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On Local and Superlinear Convergence of Secant Methods for Nonlinear Equations

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Abstract. This paper considers local convergence of secant methods for a nonlinear system of equations. The well-known local convergence theory has been developed by Broyden, Dennis and Moré (1973). They used a norm inequality such that the difference between two vectors transformed by some matrix is bounded above by an order of one of the two. Instead, in the present paper, we use an inequality that bounds the angle between the vectors. This inequality has a merit of scale invariance whereas the norm inequality does not.

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§1. Introduction

We consider iterative methods for solving the system of nonlinear equations

$$F(x) = 0, \qquad x \in \mathbb{R}^n, \tag{1.1}$$

where $F(x) = (F_1(x), \ldots, F_n(x))^\top$, and each $F_i : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable for $i = 1, \ldots, n$. The best known iterative method for solving this problem is Newton's method. The method generates a sequence $\{x_k\}$, for a given $x_0 \in \mathbb{R}^n$, by the iteration

$$x_{k+1} = x_k - F'(x_k)^{-1} F(x_k), \qquad k = 0, 1, \ldots,$$

where $F'(x_k)$ denotes the $n \times n$ Jacobian matrix of F at x_k whose (i, j)th element is $\partial F_i(x_k)/\partial x_j$. Newton's method is locally convergent and very fast; its rate of convergence is usually quadratic. However, one of the major drawbacks of this method is that the Jacobian matrix is required at every iterate.

Since $F'(x_k)$ is often expensive to compute, it is approximated by a matrix B_k, and a sequence $\{x_k\}$ is defined, for a given $x_0 \in \mathbb{R}^n$ and $B_0 \in \mathbb{R}^{n\times n}$, by

$$x_{k+1} = x_k - B_k^{-1} F(x_k), \qquad k = 0, 1, \ldots,$$

where a new approximation B_{k+1} is commonly obtained from B_k by using a low-rank (usually rank-one or two) updating formula. Because B_k is intended to be an approximation to the Jacobian $F'(x_k)$, the next approximation B_{k+1} is often required to satisfy the secant equation

$$B_{k+1} s_k = y_k,$$

where $s_k = x_{k+1} - x_k$ and $y_k = F(x_{k+1}) - F(x_k)$. A method with such an update is called a secant (or quasi-Newton) method. The choices of different updates give different methods. These methods are closely related to the ones for solving nonlinear minimization problems. Indeed, nonlinear equations can easily be found in differentiable unconstrained minimization. In this case, nonlinear equations to be solved are derived from the first-order necessary conditions, namely, F is the gradient of some function to be minimized. For more details, see a standard textbook, for example, Dennis and Schnabel [4].

Various secant methods for solving problem (1.1) and the minimization problem have been proposed, and their convergence properties have been studied by many researchers. Among those, one of the well-known and remarkable results is the one developed by Broyden, Dennis and Moré [1]. The theory includes so-called bounded deterioration properties, originally investigated by Dennis. This property plays a key role in a local convergence analysis for secant methods. They used a norm inequality such that the difference between two vectors suitably scaled by some matrix is bounded above by an order of one of the two.

In the present paper, we give a bounded deterioration property based on an inequality different from the norm inequality that Broyden, Dennis and Moré [1] used. The underlying inequality has a very simple geometric meaning that it bounds the angle between the suitably scaled vectors. Moreover, the inequality has a merit that it is invariant to the size, while the norm inequality does not. There is a case that the scale invariance enables us to simplify the proof of a local convergence property of a structured secant method.

In Section 2, we describe our notation, basic properties and some assumptions employed. In Section 3, we deal with rank-one secant updates. We show local superlinear convergence of the rank-one secant methods. Symmetry requirement for updates arises naturally in the case for smooth unconstrained minimization. However, rank-one updates are in general *not* symmetric. For this reason, in Section 4, we consider *symmetric* rank-two secant updates. We show local superlinear convergence of the rank-two secant methods.

§2. Notation, Basic Properties and Assumptions

Throughout this paper, symbol I denotes the identity matrix of order n. Superscripts $\top$ and $-\top$ denote, respectively, the transpose of a vector or a matrix and the transpose of an inverse matrix. The latter equals the inverse of a transposed matrix. Norm $\|\cdot\|$ denotes the l_2-vector norm or its induced operator norm. Norms $\|\cdot\|_{\mathrm{F}}$ and $\|\cdot\|_{M,N}$ denote the Frobenius norm and a weighted Frobenius norm defined as

$$\|A\|_{\mathrm{F}} \equiv \sqrt{\mathrm{tr}(A^\top A)} \quad \text{and} \quad \|A\|_{M,N} \equiv \|NAM^\top\|_{\mathrm{F}} \quad \text{for } A \text{ in } \mathbb{R}^{n\times n},$$

where M and N are nonsingular matrices in $\mathbb{R}^{n\times n}$. In particular, we denote $\|A\|_{M,M}$ by $\|A\|_M$ simply. We will use the following basic properties: for any matrices A and B in $\mathbb{R}^{n\times n}$,

$$\begin{aligned}
&\|A^\top\| = \|A\| \le \|A\|_{\mathrm{F}} = \|A^\top\|_{\mathrm{F}},\\
&\|A\| \le \|M^{-1}\|\|N^{-1}\|\|A\|_{M,N},\\
&\|AB\|_{\mathrm{F}} \le \min\Big\{\|A\|\|B\|_{\mathrm{F}}, \|A\|_{\mathrm{F}}\|B\|\Big\} \le \|A\|_{\mathrm{F}}\|B\|_{\mathrm{F}}.
\end{aligned}$$

For any nonsingular matrices M and N in $\mathbb{R}^{n\times n}$, we can take unique symmetric positive definite matrices $|M|$ and $|N|$ in $\mathbb{R}^{n\times n}$ such that

$$\|A\|_{|M|,|N|} = \|A\|_{M,N}, \qquad \|A\|_{|M|} = \|A\|_M,$$

where $|M|$ denotes the absolute of M, i.e., $|M| \equiv (M^\top M)^{1/2}$. However, we will not assume in this paper that M and N are symmetric positive definite.

For vectors a and c in $\mathbb{R}^n$ with $c^\top a \ne 0$, we define projection operators by

$$P(a,c) \equiv I - \frac{ac^\top}{c^\top a} \quad \text{and} \quad Q(a,c) \equiv \frac{ac^\top}{c^\top a}, \tag{2.1}$$

and we abbreviate $P(a,a)$ and $Q(a,a)$ as $P(a)$ and $Q(a)$. Thus, by $P(a)$ and $Q(a)$ for nonzero vector a in $\mathbb{R}^n$, we mean

$$P(a) \equiv I - \frac{aa^\top}{\|a\|^2} \quad \text{and} \quad Q(a) \equiv \frac{aa^\top}{\|a\|^2}. \tag{2.2}$$

Operators $P(a)$ and $Q(a)$ are both orthogonal projectors. Obviously, $P(a,c) + Q(a,c) = P(a) + Q(a) = I$. For vectors a, r and c in $\mathbb{R}^n$ with $c^\top a \ne 0$, we define two functions given by

$$\begin{aligned}
\Phi_1(a,r,c) &\equiv \frac{rc^\top}{c^\top a},\\
\Phi_2(a,r,c) &\equiv \frac{rc^\top + cr^\top}{c^\top a} - \frac{a^\top r}{(c^\top a)^2}cc^\top.
\end{aligned}$$

It is easily seen that each function $\Phi_i(a, \cdot, c)$ is linear in r, i.e., for any real numbers t_1, t_2 and for any vectors r_1, r_2 in $\mathbb{R}^n$, it holds that

$$\Phi_i(a, t_1 r_1 + t_2 r_2, c) = t_1 \Phi_i(a, r_1, c) + t_2 \Phi_i(a, r_2, c). \tag{2.3}$$

For $r = Ya$, where Y is any matrix in $\mathbb{R}^{n\times n}$, the function Φ_1 is written as

$$\Phi_1(a, Ya, c) = YQ(a, c). \tag{2.4}$$

Similarly, we can verify by a direct calculation that, for $r = Ya$, where Y is any *symmetric* matrix in $\mathbb{R}^{n\times n}$, the function Φ_2 is written as

$$\begin{aligned} \Phi_2(a, Ya, c) &= Y - \Big(I - \frac{ca^\top}{c^\top a}\Big) Y \Big(I - \frac{ac^\top}{c^\top a}\Big) \\ &= Y - P(a, c)^\top Y P(a, c). \end{aligned} \tag{2.5}$$

Furthermore, for vectors a, b and c in $\mathbb{R}^n$ with $c^\top a \neq 0$ and X in $\mathbb{R}^{n\times n}$, we define rank-one and rank-two update functions by

$$\Delta_1(a, b, c, X) \equiv \frac{(b - Xa)c^\top}{c^\top a}, \tag{2.6}$$

$$\Delta_2(a, b, c, X) \equiv \frac{(b - Xa)c^\top + c(b - Xa)^\top}{c^\top a} - \frac{a^\top(b - Xa)}{(c^\top a)^2} cc^\top. \tag{2.7}$$

Then clearly,

$$\Delta_i(a, b, c, X) = \Phi_i(a, b - Xa, c), \quad i = 1, 2. \tag{2.8}$$

Also note that Δ_2 is symmetric. Letting $X_{i+} = X + \Delta_i(a, b, c, X)$, $i = 1, 2$, we can readily verify that both of X_{i+} satisfy the 'secant' equation $X_+ a = b$. Therefore, X_{1+} and X_{2+} give, respectively, the rank-one and the symmetric rank-two secant updates. The vector c in (2.6) or (2.7) is called the *scale*. We note that Δ_i $(i = 1, 2)$ are invariant to the size of the scale c.

