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# A note on a theorem of A. Saeki and R. Ikehata

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**Abstract.** We extend the theorem of A. Saeki and R. Ikehata for the wave equation with linear dissipation.

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## 1. Introduction

In this note we consider the wave equation of the form

$$w_{tt} - \Delta w + b(x)w_t = 0 \quad \text{in} \quad (0, +\infty) \times \Omega, \tag{1.1}$$

$$w(0, x) = w_0(x), \quad w_t(0, x) = w_1(x) \quad \text{in} \quad \Omega, \tag{1.2}$$

$$w(t, x) = 0 \quad \text{on} \quad (0, +\infty) \times \partial\Omega, \tag{1.3}$$

where $N \geq 3$, $\Omega(\subset \mathbb{R}^N)$ is an unbounded domain with smooth boundary $\partial\Omega$ and $b(\cdot) \in C^1(\Omega)$ is a positive and bounded function: $0 < b_0 \leq b(x) \leq b_1$ in $\Omega$ for some constants $b_0$ and $b_1$. In the following we denote $L^2 = L^2(\Omega)$, $H_0^1 = H_0^1(\Omega)$ e.t.c.. If we assume $\{w_0, w_1\} \in H_0^1 \times L^2$ then we know the following estimate holds for the solutions $w(t, \cdot) \in C^0([0, \infty); H_0^1) \cap C^1([0, \infty); L^2)$ of (1.1) – (1.3):

$$\begin{aligned}(1+t)||w(t)||_E^2 + ||w(t)||_{L^2}^2 + \int_0^t \left\{(1+\tau)||w_t(\tau)||_{L^2}^2 + ||\nabla w(\tau)||_{L^2}^2\right\} d\tau \\ \leq C_1\{||w_0||_{H^1}^2 + ||w_1||_{L^2}^2\}\end{aligned}$$

for some positive constant $C_1$ (cf. Hirosawa–Nakazawa[1]), where

$$||w(t)||_E^2 = \frac{1}{2}\left(||w_t(t)||_{L^2}^2 + ||\nabla w(t)||_{L^2}^2\right)$$

is the energy at time $t(\geq 0)$.

Recently, A. Saeki and R. Ikehata showed the following

**Theorem 1 ([3]).** (1) *Assume* $\{w_0, w_1\} \in H_0^1 \cap L^{2,1} \times L^{2,1}$, *where*

$$L^{2,1} = \left\{ f \;\middle|\; ||f||_{L^{2,1}}^2 = \int_\Omega (1+|x|)^2 |f(x)|^2 dx < \infty \right\}.$$

*Then the solutions* $w(t,\cdot)$ *of* (1.1) − (1.3) *satisfy the following inequalities:*

$$(1+t)^2 ||w(t)||_E^2 \le C_2 ||(w_0, w_1)||_{H^1 \cap L^{2,1} \times L^{2,1}}^2, \tag{1.4}$$

$$(1+t) ||w(t)||_{L^2}^2 \le C_3 ||(w_0, w_1)||_{H^1 \cap L^{2,1} \times L^{2,1}}^2 \tag{1.5}$$

*for some positive constants* $C_2$ *and* $C_3$, *where*

$$||(w_0, w_1)||_{H^1 \cap L^{2,1} \times L^{2,1}}^2 \equiv ||w_0||_{H^1}^2 + ||w_0||_{L^{2,1}}^2 + ||w_1||_{L^{2,1}}^2.$$

(2) *Moreover assume that* $w_1 + b(x) w_0 = 0$. *Then the following inequalities hold:*

$$(1+t)^3 ||w(t)||_E^2 \le C_4 ||(w_0, w_1)||_{H^1 \cap L^{2,1} \times L^{2,1}}^2, \tag{1.6}$$

$$(1+t)^2 ||w(t)||_{L^2}^2 \le C_5 ||(w_0, w_1)||_{H^1 \cap L^{2,1} \times L^{2,1}}^2 \tag{1.7}$$

*for some positive constants* $C_4$ *and* $C_5$.

