonlinear C_0-semigroup $\{U(t)\}_{t\ge 0}$ of type α on H. Then we have

Theorem 3.7. *Let $\{U(t)\}_{t\ge0}$ be a nonlinear C_0-semigroup of type α on H generated by $-(A - \alpha)$, where A is the same as in* Theorem 3.4. *Then $K(\theta)$ is invariant under $\{U(t)\}_{t\ge0}$:*

$$U(t)K(\theta) \subset K(\theta) \ \ \forall\, t \ge 0.$$

Proof. First we note that for $u \in H$,

$$\operatorname{Re}(u, u - P_{K(\theta)}u) = \|u - P_{K(\theta)}u\|^2. \tag{3.6}$$

In fact, let $z \in \mathbb{C}$. Then we see from Lemma 3.5 that $\operatorname{Re}(z\overline{P_\Sigma z}) = |P_\Sigma z|^2$ and hence $\operatorname{Re}(z\overline{(z - P_\Sigma z)}) = |z - P_\Sigma z|^2$. Setting $z = u(x)$ and integrating it over Ω, we can obtain (3.6). Next let $u \in D(A)$. Then (3.5) and (3.6) yield that

$$\operatorname{Re}((A-\alpha)u, u - P_{K(\theta)}u) \ge -\alpha\|u - P_{K(\theta)}u\|^2.$$

Therefore the assertion follows from Theorem 2.4 (ii) ⇒ (v). □

Remark 3. Let A and $\{U(t)\}_{t\ge0}$ be the same as above. Since $A = \partial(\phi+\psi)$ by (3.2), we can prove the smoothing effect such that $U(t) : H = \overline{D(A)} \to D(A)$ for every $t > 0$ in the same way as in the real space case [**4**]. Hence for every $u_0 \in H$, $u(t) := U(t)u_0$ is a unique strong solution to the initial-boundary value problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}(|\nabla u|^{p-2}\nabla u) + g(|u|^2)u - \alpha u = 0, & (x,t) \in \Omega \times (0,\infty), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases}$$

Theorem 3.7 implies that $u(t) \in K(\theta)$ if $u_0 \in K(\theta)$.

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