An ε-neighborhood of $a \in \mathbb{R}^n$, denoted by $\mathcal{N}(a; \varepsilon)$, is an open ball centered at a with radius ε, i.e., $\mathcal{N}(a; \varepsilon) \equiv \{ x \in \mathbb{R}^n : \|x - a\| < \varepsilon \}$. Analogously, a δ-neighborhood of $A \in \mathbb{R}^{n\times n}$ induced by $\|\cdot\|_{M,N}$ is an open ellipsoid centered at A with radius δ measured by $\|\cdot\|_{M,N}$ and is denoted by $\mathcal{N}_{M,N}(A; \delta)$, i.e., $\mathcal{N}_{M,N}(A; \delta) \equiv \{ X \in \mathbb{R}^{n\times n} : \|X - A\|_{M,N} < \delta \}$. A δ-neighborhood $\mathcal{N}_M(A; \delta)$ refers to $\mathcal{N}_{M,M}(A; \delta)$.

We make the following standard assumptions regarding F.

(A1) The function $F : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ is differentiable in an open convex subset Ω.

(A2) There exists a point x_* in Ω such that $F(x_*) = 0$.

(A3) The Jacobian matrix $F'(x_*) \in \mathbb{R}^{n\times n}$ is nonsingular.

(A4) The derivative F' is locally Hölder continuous at x_*, i.e., there exist constants $\xi \geq 0$, $p \in (0, 1]$ and $\varepsilon_c > 0$ such that for all $x \in \mathcal{N}(x_*; \varepsilon_c) \subset \Omega$,

$$\|F'(x) - F'(x_*)\| \leq \xi\|x - x_*\|^p. \tag{2.9}$$

It follows from (A1) and (A4) that for all u and v in $\mathcal{N}(x_*; \varepsilon_c)$,

$$\|F(u) - F(x_*) - F'(x_*)(u - x_*)\| \leq \frac{\xi}{p+1}\|u - x_*\|^{p+1}, \tag{2.10}$$

$$\|F(u) - F(v) - F'(x_*)(u - v)\| \leq \xi\sigma(u, v)^p\|u - v\|, \tag{2.11}$$

where $\sigma(u, v) \equiv \max\{\|u - x_*\|, \|v - x_*\|\}$. See Lemma 4.1.12 in Dennis and Schnabel [4] and Lemma 3.1 of Broyden, Dennis and Moré [1]. We write $\sigma(x_k, x_{k+1})$ as σ_k for short.

§3. Superlinear Convergence of Rank-one Secant Methods

In this section, we prove local superlinear convergence of rank-one secant methods. We begin with a form convenient to estimate the difference between an updated matrix and a target matrix by the weighted Frobenius norm.

Lemma 1. *Let $X \in \mathbb{R}^{n\times n}$, and let a, b and c be vectors in $\mathbb{R}^n$ with $c^\top a \neq 0$. Let $\overline{X}$ be defined by the rank-one secant update*

$$\overline{X} = X + \Delta_1(a, b, c, X). \tag{3.1}$$

Then, for any A, $N \in \mathbb{R}^{n\times n}$ and any nonsingular $M \in \mathbb{R}^{n\times n}$, there hold

(a) $$\overline{E} = EP(\widehat{a}) + E\widehat{a}\left[\frac{\widehat{a}}{\|\widehat{a}\|^2} - \frac{\widehat{c}}{\widehat{c}^\top\widehat{a}}\right]^\top + \frac{N(b - Aa)(Mc)^\top}{c^\top a},$$

(b) $$\overline{E} = EP(\widehat{c}) + E\left[\frac{\widehat{c}}{\|\widehat{c}\|^2} - \frac{\widehat{a}}{\widehat{c}^\top\widehat{a}}\right]\widehat{c}^\top + \frac{N(b - Aa)(Mc)^\top}{c^\top a},$$

where $\overline{E} := N(\overline{X} - A)M^\top$, $E := N(X - A)M^\top$, $\widehat{a} := M^{-\top}a$ and $\widehat{c} := Mc$.

Proof. From (3.1), we have

$$\begin{aligned}
\overline{X} - A &= X - A + \Delta_1(a, b, c, X) \\
&= X - A + \Phi_1(a, b - Xa, c) && \text{(by (2.8))} \\
&= X - A + \Phi_1(a, -(X - A)a + b - Aa, c) \\
&= X - A - \Phi_1(a, (X - A)a, c) + \Phi_1(a, b - Aa, c) && \text{(by (2.3))} \\
&= X - A - (X - A)Q(a, c) + \Delta_1(a, b, c, A) && \text{(by (2.4), (2.8))} \\
&= (X - A)P(a, c) + \Delta_1, && \text{(by (2.1))}
\end{aligned}$$

where we have written $\Delta_1(a,b,c,A)$ as Δ_1 for short. Note that since $\widehat{c}^\top \widehat{a} = (Mc)^\top (M^{-\top} a) = c^\top a \neq 0$, the vectors $\widehat{a}$ and $\widehat{c}$ are both nonzero. Substituting the above expression and noting that $M^{-\top} P(a,c) M^\top = P(M^{-\top} a, Mc) = P(\widehat{a}, \widehat{c})$, we obtain

$$\begin{aligned}
\overline{E} &= N(\overline{X} - A)M^\top \\
&= N[(X - A)P(a,c) + \Delta_1]M^\top \\
&= N(X - A)M^\top \cdot M^{-\top} P(a,c) M^\top + N\Delta_1 M^\top \\
&= EP(\widehat{a}, \widehat{c}) + N\Delta_1 M^\top \\
&= EP(\widehat{a}) + E[P(\widehat{a}, \widehat{c}) - P(\widehat{a})] + N\Delta_1 M^\top \\
&= EP(\widehat{a}) + E\Big[-\frac{\widehat{a}\widehat{c}^\top}{\widehat{c}^\top \widehat{a}} + \frac{\widehat{a}\widehat{a}^\top}{\|\widehat{a}\|^2}\Big] + N\Delta_1 M^\top. \quad \text{(by (2.1) and (2.2))}
\end{aligned}$$

This proves part (a). Adding and subtracting $EP(\widehat{c})$ in place of $EP(\widehat{a})$ on the right-hand side of the second last equation gives part (b). □

The next lemma gives basic estimates for later use, and corresponds to Lemma 4.2 of Broyden, Dennis and Moré [1].

Lemma 2. *Let $M \in \mathbb{R}^{n\times n}$ be a nonsingular matrix. Let a and c be vectors in $\mathbb{R}^n$, and put $\widehat{a} := M^{-\top} a$ and $\widehat{c} := Mc$. If the inequality*

$$1 - \left(\frac{\widehat{c}^\top \widehat{a}}{\|\widehat{c}\|\|\widehat{a}\|}\right)^2 \leq \beta^2 \tag{3.2}$$

holds for some $\beta \in [0,1)$, then

(a) $$\left\|\widehat{a}\Big[\frac{\widehat{a}}{\|\widehat{a}\|^2} - \frac{\widehat{c}}{\widehat{c}^\top \widehat{a}}\Big]^\top\right\|_{\mathrm{F}} \leq \frac{\beta}{\sqrt{1-\beta^2}},$$

(b) $$\left\|\Big[\frac{\widehat{c}}{\|\widehat{c}\|^2} - \frac{\widehat{a}}{\widehat{c}^\top \widehat{a}}\Big]\widehat{c}^\top\right\|_{\mathrm{F}} \leq \frac{\beta}{\sqrt{1-\beta^2}}.$$

Moreover, for any $A \in \mathbb{R}^{n\times n}$ and any $b \in \mathbb{R}^n$,

(c) $$\left\|\frac{(b - Aa)(Mc)^\top}{c^\top a}\right\|_{\mathrm{F}} \leq \frac{\|M\|}{\sqrt{1-\beta^2}} \frac{\|b - Aa\|}{\|a\|}.$$