In this theorem, essential lemma is the following:

**Lemma 2 ([3]).** *Under the same assumption as in Theorem 1* (1),

$$||w(t)||_{L^2}^2 + \int_0^t ||w(\tau)||_{L^2}^2 d\tau \le C_6 ||(w_0, w_1)||_{H^1 \cap L^{2,1} \times L^{2,1}}^2 \tag{1.8}$$

*for some positive constant* $C_6$.

We shall give the simple proof of Theorem 1 and its extension:

**Theorem 3.** (1) *Under the same condition as in Theorem 1* (1),

(1.9)

$$(1+t)^2 ||w(t)||_E^2 + \int_0^t \left\{ (1+\tau)^2 ||w_t(\tau)||_{L^2}^2 + (1+\tau) ||\nabla w(t)||_{L^2}^2 \right\} d\tau \le C_7 ||(w_0, w_1)||_{H^1 \cap L^{2,1} \times L^{2,1}}^2,$$

$$\lim_{t \to +\infty} (1+t)^2 ||w(t)||_E^2 = 0, \tag{1.10}$$

$$\lim_{t \to +\infty} (1+t) ||w(t)||_{L^2}^2 = 0, \tag{1.11}$$

*hold for some positive constant* $C_7$.

(2) *Under the same assumption as in Theorem 1* (2),

(1.12)
$$(1+t)^4||w(t)||_E^2 + \int_0^t \left\{(1+\tau)^4||w_t(\tau)||_{L^2}^2 + (1+\tau)^3||\nabla w(t)||_{L^2}^2\right\} d\tau \le C_8||(w_0,w_1)||^2_{H^1\cap L^{2,1}\times L^{2,1}},$$

(1.13)
$$(1+t)^2||w(t)||_{L^2}^2 + \int_0^t (1+\tau)^2||w(\tau)||_{L^2}^2 d\tau \le C_9||(w_0,w_1)||^2_{H^1\cap L^{2,1}\times L^{2,1}},$$

(1.14) $$\lim_{t\to+\infty}(1+t)^4||w(t)||_E^2 = 0,$$

(1.15) $$(1+t)^3||w(t)||_{L^2}^2 \le C_{10}||(w_0,w_1)||^2_{H^1\cap L^{2,1}\times L^{2,1}},$$

(1.16) $$\lim_{t\to+\infty}(1+t)^3||w(t)||_{L^2}^2 = 0,$$

*hold for some positive constants* $C_8$, $C_9$ *and* $C_{10}$.

*Remark.* Obviously, we can treat (1.1) – (1.3) with $N = 2$ and the Cauchy problem in $\mathbb{R}^N$ with $N \ge 3$ ([3]).

## 2. Proof of Theorem 3 (1)

We assume that $\varphi(t)$ and $\psi(t)$ are the smooth, non–decreasing and non–negative functions of $t$. Multiplying $\{\varphi w_t + \psi w\}$ the both sides of (1.1), and integrating on $\Omega$, we obtain (c.f., Mochizuki–Nakazawa [2])

(2.1) $$\frac{d}{dt}\int_\Omega X(t,x)dx + \int_\Omega Z(t,x)dx = 0,$$

where

(2.2)
$$X(t,x) = \frac{\varphi(t)}{2}\left\{w_t^2(t,x) + |\nabla w(t,x)|^2\right\} + \psi(t)w_t(t,x)w(t,x) + \frac{b(x)\psi(t)-\psi_t(t)}{2}w(t,x)^2,$$

(2.3)
$$Z(t,x) = \left\{b(x)\varphi(t) - \frac{\varphi_t(t)}{2} - \psi(t)\right\} w_t(t,x)^2 + \left\{\psi(t) - \frac{\varphi_t(t)}{2}\right\}|\nabla w(t,x)|^2 + \frac{\psi_{tt}(t)-b(x)\psi_t(t)}{2}w(t,x)^2.$$

Firstly we shall show (1.9). Put $\varphi(t) = 2(\frac{3}{b_0} + t)^2$ and $\psi(t) = 3(\frac{3}{b_0} + t)$. Then easy computations give

$$\begin{aligned}
&b(x)\varphi(t) - \frac{\varphi_t(t)}{2} - \psi(t) \geq C_{11}(1+t)^2,\\
&\psi(t) - \frac{\varphi_t(t)}{2} \geq C_{12}(1+t),\\
&\frac{\psi_{tt}(t) - b(x)\psi_t(t)}{2} \geq -\frac{b(x)\psi_t(t)}{2} \geq -C_{13}
\end{aligned}$$

for some positive constants $C_{11}$, $C_{12}$ and $C_{13}$. Thus we have

(2.4)
$$\int_\Omega Z(t,x)dx \geq C_{14}\left\{(1+t)^2||w_t(t)||^2_{L^2} + (1+t)||\nabla w(t)||^2_{L^2} - ||w(t)||^2_{L^2}\right\}$$

for some positive constant $C_{14}$. Next choosing $\eta$ as $0 < \eta < 1$, we find

$$\begin{aligned}
X(t,x) \geq\ &\frac{(1-\eta)}{2}\varphi(t)\left\{w_t(t,x)^2 + |\nabla w(t,x)|^2\right\}\\
&+ \left\{\frac{b(x)\psi(t) - \psi_t(t)}{2} - \frac{\psi(t)^2}{2\eta\varphi(t)}\right\} w(t,x)^2.
\end{aligned}$$

As is easily seen, there exists a positive constant $C_{15}$ such that

$$\frac{b(x)\psi(t) - \psi_t(t)}{2} - \frac{\psi(t)^2}{2\eta\varphi(t)} \geq -\frac{\psi(t)^2}{2\eta\varphi(t)} \geq -C_{15}$$

holds. From this, we have

$$\int_\Omega X(t,x)dx \geq C_{16}\left\{(1+t)^2||w(t)||^2_E - ||w(t)||^2_{L^2}\right\}. \tag{2.5}$$

On the other hand,

$$\int_\Omega X(0,x)dx \leq C_{17}\left(||w_0||^2_{H^1} + ||w_1||^2_{L^2}\right) \tag{2.6}$$

Integrating (2.1) over $[0,t]$ and using (2.4), (2.5), (2.6) and Lemma 2, we obtain (1.9).

Next we shall show (1.10). By (1.9), we find

$$\liminf_{t\to+\infty}(1+t)^2||w(t)||^2_E = 0.$$

Integration the both sides of

$$\frac{d}{dt}\left\{(1+t)^2||w(t)||^2_E\right\} = 2(1+t)||w(t)||^2_E - (1+t)^2\int_\Omega b(x)w_t(t,x)^2dx,$$

where we have used

$$\frac{d}{dt}||w(t)||_E^2 = -\int_\Omega b(x) w_t(t,x)^2 dx,$$

on $[t_1, t_2]$ $(0 \le t_1 \le t_2 < +\infty)$ gives

$$\begin{aligned} &\left|(1+t_2)^2||w(t_2)||_E^2 - (1+t_1)^2||w(t_1)||_E^2\right| \\ &\le 2\int_{t_1}^{t_2}(1+\tau)||w(\tau)||_E^2 d\tau + b_1\int_{t_1}^{t_2}(1+\tau)^2||w_t(\tau)||_{L^2}^2 d\tau. \end{aligned} \tag{2.7}$$

Using (1.9), we find the right hand side of (2.7) tends to 0 as $t_1$ and $t_2 \to +\infty$ and conclude (1.10).