Proof. We first note that since (3.2) holds for $\beta \in [0,1)$, $\widehat{c}^\top \widehat{a} = c^\top a$ is nonzero. By using $\|xy^\top\|_{\mathrm{F}} = \|x\|\|y\|$ and (3.2), we have

$$\left\|\widehat{a}\Big[\frac{\widehat{a}}{\|\widehat{a}\|^2} - \frac{\widehat{c}}{\widehat{c}^\top \widehat{a}}\Big]^\top\right\|_{\mathrm{F}}^2 = \|\widehat{a}\|^2 \left\|\frac{\widehat{a}}{\|\widehat{a}\|^2} - \frac{\widehat{c}}{\widehat{c}^\top \widehat{a}}\right\|^2$$

$$= \left\| \frac{\widehat{a}}{\|\widehat{a}\|} - \frac{\|\widehat{a}\|\widehat{c}}{\widehat{c}^\top \widehat{a}} \right\|^2$$
$$= \left(\frac{\|\widehat{c}\|\|\widehat{a}\|}{\widehat{c}^\top \widehat{a}} \right)^2 - 1$$
$$\leq \frac{1}{1-\beta^2} - 1$$
$$= \frac{\beta^2}{1-\beta^2}.$$

This proves part (a). Part (b) follows in the same way. For part (c), note again that it holds from (3.2)

$$\frac{\|\widehat{c}\|\|\widehat{a}\|}{|\widehat{c}^\top \widehat{a}|} = \frac{\|Mc\|\|M^{-\top}a\|}{|c^\top a|} \leq \frac{1}{\sqrt{1-\beta^2}}. \tag{3.3}$$

Using this and $\|a\| \leq \|M^\top\|\|M^{-\top}a\| = \|M\|\|M^{-\top}a\|$, we obtain

$$\begin{aligned}\left\| \frac{(b-Aa)(Mc)^\top}{c^\top a} \right\|_{\mathrm{F}} &= \|b - Aa\| \cdot \frac{\|Mc\|}{|c^\top a|} \\ &\leq \|b - Aa\| \cdot \frac{1}{\sqrt{1-\beta^2}\|M^{-\top}a\|} \\ &\leq \frac{\|b-Aa\|}{\sqrt{1-\beta^2}} \cdot \frac{\|M\|}{\|a\|},\end{aligned}$$

which is the desired result. □

Remark 1. We note that parts (a) and (b) are, in fact, equivalent because one becomes the other by interchanging $\widehat{a}$ and $\widehat{c}$ provided inequality (3.2) holds.

Remark 2. It is clear that inequality (3.2) is equivalent to the following (see also (3.3)):

$$|\cos\theta| \equiv \frac{|\widehat{c}^\top \widehat{a}|}{\|\widehat{c}\|\|\widehat{a}\|} \geq \sqrt{1-\beta^2},$$

where θ is the angle between $\widehat{c}$ and $\widehat{a}$. Hence, (3.2) states simply $|\sin\theta| \leq \beta$.

Remark 3. In Broyden, Dennis and Moré [1], the following was assumed instead of (3.2): for nonsingular symmetric matrix M, vectors a, c with $a \neq 0$ and $\beta \in [0, \frac{1}{3}]$, it holds

$$\|\widehat{c} - \widehat{a}\| \leq \beta\|\widehat{a}\|, \quad \text{i.e.,} \quad \|Mc - M^{-1}a\| \leq \beta\|M^{-1}a\|. \tag{3.4}$$

It should be noted that inequality (3.4) implies (3.2). Indeed, we see

$$\|\widehat{c} - \widehat{a}\| \leq \beta\|\widehat{a}\| \quad \Leftrightarrow \quad \|\widehat{c}\|^2 - 2\widehat{c}^\top\widehat{a} + \|\widehat{a}\|^2 \leq \beta^2\|\widehat{a}\|^2$$

$$
\begin{aligned}
\Leftrightarrow \quad & 2\widehat{c}^\top \widehat{a} \geq \|\widehat{c}\|^2 + (1-\beta^2)\|\widehat{a}\|^2 \\
\Rightarrow \quad & 2\widehat{c}^\top \widehat{a} \geq 2\sqrt{1-\beta^2}\|\widehat{c}\|\|\widehat{a}\| \\
\Rightarrow \quad & \frac{|\widehat{c}^\top \widehat{a}|}{\|\widehat{c}\|\|\widehat{a}\|} \geq \sqrt{1-\beta^2},
\end{aligned}
$$

where the third implication follows from the arithmetic-geometric mean inequality. It is clear from the above that (3.4) is not implied by (3.2).

Note also that in (3.2) the roles of $\widehat{c}$ and $\widehat{a}$ are equal, whereas in (3.4) they are not. Moreover, (3.2) has a favorable property that it is invariant under the sizing of a, c and M, i.e., a multiple of those, while (3.4) is not. In proving some local convergence properties, if (3.2) is used in place of (3.4), then the proof may be somewhat simplified. We point out here one such case. Huschens [5, Theorem 3.1] proved that his structured quasi-Newton method for nonlinear least squares problems converges quadratically to a solution in the zero-residual case. In the proof, he used a sized weighting matrix M_τ (see (3.7) of [5, p. 114]) and gave an estimate of the form

$$\|M_\tau c - M_\tau^{-1} a\| \leq K\sigma(x, x_+)\|M_\tau^{-1} a\|,$$

where K is a positive constant (see (3.19) of [5, p. 116]). However, by using an estimate such as (3.2), we will be able to remove the sizing of the weighting matrix, and consequently, able to simplify the proof. Details are now in preparation.

The following lemma states a so-called "bounded deterioration" property of a rank-one secant update.

Lemma 3. *Let $M \in \mathbb{R}^{n\times n}$ be a nonsingular matrix, and let a and c be vectors in $\mathbb{R}^n$ such that inequality* (3.2) *holds for some $\beta \in [0,1)$. Let $X \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$. Then, the rank-one secant update $\overline{X}$ given by* (3.1) *is well-defined, and for any $A \in \mathbb{R}^{n\times n}$ and any nonsingular $N \in \mathbb{R}^{n\times n}$, it holds that*

$$\text{(a)} \qquad \|\overline{X} - A\|_{M,N} \leq \left(1 + \frac{\beta}{\sqrt{1-\beta^2}}\right)\|X - A\|_{M,N} + \frac{\|M\|\|N\|}{\sqrt{1-\beta^2}}\frac{\|b - Aa\|}{\|a\|}.$$

Moreover, if $\|X - A\|_{M,N} \leq \rho$ for some $\rho > 0$, then the following hold:

$$\text{(b)} \qquad \|\overline{X} - A\|_{M,N} \leq \left(1 + \frac{\beta}{\sqrt{1-\beta^2}}\right)\|X - A\|_{M,N} + \frac{\|M\|\|N\|}{\sqrt{1-\beta^2}}\frac{\|b - Aa\|}{\|a\|} - \frac{1}{2\rho\|M^{-1}\|^2\|N^{-1}\|^2}\left(\frac{\|(X-A)a\|}{\|a\|}\right)^2,$$

$$\text{(c)} \qquad \|\overline{X} - A\|_{M,N} \leq \left(1 + \frac{\beta}{\sqrt{1-\beta^2}}\right)\|X - A\|_{M,N} + \frac{\|M\|\|N\|}{\sqrt{1-\beta^2}}\frac{\|b - Aa\|}{\|a\|}$$

$$- \frac{1}{2\rho}\left(\frac{\|N(X-A)M^\top Mc\|}{\|Mc\|}\right)^2.$$

Proof. As noted at the beginning of the proof of Lemma 2, $c^\top a$ is nonzero, and so $\overline{X}$ is well-defined and the expressions given in Lemma 1 can be used. We prove part (b) only. Part (c) follows similarly from Lemma 1(b) and Lemma 2(b)–(c). Part (a) follows more easily. Since, for any nonzero matrix W in $\mathbb{R}^{n\times n}$ and any nonzero vector u in $\mathbb{R}^n$,

$$\begin{aligned}\left\|W\left[I - \frac{uu^\top}{\|u\|^2}\right]\right\|_{\mathrm{F}}^2 &= \|W\|_{\mathrm{F}}^2 - 2\mathrm{tr}\left(W^\top \frac{Wuu^\top}{\|u\|^2}\right) + \frac{\|Wuu^\top\|_{\mathrm{F}}^2}{\|u\|^4}\\ &= \|W\|_{\mathrm{F}}^2 - \frac{\|Wu\|^2}{\|u\|^2}\\ &\le \left(\|W\|_{\mathrm{F}} - \frac{\|Wu\|^2}{2\|W\|_{\mathrm{F}}\|u\|^2}\right)^2,\end{aligned}$$

and since the quantity in the last parentheses is positive by $\|Wu\| \le \|W\|_{\mathrm{F}}\|u\|$, we have that for W with $0 < \|W\|_{\mathrm{F}} \le \rho$,

$$\|WP(u)\|_{\mathrm{F}} = \left\|W\left[I - \frac{uu^\top}{\|u\|^2}\right]\right\|_{\mathrm{F}} \le \|W\|_{\mathrm{F}} - \frac{\|Wu\|^2}{2\rho\|u\|^2}, \tag{3.5}$$

which also holds for $W = 0$.
Note that $\|\overline{X}-A\|_{M,N} = \|\overline{E}\|_{\mathrm{F}}$ and $\|X-A\|_{M,N} = \|E\|_{\mathrm{F}} \le \rho$ with notations of Lemma 1. Thus, all we need is to estimate the right-hand side of Lemma 1(a). By using (3.5) for $W = E$ and $u = \widehat{a}$ and the estimates of Lemma 2(a) and Lemma 2(c), we have

$$\begin{aligned}\|\overline{E}\|_{\mathrm{F}} &\le \|EP(\widehat{a})\|_{\mathrm{F}} + \left\|E\widehat{a}\left[\frac{\widehat{a}}{\|\widehat{a}\|^2} - \frac{\widehat{c}}{\widehat{c}^\top\widehat{a}}\right]^\top\right\|_{\mathrm{F}} + \left\|\frac{N(b-Aa)(Mc)^\top}{c^\top a}\right\|_{\mathrm{F}}\\ &\le \|E\|_{\mathrm{F}} - \frac{\|E\widehat{a}\|^2}{2\rho\|\widehat{a}\|^2} + \|E\|_{\mathrm{F}}\frac{\beta}{\sqrt{1-\beta^2}} + \|N\|\frac{\|M\|}{\sqrt{1-\beta^2}}\frac{\|b-Aa\|}{\|a\|}\\ &= \left(1 + \frac{\beta}{\sqrt{1-\beta^2}}\right)\|E\|_{\mathrm{F}} - \frac{\|E\widehat{a}\|^2}{2\rho\|\widehat{a}\|^2} + \frac{\|M\|\|N\|}{\sqrt{1-\beta^2}}\frac{\|b-Aa\|}{\|a\|}.\end{aligned} \tag{3.6}$$

But since

$$\|N(X-A)a\| \ge \frac{\|(X-A)a\|}{\|N^{-1}\|}$$

and $\|M^{-\top}a\| \le \|M^{-\top}\|\|a\| = \|M^{-1}\|\|a\|$, we get

$$\begin{aligned}\frac{\|E\widehat{a}\|}{\|\widehat{a}\|} = \frac{\|N(X-A)a\|}{\|M^{-\top}a\|} &\ge \frac{\|(X-A)a\|}{\|N^{-1}\|}\frac{1}{\|M^{-1}\|\|a\|}\\ &= \frac{1}{\|M^{-1}\|\|N^{-1}\|}\frac{\|(X-A)a\|}{\|a\|},\end{aligned} \tag{3.7}$$

and therefore, by substitution into (3.6), part (b) follows. □

The estimate (b) will be used later when showing superlinear convergence of iterates. In the proof, the limit like $\lim \|(X-A)a\|/\|a\| = 0$ follows, and this property implies superlinearity of convergence. (See the last paragraph in the proof of Theorem 2, and in particular, (3.19).) We could alternatively use the estimate (c). To see this, first note that inequality (3.2) yields

$$\begin{aligned}\left\|\frac{\widehat{c}}{\|\widehat{c}\|} - \operatorname{sgn}(\widehat{c}^\top \widehat{a})\frac{\widehat{a}}{\|\widehat{a}\|}\right\|^2 &= 2\Big(1 - \frac{|\widehat{c}^\top \widehat{a}|}{\|\widehat{c}\|\|\widehat{a}\|}\Big) \\ &\le 2(1-\sqrt{1-\beta^2}) = \frac{2\beta^2}{1+\sqrt{1-\beta^2}} \\ &\le 2\beta^2,\end{aligned}$$

where $\operatorname{sgn}(\widehat{c}^\top \widehat{a})$ denotes the sign of $\widehat{c}^\top \widehat{a}$, so that

$$\left\|\frac{\widehat{c}}{\|\widehat{c}\|} - \operatorname{sgn}(\widehat{c}^\top \widehat{a})\frac{\widehat{a}}{\|\widehat{a}\|}\right\| \le \sqrt{2}\beta.$$

Therefore, from this and $\|E\| \le \|E\|_{\mathrm{F}} \le \rho$, we have

$$\begin{aligned}\frac{\|E\widehat{a}\|}{\|\widehat{a}\|} &= \left\|E\Big[\frac{\widehat{c}}{\|\widehat{c}\|} - \Big(\frac{\widehat{c}}{\|\widehat{c}\|} - \operatorname{sgn}(\widehat{c}^\top \widehat{a})\frac{\widehat{a}}{\|\widehat{a}\|}\Big)\Big]\right\| \\ &\le \left\|\frac{E\widehat{c}}{\|\widehat{c}\|}\right\| + \|E\|\left\|\frac{\widehat{c}}{\|\widehat{c}\|} - \operatorname{sgn}(\widehat{c}^\top \widehat{a})\frac{\widehat{a}}{\|\widehat{a}\|}\right\| \\ &\le \frac{\|E\widehat{c}\|}{\|\widehat{c}\|} + \sqrt{2}\rho\beta.\end{aligned}$$

Thus, we know that if

$$\frac{\|N(X-A)M^\top Mc\|}{\|Mc\|} = \frac{\|E\widehat{c}\|}{\|\widehat{c}\|} \to 0$$

and if $\beta \to 0$ (β will be taken to be such a quantity; see the proof of Theorem 1 below), then it holds that

$$\frac{\|N(X-A)a\|}{\|M^{-\top}a\|} = \frac{\|E\widehat{a}\|}{\|\widehat{a}\|} \to 0,$$

and hence, by (3.7),

$$\frac{\|(X-A)a\|}{\|a\|} \to 0.$$

However, we prefer the estimate (b) because it is applicable more directly to the proof of superlinear convergence, while (c) is rather complicated due to the presence of the *scale* vector c unspecified.

The next proposition is essentially the same as Lemma 3.3 of Dennis and Moré [2]. However, we give here a proof different from theirs; somewhat direct and simpler one.

Proposition 1. *Let $\{a_k\}$ and $\{b_k\}$ be two sequences of nonnegative numbers such that*

$$a_{k+1} \leq (1 + \alpha_1 b_k)a_k + \alpha_2 b_k, \qquad \forall k \geq 0, \tag{3.8}$$

where α_1 and α_2 are nonnegative constants that are not both zero. Then, the sequence $\{(1 + a_k)/c_k\}$ converges to some limit $\mu \geq 0$, where $c_k := \prod_{j=0}^{k-1}(1 + \alpha b_j)$ $(k \geq 1)$, $c_0 := 1$ and $\alpha := \max\{\alpha_1, \alpha_2\}$. If $\sum_{k=0}^{\infty} b_k < \infty$, then $\mu > 0$ and $\{a_k\}$ converges. If $\sum_{k=0}^{\infty} b_k = \infty$ and $\mu > 0$, then $\{a_k\}$ diverges to ∞. If $\sum_{k=0}^{\infty} b_k = \infty$ and $\mu = 0$, then $\{a_k\}$ may converge or diverge.

Proof. From (3.8), it follows that $a_{k+1} \leq (1 + \alpha b_k)a_k + \alpha b_k$. Adding 1 to both sides yields $1 + a_{k+1} \leq (1 + \alpha b_k)(1 + a_k)$, and dividing by c_{k+1} gives $(1 + a_{k+1})/c_{k+1} \leq (1 + a_k)/c_k$. Namely, $\{(1 + a_k)/c_k\}$ is a nonincreasing sequence of positive numbers. Since it is of course bounded below by zero, it converges to $\mu \geq 0$, say. For notational simplicity, we let $d_k := \sum_{j=0}^{k-1} b_j$ $(k \geq 1)$ and $d_0 := 0$. Note first that $1 + \alpha d_k \leq c_k \leq \exp(\alpha d_k)$. Indeed, the left inequality follows from $(1+\alpha b_0)(1+\alpha b_1)\cdots(1+\alpha b_{k-1}) = 1+\alpha(b_0+b_1+\cdots+b_{k-1})+(\alpha^2 b_0 b_1+\cdots)$, and the right inequality follows from the inequality $1 + x \leq e^x$ for $x \geq 0$. Since $\alpha > 0$ and since the sequences $\{c_k\}$ and $\{d_k\}$ are both nondecreasing, it follows that they either both converge or both diverge to ∞.