Finally we shall show (1.11). Lemma 2 gives

$$\liminf_{t\to+\infty}(1+t)||w(t)||_{L^2}^2 = 0.$$

From

$$\frac{d}{dt}\left\{(1+t)||w(t)||_{L^2}^2\right\} = ||w(t)||_{L^2}^2 + 2(1+t)(w(t), w_t(t))_{L^2},$$

where $(\cdot,\cdot)_{L^2}$ denotes the inner product in $L^2$, we obtain

$$\begin{aligned} &\left|(1+t_2)||w(t_2)||_{L^2}^2 - (1+t_1)||w(t_1)||_{L^2}^2\right| \\ &\le 2\int_{t_1}^{t_2}||w(\tau)||_{L^2}^2 d\tau + \int_{t_1}^{t_2}(1+\tau)^2||w_t(\tau)||_{L^2}^2 d\tau \to 0 \end{aligned}$$

as $t_1$ and $t_2 \to +\infty$ by Lemma 2 and (1.9), where we have used

$$|2(1+t)(w(t), w_t(t))_{L^2}| \le ||w(t)||_{L^2}^2 + (1+t)^2||w_t(t)||_{L^2}^2.$$

## 3. Proof of Theorem 3 (2)

Firstly we shall show (1.12) and (1.13). We put $\varphi(t) = (\frac{13}{b_0}+t)^4$ and $\psi(t) = 3(\frac{13}{b_0}+t)^3$. Easy computations give

$$\begin{aligned} &b(x)\varphi(t) - \frac{\varphi_t(t)}{2} - \psi(t) \ge C_{18}(1+t)^4, \\ &\psi(t) - \frac{\varphi_t(t)}{2} \ge C_{19}(1+t)^3, \\ &\frac{\psi_{tt}(t) - b(x)\psi_t(t)}{2} \ge -C_{20}(1+t)^2 \end{aligned}$$

for some positive constants $C_{18}$, $C_{19}$ and $C_{20}$. Thus we have

$$\begin{aligned}(3.1)\quad &\int_\Omega Z(t,x)dx \\ &\geq C_{21}\left\{(1+t)^4||w_t(t)||_{L^2}^2 + (1+t)^3||\nabla w(t)||_{L^2}^2 - (1+t)^2||w(t)||_{L^2}^2\right\}\end{aligned}$$

for some positive constant $C_{21}$.

Similarly as in section 2, there exists a positive constant $C_{22}$ such that

$$\frac{b(x)\psi(t)-\psi_t(t)}{2} - \frac{\psi(t)^2}{2\eta\varphi(t)} \geq -C_{22}(1+t)^2$$

holds. From this, we have

$$(3.2)\qquad \int_\Omega X(t,x)dx \geq C_{23}\left\{(1+t)^4||w(t)||_E^2 - (1+t)^2||w(t)||_{L^2}^2\right\}.$$

Integrating (2.1) over $[0,t]$ and using (3.1), (3.2) and (2.6), we obtain

$$\begin{aligned}(3.3)\quad &(1+t)^4||w(t)||_E^2 + \int_0^t \left\{(1+\tau)^4||w_t(\tau)||_{L^2}^2 + (1+\tau)^3||\nabla w(\tau)||_{L^2}^2\right\}d\tau \\ &\leq C_{24}\left(||w_0||_{H^1}^2 + ||w_1||_{L^2}^2 + (1+t)^2||w(t)||_{L^2}^2 + \int_0^t (1+\tau)^2||w(\tau)||_{L^2}^2 d\tau\right).\end{aligned}$$

Put

$$u(t,x) = \int_0^t w(\tau,x)d\tau.$$

Noting the assumption $w_1(x)+b(x)w_0(x)=0$, we find $u$ satisfies (1.1) – (1.3) with $u(0,x)=0$ and $u_t(0,x)=w_0(x)$. Applying Theorem 3 (1), (1.9) and noting $u_t(t,x)=w(t,x)$, we obtain

$$(3.4)\quad (1+t)^2||w(t)||_{L^2}^2 + \int_0^t (1+\tau)^2||w(\tau)||_{L^2}^2 d\tau \leq C_{25}||(w_0,w_1)||_{H^1\cap L^{2,1}\times L^{2,1}}^2.$$

(3.3) and (3.4) give (1.12) and (1.13).