Assume now that $\sum_{k=0}^{\infty} b_k < \infty$. Then $\{d_k\}$ converges, and so does $\{c_k\}$. Let $\lim_{k\to\infty} c_k = c \geq 1$. Since $(1 + a_k)/c_k \geq 1/c_k$, by letting $k \to \infty$, this yields $\mu \geq 1/c > 0$. Therefore, by the ratio test, $\{1 + a_k\}$ converges, and so does $\{a_k\}$.

Assume next that $\sum_{k=0}^{\infty} b_k = \infty$, i.e., $\lim_{k\to\infty} d_k = \infty$. Then, $\lim_{k\to\infty} c_k = \infty$. If $\mu > 0$, then again by the ratio test, $\lim_{k\to\infty}(1 + a_k) = \infty$, and so $\lim_{k\to\infty} a_k = \infty$. If $\mu = 0$, then we cannot conclude whether $\{a_k\}$ converges or not. □

Remark 4. If $\alpha_1 = \alpha_2 = 0$ in (3.8), then it is obvious that $\{a_k\}$ is convergent.

Remark 5. As stated at the end of the proposition, no conclusion on the convergence or divergence of $\{a_k\}$ can be drawn from the fact that $\sum_{k=0}^{\infty} b_k = \infty$ and $\mu = 0$. The following example shows that either case can actually occur.

Example. Let $\alpha_1 = \alpha_2 = 1$ and $b_k \equiv 1$ $(k \geq 0)$. Let $a_{1,k} = 1/k$ and $a_{2,k} = k$ for $k \geq 1$, and let $a_{1,0}$ and $a_{2,0}$ be any nonnegative numbers. Then, it is easy to check that both $\{a_{i,k}\}$ satisfy (3.8). Since $c_k = 2^k$, that $(1 + a_{i,k})/c_k \to 0 = \mu$ is obvious, and of course $\sum_{k=0}^{\infty} b_k = \infty$. The sequence $\{a_{1,k}\}$ converges to zero, while the sequence $\{a_{2,k}\}$ diverges to ∞.

Suppose that $F : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ satisfies assumption (A1). Let $x_* \in \Omega$ (not necessarily $F(x_*) = 0$), and let $\mathcal{D}_{\rm s}$ and $\mathcal{D}_{\rm a}$ be (open) neighborhoods of x_*. We say that v satisfies the *scale condition* (SC) in a on $\mathcal{D}_{\rm s}$ if:

(SC) there exist constants $K_{\rm s} \geq 0$, $q \in (0, 1]$ and a nonsingular matrix $M \in \mathbb{R}^{n\times n}$ such that, for all distinct $x, x_+ \in \mathcal{D}_{\rm s}$, it holds that

$$\sqrt{1 - \left(\frac{v^\top a}{\|Mv\|\|M^{-\top}a\|}\right)^2} \leq K_{\rm s}\sigma(x, x_+)^q. \tag{3.9}$$

We say that y satisfies the *approximation condition* (AC) on $\mathcal{D}_{\rm a}$ if:

(AC) there exist constants $K_{\rm a} \geq 0$ and $r \in (0, 1]$ such that, for all distinct $x, x_+ \in \mathcal{D}_{\rm a}$ and $s = x_+ - x$, it holds that

$$\|y - F'(x_*)s\| \leq K_{\rm a}\sigma(x, x_+)^r\|s\|. \tag{3.10}$$

Typically, v is a function of s, y and B, e.g., $v = s$, and y is a function of x and x_+, e.g., $y = F(x_+) - F(x)$. It should be noted that if $v = a$, then (3.9) holds for $M = I$. Also note that if $y = F(x_+) - F(x)$ as usual, then (3.10) holds on $\mathcal{N}(x_*; \varepsilon_{\rm c})$ from (2.11).

We show a bounded deterioration inequality for a rank-one secant update, which plays an important role in proving Q-linear convergence of iterates in Theorem 2.

Theorem 1. *Suppose that $F : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ satisfies assumption* (A1)*. Let $\mathcal{D}_{\rm s}$ and $\mathcal{D}_{\rm a}$ be neighborhoods of x_* in Ω. Assume that v satisfies the scale condition* (SC) *in s with $M \in \mathbb{R}^{n\times n}$ and $q \in (0, 1]$ on $\mathcal{D}_{\rm s}$, and that y satisfies the approximation condition* (AC) *with $r \in (0, 1]$ on $\mathcal{D}_{\rm a}$. Let B_+ be a rank-one secant update defined by*

$$B_+ = B + \Delta_1(s, y, v, B),$$

where Δ_1 is given by (2.6)*. Then, there exists a neighborhood $\mathcal{D}_{\rm b}$ of x_* such that, for all distinct $x, x_+ \in \mathcal{D}_{\rm b} \subset \mathcal{D}_{\rm s} \cap \mathcal{D}_{\rm a}$, B_+ is well-defined and the following bounded deterioration inequality holds:*

$$\begin{aligned} \|B_+ - F'(x_*)\|_{M,N} \\ \leq (1 + \alpha_{\rm s}\sigma(x, x_+)^q)\|B - F'(x_*)\|_{M,N} + \alpha_{\rm a}\sigma(x, x_+)^r, \end{aligned} \tag{3.11}$$

where $N \in \mathbb{R}^{n\times n}$ is any nonsingular matrix, and $\alpha_{\rm s}$ and $\alpha_{\rm a}$ are some nonnegative constants.

Proof. Since v satisfies the scale condition (SC) in s on $\mathcal{D}_{\rm s}$, there exists a constant $K_{\rm s} \geq 0$ satisfying (3.9) with s replacing a. Also since y satisfies the approximation condition (AC) on $\mathcal{D}_{\rm a}$, there exists a constant $K_{\rm a} \geq 0$ satisfying (3.10). Let $\beta' \in (0,1)$ be a constant. Choose an $\varepsilon_{\rm b} > 0$ so small that $K_{\rm s}\varepsilon_{\rm b}^q \leq \beta'$ and $\mathcal{N}(x_*;\varepsilon_{\rm b}) \subset \mathcal{D}_{\rm s}$. Set $\mathcal{D}_{\rm b} := \mathcal{N}(x_*;\varepsilon_{\rm b}) \cap \mathcal{D}_{\rm a}$. Then, $\mathcal{D}_{\rm b}$ is a neighborhood of x_* and $\mathcal{D}_{\rm b} \subset \mathcal{D}_{\rm s} \cap \mathcal{D}_{\rm a}$. Now, let $x, x_+ \in \mathcal{D}_{\rm b}$ with $x \neq x_+$. For brevity, let $\sigma = \sigma(x, x_+)$. Note that $K_{\rm s}\sigma^q \leq K_{\rm s}\varepsilon_{\rm b}^q \leq \beta' < 1$. From this and (3.9), inequality (3.2) holds for $a = s$, $c = v$ and $\beta = K_{\rm s}\sigma^q$. Hence, we can apply Lemma 3 to $\overline{X} = B_+$, $X = B$, $b = y$ and $A = F'(x_*)$. (For $N \in \mathbb{R}^{n\times n}$, we can take any nonsingular matrix.) It follows from Lemma 3(a) and (3.10) that

$$
\begin{aligned}
&\|B_+ - F'(x_*)\|_{M,N} \\
&\quad\leq \left(1 + \frac{K_{\rm s}\sigma^q}{\sqrt{1-(K_{\rm s}\sigma^q)^2}}\right)\|B - F'(x_*)\|_{M,N} + \frac{\|M\|\|N\|}{\sqrt{1-(K_{\rm s}\sigma^q)^2}}\frac{\|y - F'(x_*)s\|}{\|s\|} \\
&\quad\leq \left(1 + \frac{K_{\rm s}\sigma^q}{\sqrt{1-\beta'^2}}\right)\|B - F'(x_*)\|_{M,N} + \frac{\|M\|\|N\|}{\sqrt{1-\beta'^2}}K_{\rm a}\sigma^r.
\end{aligned}
$$

Therefore, we have (3.11) with nonnegative constants

$$
\alpha_{\rm s} = \frac{K_{\rm s}}{\sqrt{1-\beta'^2}} \quad \text{and} \quad \alpha_{\rm a} = \frac{K_{\rm a}\|M\|\|N\|}{\sqrt{1-\beta'^2}}. \qquad \square
$$

We are ready to prove local superlinear convergence of rank-one secant methods.

Theorem 2. *Suppose that $F : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ satisfies assumptions* (A1)–(A4). *Assume that for some neighborhoods $\mathcal{D}_{\rm s}$, $\mathcal{D}_{\rm a}$ of x_*, v and y satisfy, respectively, the scale condition* (SC) *in s with $M \in \mathbb{R}^{n\times n}$ on $\mathcal{D}_{\rm s}$, and the approximation condition* (AC) *on $\mathcal{D}_{\rm a}$. Then, there exist constants ε, $\delta > 0$ such that, for $x_0 \in \mathcal{N}(x_*;\varepsilon)$ and $B_0 \in \mathcal{N}_{M,N}(F'(x_*);\delta)$, where $N \in \mathbb{R}^{n\times n}$ is any constant nonsingular matrix, the sequence $\{x_k\}$ generated by*

$$
x_{k+1} = x_k - B_k^{-1}F(x_k), \quad B_{k+1} = B_k + \Delta_1(s_k, y_k, v_k, B_k), \quad k = 0, 1, \ldots
$$

is well-defined, and converges Q-superlinearly to x_*.

Proof. We follow essentially the same arguments as those used in Theorem 3.2 of Broyden, Dennis and Moré [1]. We first note from assumptions that the conclusions of Theorem 1 hold. Let $N \in \mathbb{R}^{n\times n}$ be a given nonsingular matrix, and let $\lambda \in (0,1)$ be fixed. Set $\gamma \geq \|F'(x_*)^{-1}\|$ and $\eta \geq \|M^{-1}\|\|N^{-1}\|$. By assumption (A4), there exist constants $\xi \geq 0$, $p \in (0,1]$ and $\varepsilon_{\rm c} > 0$ such that for all $x \in \mathcal{N}(x_*;\varepsilon_{\rm c}) \subset \Omega$, inequality (2.9) holds. Choose positive numbers ε and δ such that

$$\varepsilon \leq \min\{1, \varepsilon_{\rm c}\}, \tag{3.12}$$

$$\gamma(1+\lambda)\Big(\frac{\xi}{p+1}\varepsilon^p + 2\eta\delta\Big) \leq \lambda, \tag{3.13}$$

$$\frac{2\delta\alpha_{\rm s}\varepsilon^q}{1-\lambda^q} + \frac{\alpha_{\rm a}\varepsilon^r}{1-\lambda^r} \leq \delta, \tag{3.14}$$

where constants $\alpha_{\rm s}$ and $\alpha_{\rm a}$ are given in Theorem 1. Set $\mathcal{N}_1 = \mathcal{N}(x_*;\varepsilon)$ and $\mathcal{N}_2 = \mathcal{N}_{M,N}(F'(x_*);2\delta)$. From (3.12), we have $\mathcal{N}_1 \subset \mathcal{N}(x_*;\varepsilon_{\rm c})$. If necessary further restrict ε so that $\mathcal{N}_1 \subset \mathcal{D}_{\rm b}$, where $\mathcal{D}_{\rm b}$ is given in Theorem 1. Any $B \in \mathcal{N}_2$ is nonsingular. Indeed, we first note that

$$\|B - F'(x_*)\| \leq \|M^{-1}\|\|N^{-1}\|\|B - F'(x_*)\|_{M,N} < 2\eta\delta. \tag{3.15}$$

From this, and by noting from (3.13) that $2(1+\lambda)\gamma\eta\delta \leq \lambda$ holds, it follows that

$$\|F'(x_*)^{-1}[B - F'(x_*)]\| \leq \|F'(x_*)^{-1}\|\|B - F'(x_*)\| < 2\gamma\eta\delta \leq \frac{\lambda}{1+\lambda} < 1.$$

Thus, by the Banach perturbation lemma, B is nonsingular and

$$\|B^{-1}\| \leq \frac{\|F'(x_*)^{-1}\|}{1 - \|F'(x_*)^{-1}[B - F'(x_*)]\|} \leq \gamma(1+\lambda). \tag{3.16}$$

We next show that $(x,B) \in \mathcal{N}_1 \times \mathcal{N}_2$ implies $\|x_+ - x_*\| \leq \lambda\|x - x_*\|$, and thus $x_+ \in \mathcal{N}_1$, where $x_+ := x - B^{-1}F(x)$. Since $x \in \mathcal{N}_1 \subset \Omega$ and $B \in \mathcal{N}_2$, x_+ is well-defined and

$$\begin{aligned} x_+ - x_* &= x - x_* - B^{-1}F(x) \\ &= -B^{-1}\{F(x) - F(x_*) - F'(x_*)(x - x_*) - [B - F'(x_*)](x - x_*)\}. \end{aligned}$$

By (3.16), (2.10), (3.15) and (3.13), it holds that

$$\begin{aligned} \|x_+ - x_*\| &\leq \|B^{-1}\|\{\|F(x) - F(x_*) - F'(x_*)(x - x_*)\| \\ &\qquad\qquad + \|B - F'(x_*)\|\|x - x_*\|\} \\ &\leq \gamma(1+\lambda)\Big(\frac{\xi}{p+1}\|x - x_*\|^{p+1} + 2\eta\delta\|x - x_*\|\Big) \\ &\leq \gamma(1+\lambda)\Big(\frac{\xi}{p+1}\varepsilon^p + 2\eta\delta\Big)\|x - x_*\| \\ &\leq \lambda\|x - x_*\| < \lambda\varepsilon < \varepsilon. \end{aligned}$$

Thus, $x_+ \in \mathcal{N}_1$. Moreover, if $x_+ = x$, then $x = x_* = x_+$.

Now, we prove by induction on k that if $(x_0, B_0) \in \mathcal{N}_1 \times \mathcal{N}_{M,N}(F'(x_*); \delta)$, then, for all $k \geq 0$, $(x_k, B_k) \in \mathcal{N}_1 \times \mathcal{N}_2$ whenever $x_k \neq x_*$. (If finite convergence occurs, then the algorithm will terminate successfully, and the matrix B_k will not be generated any more.) This obviously holds for $k = 0$ from the choice of the initial point and matrix. Assume that $(x_k, B_k) \in \mathcal{N}_1 \times \mathcal{N}_2$ and $x_k \neq x_*$ for $k = 0, \ldots, j$. We want to prove that $(x_{j+1}, B_{j+1}) \in \mathcal{N}_1 \times \mathcal{N}_2$. We have already seen above that $x_{j+1} \in \mathcal{N}_1$ follows from the assumption $(x_j, B_j) \in \mathcal{N}_1 \times \mathcal{N}_2$. So, we need only to prove that $B_{j+1} \in \mathcal{N}_2$. From the hypotheses of induction, it follows that $\|x_{k+1} - x_*\| \leq \lambda \|x_k - x_*\|$ and $x_{k+1} \neq x_k$ for $k = 0, \ldots, j$. Hence, by Theorem 1, it follows that for all $k = 0, \ldots, j$, B_{k+1} is well-defined and the following bounded deterioration inequality holds:

$$\|B_{k+1} - F'(x_*)\|_{M,N} \leq (1 + \alpha_{\mathrm{s}} \sigma_k^q) \|B_k - F'(x_*)\|_{M,N} + \alpha_{\mathrm{a}} \sigma_k^r. \tag{3.17}$$

Since $B_k \in \mathcal{N}_2$ and $\sigma_k = \|x_k - x_*\| \leq \lambda^k \|x_0 - x_*\| < \lambda^k \varepsilon$, we have

$$\begin{aligned} \|B_{k+1} - F'(x_*)\|_{M,N} &\leq \|B_k - F'(x_*)\|_{M,N} + 2\delta \alpha_{\mathrm{s}} \sigma_k^q + \alpha_{\mathrm{a}} \sigma_k^r \\ &\leq \|B_k - F'(x_*)\|_{M,N} + 2\delta \alpha_{\mathrm{s}} \lambda^{kq} \varepsilon^q + \alpha_{\mathrm{a}} \lambda^{kr} \varepsilon^r. \end{aligned}$$

Summing both sides from $k = 0$ to j and using (3.14), we obtain

$$\begin{aligned} \|B_{j+1} - F'(x_*)\|_{M,N} &\leq \|B_0 - F'(x_*)\|_{M,N} + 2\delta \alpha_{\mathrm{s}} \varepsilon^q \sum_{k=0}^{j} \lambda^{qk} + \alpha_{\mathrm{a}} \varepsilon^r \sum_{k=0}^{j} \lambda^{rk} \\ &< \delta + \frac{2\delta \alpha_{\mathrm{s}} \varepsilon^q}{1 - \lambda^q} + \frac{\alpha_{\mathrm{a}} \varepsilon^r}{1 - \lambda^r} \\ &\leq 2\delta. \end{aligned}$$

Accordingly, $B_{j+1} \in \mathcal{N}_2$. At the same time, we have shown that the sequence $\{x_k\}$ is well-defined and converges at least Q-linearly to x_* with rate λ.