Next, we shall show (1.14). (1.12) gives

$$\liminf_{t\to+\infty}(1+t)^4||w(t)||_E^2 = 0.$$

Integration the both sides of

$$\begin{aligned}&\frac{d}{dt}\left\{(1+t)^4||w(t)||_E^2\right\} \\ &= 4(1+t)^3||w(t)||_E^2 - (1+t)^4\int_{\mathbb{R}^N} b(x)w_t(t,x)^2dx\end{aligned}$$

over $[t_1, t_2]$ $(0 \le t_1 \le t_2 < \infty)$ gives

$$\begin{aligned}&\left|((1+t_2)^4||w(t_2)||_E^2 - (1+t_1)^4||w(t_1)||_E^2\right|\\ &\le 4\int_{t_1}^{t_2}(1+\tau)^3||w(\tau)||_E^2 d\tau + b_1\int_{t_1}^{t_2}(1+\tau)^4||w_t(\tau)||_{L^2}^2 d\tau \to 0\end{aligned}$$

as $t_1$ and $t_2 \to +\infty$ by (1.12) and conclude (1.14).

Next we shall show (1.15). Put $\varphi(t) = 0$ and $\psi(t) = (1+t)^3$. Then we have

$$\begin{aligned}&|\psi(t)(w_t(t), w(t))_{L^2}| \le C_{26}\left\{(1+t)^4||w_t(t)||_{L^2}^2 + (1+t)^2||w(t)||_{L^2}^2\right\},\\ &\frac{b(x)\psi(t) - \psi_t(t)}{2} \ge C_{27}(1+t)^3 - C_{28}(1+t)^2.\end{aligned}$$

From these we obtain

$$\begin{aligned}&\int_\Omega X(t,x)dx\\ &\ge C_{29}(1+t)^3||w(t)||_{L^2}^2 - C_{30}\left\{(1+t)^4||w_t(t)||_{L^2}^2 + (1+t)^2||w(t)||_{L^2}^2\right\}\end{aligned} \tag{3.5}$$

Moreover,

$$\int_\Omega Z(t,x)dx \ge -C_{31}\left\{(1+t)^3||w_t(t)||_{L^2}^2 + (1+t)^2||w(t)||_{L^2}^2\right\} \tag{3.6}$$

Integrating the both sides of (2.1) over $[0, t]$ and using (3.5), (3.6) and (2.6), we obtain

$$\begin{aligned}&(1+t)^3||w(t)||_{L^2}^2\\ &\le C_{32}\Big\{(1+t)^4||w_t(t)||_{L^2}^2 + (1+t)^2||w(t)||_{L^2}^2\\ &+\int_0^t(1+\tau)^3||w_t(\tau)||_{L^2}^2 d\tau + \int_0^t(1+\tau)^2||w(\tau)||_{L^2}^2 d\tau + ||w_0||_{H^1}^2 + ||w_1||_{L^2}^2\Big\}.\end{aligned}$$

Noting (1.12) and (1.13), we have (1.15).

Finally, we shall show (1.16). By (1.13), we find

$$\liminf_{t\to+\infty}(1+t)^3||w(t)||_{L^2}^2 = 0.$$

Integration the both sides of

$$\begin{aligned}&\frac{d}{dt}\left\{(1+t)^3||w(t)||_{L^2}^2\right\}\\ &= 3(1+t)^2||w(t)||_{L^2}^2 + 2(1+t)^3(w_t(t), w(t))_{L^2}\end{aligned}$$

over $[t_1, t_2]$ $(0 \le t_1 \le t_2 < +\infty)$ gives

$$\begin{aligned}&\left|(1+t_2)^3||w(t_2)||_{L^2}^2 - (1+t_1)^3||w(t_1)||_{L^2}^2\right|\\ &\le 4\int_{t_1}^{t_2}(1+\tau)^2||w(\tau)||_{L^2}^2 d\tau + \int_{t_1}^{t_2}(1+\tau)^4||w_t(\tau)||_{L^2}^2 d\tau \to 0\end{aligned}$$

as $t_1$ and $t_2 \to +\infty$ by (1.12) and (1.13).

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