We finally show superlinearity of the convergence. As we obtained (3.17) from (3.11) of Theorem 1, if we use a strong bounded deterioration inequality obtained by applying Lemma 3(b) with $\rho = 2\delta$ instead of Lemma 3(a) in the proof of Theorem 1, then we have

$$\begin{aligned} &\|B_{k+1} - F'(x_*)\|_{M,N} \\ &\quad \leq (1 + \alpha_{\mathrm{s}} \sigma_k^q) \|B_k - F'(x_*)\|_{M,N} + \alpha_{\mathrm{a}} \sigma_k^r \\ &\qquad - \frac{1}{4\delta \|M^{-1}\|^2 \|N^{-1}\|^2} \left(\frac{\|(B_k - F'(x_*)) s_k\|}{\|s_k\|} \right)^2. \end{aligned} \tag{3.18}$$

Since $\sigma_k < \lambda^k \varepsilon \leq 1$ for all $k \geq 0$, we know that

$$\sum_{k=0}^{\infty} \max\{\sigma_k^q, \sigma_k^r\} = \sum_{k=0}^{\infty} \sigma_k^{\min\{q,r\}} < +\infty.$$

Hence, by setting $a_k = \|B_k - F'(x_*)\|_{M,N}$, $b_k = \max\{\sigma_k^q, \sigma_k^r\}$, $\alpha_1 = \alpha_{\mathrm{s}}$ and $\alpha_2 = \alpha_{\mathrm{a}}$ in Proposition 1, it follows that $\|B_k - F'(x_*)\|_{M,N}$ converges. Rearranging terms of (3.18) and using $\eta \geq \|M^{-1}\|\|N^{-1}\|$, we have

$$\frac{1}{4\delta\eta^2}\left(\frac{\|(B_k - F'(x_*))s_k\|}{\|s_k\|}\right)^2 \\ \leq \|B_k - F'(x_*)\|_{M,N} - \|B_{k+1} - F'(x_*)\|_{M,N} + 2\delta\alpha_{\mathrm{s}}\sigma_k^q + \alpha_{\mathrm{a}}\sigma_k^r.$$

By passing to the limit $k \to \infty$, we conclude that

$$\lim_{k\to\infty} \frac{\|(B_k - F'(x_*))s_k\|}{\|s_k\|} = 0, \tag{3.19}$$

which is the Dennis-Moré condition [2] for a convergent sequence $\{x_k\}$ to converge Q-superlinearly to x_*. □

Remark 6. We observe from the arguments through this section that the right weighting matrix N appearing in the definition of the norm $\|\cdot\|_{M,N}$ does not affect our convergence analysis at all.

§4. Superlinear Convergence of Rank-two Secant Methods

In the context of unconstrained minimization, the Jacobian F' to be approximated becomes the Hessian of the objective function in question, and is usually symmetric. Hence, it is reasonable that approximate Hessian matrices are also generated so as to be symmetric. This leads us to imposing symmetry on update matrices. In this section, we deal with *symmetric* rank-two secant updates. As in the previous section, we start our analysis by giving a form convenient to bound an error of a rank-two secant update.

Lemma 4. *Let $X \in \mathbb{R}^{n\times n}$, and let a, b and c be vectors in $\mathbb{R}^n$ with $c^\top a \neq 0$. Let $\overline{X}$ be defined by the rank-two secant update*

$$\overline{X} = X + \Delta_2(a, b, c, X). \tag{4.1}$$

Then, for any $A \in \mathbb{R}^{n\times n}$ such that $X - A$ is symmetric, $\overline{X} - A$ is also symmetric, and it holds that

$$\overline{E} = PEP + PED + D^\top EP + D^\top ED + M\Delta_2(a, b, c, A)M^\top, \tag{4.2}$$

where $M \in \mathbb{R}^{n\times n}$ is any nonsingular matrix and

(a) $\quad P = P(\widehat{a}), \quad D = Q(\widehat{a}) - Q(\widehat{a}, \widehat{c}) = \widehat{a}\left[\frac{\widehat{a}}{\|\widehat{a}\|^2} - \frac{\widehat{c}}{\widehat{c}^\top\widehat{a}}\right]^\top$

or

(b) $\quad P = P(\widehat{c}), \quad D = Q(\widehat{c}) - Q(\widehat{a}, \widehat{c}) = \left[\dfrac{\widehat{c}}{\|\widehat{c}\|^2} - \dfrac{\widehat{a}}{\widehat{c}^\top \widehat{a}}\right]\widehat{c}^\top$

with $\overline{E} := M(\overline{X} - A)M^\top$, $E := M(X - A)M^\top$, $\widehat{a} := M^{-\top}a$ *and* $\widehat{c} := Mc$.

Remark 7. Equation (4.2) can also be written in the simpler forms as follows:

$$\text{(i)} \quad \overline{E} = PEP + ED + D^\top E + \widehat{a}^\top E\widehat{a}\left[\frac{\widehat{c}\widehat{c}^\top}{(\widehat{c}^\top \widehat{a})^2} - \frac{\widehat{a}\widehat{a}^\top}{\|\widehat{a}\|^4}\right] + M\Delta_2 M^\top,$$

$$\text{(ii)} \quad \overline{E} = PEP + ED + D^\top E + \left(\frac{\widehat{a}^\top E\widehat{a}}{(\widehat{c}^\top \widehat{a})^2} - \frac{\widehat{c}^\top E\widehat{c}}{\|\widehat{c}\|^4}\right)\widehat{c}\widehat{c}^\top + M\Delta_2 M^\top,$$

where P and D are given by (a) for (i) or (b) for (ii), and $\Delta_2 = \Delta_2(a, b, c, A)$. To see this, we have only to note that

$$-QED - D^\top EQ + D^\top ED = Q(\widehat{a}, \widehat{c})^\top EQ(\widehat{a}, \widehat{c}) - QEQ,$$

where $Q = Q(\widehat{a})$ or $Q(\widehat{c})$. However, the form (4.2) will be convenient for the norm estimation because the estimate of $D^\top ED$ in (4.2) can be obtained more easily and directly than that of the last but one term in (i) or (ii).

Proof of Lemma 4. It is obvious that for any nonsingular matrix M, $\widehat{c} = Mc$ and $\widehat{a} = M^{-\top}a$ are both nonzero since $\widehat{c}^\top \widehat{a} = c^\top a \neq 0$. Since by assumption $X - A$ is symmetric, we have

$$\begin{aligned}
\overline{X} - A &= X - A + \Delta_2(a, b, c, X) \\
&= X - A + \Phi_2(a, b - Xa, c) && \text{(by (2.8))} \\
&= X - A + \Phi_2(a, -(X - A)a + b - Aa, c) \\
&= X - A - \Phi_2(a, (X - A)a, c) + \Phi_2(a, b - Aa, c) && \text{(by (2.3))} \\
&= P(a, c)^\top (X - A)P(a, c) + \Delta_2(a, b, c, A). && \text{(by (2.5), (2.8))}
\end{aligned}$$

By substituting this and using $M^{-\top}P(a, c)M^\top = P(\widehat{a}, \widehat{c})$, it follows that

$$\begin{aligned}
\overline{E} &= M(\overline{X} - A)M^\top \\
&= M[P(a, c)^\top M^{-1} \cdot M(X - A)M^\top \cdot M^{-\top}P(a, c) + \Delta_2(a, b, c, A)]M^\top \\
&= P(\widehat{a}, \widehat{c})^\top EP(\widehat{a}, \widehat{c}) + M\Delta_2 M^\top \\
&= [P(\widehat{a}) + Q(\widehat{a}) - Q(\widehat{a}, \widehat{c})^\top]E[P(\widehat{a}) + Q(\widehat{a}) - Q(\widehat{a}, \widehat{c})] + M\Delta_2 M^\top \\
&= (P + D^\top)E(P + D) + M\Delta_2 M^\top \\
&= PEP + PED + D^\top EP + D^\top ED + M\Delta_2 M^\top,
\end{aligned}$$

where $\Delta_2 = \Delta_2(a, b, c, A)$ as before. This yields case (a). Replacing $P(\widehat{a}) + Q(\widehat{a})$ by $P(\widehat{c}) + Q(\widehat{c})$ in the third last equation gives case (b). □

The next lemma is a result analogous to Lemma 3 for a rank-two secant update.

Lemma 5. *Let $M \in \mathbb{R}^{n\times n}$ be a nonsingular matrix, let $\beta \in [0,1)$, and let vectors a and c in $\mathbb{R}^n$ such that inequality* (3.2) *holds. Let $X \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$, and define $\overline{X}$ by the rank-two secant update* (4.1)*. Then, for any $A \in \mathbb{R}^{n\times n}$ such that $X - A$ is symmetric, it holds that*

$$\text{(a)} \qquad \|\overline{X} - A\|_M \leq \left(1 + \frac{\beta}{\sqrt{1-\beta^2}}\right)^2 \|X - A\|_M + \frac{2\|M\|^2}{1-\beta^2}\frac{\|b - Aa\|}{\|a\|}.$$

Moreover, if $\|X - A\|_M \leq \rho$ for some $\rho > 0$, then the following hold:

$$\text{(b)} \qquad \|\overline{X} - A\|_M \leq \left(1 + \frac{\beta}{\sqrt{1-\beta^2}}\right)^2 \|X - A\|_M + \frac{2\|M\|^2}{1-\beta^2}\frac{\|b - Aa\|}{\|a\|} - \frac{1}{2\rho\|M^{-1}\|^4}\left(\frac{\|(X-A)a\|}{\|a\|}\right)^2,$$

$$\text{(c)} \qquad \|\overline{X} - A\|_M \leq \left(1 + \frac{\beta}{\sqrt{1-\beta^2}}\right)^2 \|X - A\|_M + \frac{2\|M\|^2}{1-\beta^2}\frac{\|b - Aa\|}{\|a\|} - \frac{1}{2\rho}\left(\frac{\|M(X-A)M^\top Mc\|}{\|Mc\|}\right)^2.$$

Proof. We use the same notations as in Lemma 4. We prove part (b) only. Parts (a) and (c) follow similarly, actually more easily. By noting that $\|P\| = 1$ because P is an orthogonal projector, it follows from (4.2) that

$$\text{(4.3)} \qquad \|\overline{E}\|_{\mathrm{F}} \leq \|EP\|_{\mathrm{F}} + 2\|D\|\|E\|_{\mathrm{F}} + \|D\|^2\|E\|_{\mathrm{F}} + \|M\Delta_2 M^\top\|_{\mathrm{F}}.$$

Similar to Lemma 3(b), the first term on the right-hand side of (4.3) can be estimated by using (3.5) as follows:

$$\text{(4.4)} \qquad \|EP\|_{\mathrm{F}} \leq \|E\|_{\mathrm{F}} - \frac{1}{2\rho\|M^{-1}\|^4}\left(\frac{\|(X-A)a\|}{\|a\|}\right)^2.$$

To estimate the last term $M\Delta_2 M^\top$, note that Δ_2 can be written as

$$\Delta_2(a,b,c,A) = \frac{(b-Aa)c^\top}{c^\top a} + \frac{c(b-Aa)^\top}{c^\top a}P(a,c).$$

Since $M^{-\top}P(a,c)M^\top = P(\widehat{a},\widehat{c})$, we have from Lemma 2(c) and (3.3) that

$$\begin{aligned}
\|M\Delta_2 M^\top\|_{\mathrm{F}} &\leq \left\|M\frac{(b-Aa)(Mc)^\top}{c^\top a}\right\|_{\mathrm{F}} + \left\|\frac{Mc(b-Aa)^\top}{c^\top a}MP(\widehat{a},\widehat{c})\right\|_{\mathrm{F}} \\
&\leq \|M\|\left\|\frac{(b-Aa)(Mc)^\top}{c^\top a}\right\|_{\mathrm{F}}(1 + \|P(\widehat{a},\widehat{c})\|) \\
&\leq \frac{\|M\|^2}{\sqrt{1-\beta^2}}\frac{\|b-Aa\|}{\|a\|}\left(1 + \frac{\|\widehat{c}\|\|\widehat{a}\|}{|\widehat{c}^\top\widehat{a}|}\right)
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{\|M\|^2}{\sqrt{1-\beta^2}}\left(1+\frac{1}{\sqrt{1-\beta^2}}\right)\frac{\|b-Aa\|}{\|a\|}\\
&\le \frac{2\|M\|^2}{1-\beta^2}\frac{\|b-Aa\|}{\|a\|},
\end{aligned}
$$

where in the third inequality we used $\|P(a,c)\| = \|a\|\|c\|/|a^\top c|$ (see Lemma 8.5 of Dennis and Moré [3]). Substituting this and (4.4) into (4.3), and using Lemma 2(a) (or 2(b)), we have

$$
\begin{aligned}
\|\overline{E}\|_{\mathrm{F}} &\le (1+\|D\|)^2\|E\|_{\mathrm{F}} - \frac{1}{2\rho\|M^{-1}\|^4}\left(\frac{\|(X-A)a\|}{\|a\|}\right)^2 + \|M\Delta_2 M^\top\|_{\mathrm{F}}\\
&\le \left(1+\frac{\beta}{\sqrt{1-\beta^2}}\right)^2\|E\|_{\mathrm{F}} - \frac{1}{2\rho\|M^{-1}\|^4}\left(\frac{\|(X-A)a\|}{\|a\|}\right)^2\\
&\qquad\qquad + \frac{2\|M\|^2}{1-\beta^2}\frac{\|b-Aa\|}{\|a\|},
\end{aligned}
$$

which is the desired result. □

As in the rank-one case, this lemma yields a bounded deterioration inequality for a symmetric rank-two secant update as follows.

Theorem 3. *Suppose that $F:\Omega\subset\mathbb{R}^n\to\mathbb{R}^n$ satisfies assumption* (A1)*, and that $F'(x_*)$ is symmetric. Let $\mathcal{D}_{\mathrm{s}}$ and $\mathcal{D}_{\mathrm{a}}$ be neighborhoods of x_* in Ω. Assume that v satisfies the scale condition* (SC) *in s with $M\in\mathbb{R}^{n\times n}$ and $q\in(0,1]$ on $\mathcal{D}_{\mathrm{s}}$, and that y satisfies the approximation condition* (AC) *with $r\in(0,1]$ on $\mathcal{D}_{\mathrm{a}}$. Let B_+ be a rank-two secant update defined by*

$$B_+ = B + \Delta_2(s,y,v,B),$$

where B is symmetric and Δ_2 is given by (2.7)*. Then, there exists a neighborhood $\mathcal{D}_{\mathrm{b}}$ of x_* such that, for all distinct $x, x_+\in\mathcal{D}_{\mathrm{b}}\subset\mathcal{D}_{\mathrm{s}}\cap\mathcal{D}_{\mathrm{a}}$, B_+ is well-defined and the following bounded deterioration inequality holds:*

$$\|B_+ - F'(x_*)\|_M \le (1+\alpha_{\mathrm{s}}\sigma(x,x_+)^q)\|B-F'(x_*)\|_M + \alpha_{\mathrm{a}}\sigma(x,x_+)^r,$$

where α_{s} and α_{a} are some nonnegative constants.

Proof. The proof is almost the same as the one given for Theorem 1. Fix a scalar $\beta'\in(0,1)$, and take $\mathcal{D}_{\mathrm{b}}$ as in the proof of Theorem 1. Let x, $x_+\in\mathcal{D}_{\mathrm{b}}$ with $x\ne x_+$. Since $K_{\mathrm{s}}\sigma^q\le\beta'<1$ where $\sigma=\sigma(x,x_+)$, inequality (3.9) implies that (3.2) holds for $a=s$, $c=v$ and $\beta=K_{\mathrm{s}}\sigma^q$. This allows us to apply Lemma 5 to $\overline{X}=B_+$, $X=B$, $b=y$ and $A=F'(x_*)$. From Lemma 5(a) and (3.10), we obtain the desired inequality with

$$\alpha_{\mathrm{s}} = K_{\mathrm{s}}\left(\frac{2}{\sqrt{1-\beta'^2}}+\frac{\beta'}{1-\beta'^2}\right)\quad\text{and}\quad \alpha_{\mathrm{a}} = \frac{2K_{\mathrm{a}}\|M\|^2}{1-\beta'^2}.$$ □

Finally, we present a superlinear convergence result for the symmetric rank-two secant update methods.

Theorem 4. *Suppose that* $F : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ *satisfies assumptions* (A1)–(A4)*, and that* $F'(x_*)$ *is symmetric. Assume that for some neighborhoods* $\mathcal{D}_{\mathrm{s}}$, $\mathcal{D}_{\mathrm{a}}$ *of* x_**,* v *and* y *satisfy, respectively, the scale condition* (SC) *in* s *with* $M \in \mathbb{R}^{n\times n}$ *on* $\mathcal{D}_{\mathrm{s}}$*, and the approximation condition* (AC) *on* $\mathcal{D}_{\mathrm{a}}$*. Then, there exist constants* ε*,* $\delta > 0$ *such that, for* $x_0 \in \mathcal{N}(x_*;\varepsilon)$ *and symmetric* $B_0 \in \mathcal{N}_M(F'(x_*);\delta)$*, the sequence* $\{x_k\}$ *generated by*

$$x_{k+1} = x_k - B_k^{-1}F(x_k), \quad B_{k+1} = B_k + \Delta_2(s_k, y_k, v_k, B_k), \quad k = 0, 1, \dots,$$

is well-defined, and converges Q-superlinearly to x_**.*

Proof. We can prove the theorem in a way similar to the proof of Theorem 2 by using Theorem 3 instead of Theorem 1, so we omit the proof. □

§5. Concluding Remarks

In this paper, we have shown that a certain inequality can take the place of a norm inequality used in Broyden, Dennis and Moré [1] in proving local superlinear convergence of rank-one and rank-two secant methods. Our inequality requires that the angle between the scale c and a step a suitably scaled by some matrix M approaches zero. It has a merit of invariance to scalar sizing, while the norm inequality does not.

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