

On certain bases for Ariki-Koike algebras arising from canonical bases for $U_v(\mathfrak{sl}_m)$

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Abstract. Frenkel, Khovanov and Kirillov showed that the parabolic Kazhdan-Lusztig basis of Iwahori-Hecke algebra associated to \mathfrak{S}_n can be obtained as the canonical basis of a weight subspace of $V^{\otimes n}$, where V is the vector representation of the quantum group $U_v(\mathfrak{sl}_m)$. In this paper, a similar problem for the case of Ariki-Koike algebra $\mathcal{H}_{n,r}$ is discussed. We construct a certain basis of $\mathcal{H}_{n,r}$, which is fixed by the involution and is closely related to the canonical basis of $V^{\otimes n}$, by making use of the representation of $\mathcal{H}_{n,r}$ on $V^{\otimes n}$. In the case where $r = 2$, i.e., in the case of Iwahori-Hecke algebra of type B_n , this gives a basis different from the Kazhdan-Lusztig basis of $\mathcal{H}_{n,r}$.

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§0. Introduction

Let $U_v = U_v(\mathfrak{sl}_m)$ be the quantum group associated to the Lie algebra \mathfrak{sl}_m , and V the vector representation of U_v . Let \mathcal{H}_n be the Iwahori-Hecke algebra associated to the symmetric group \mathfrak{S}_n . Then the n -fold tensor space $V^{\otimes n}$

turns out to be a $U_v \otimes \mathcal{H}_n$ -module. Each weight subspace $V_\lambda^{\otimes n}$ of $V^{\otimes n}$ is \mathcal{H}_n -stable, and is naturally isomorphic to an induced module M_J from a linear representation of some parabolic subalgebra \mathcal{H}_J of \mathcal{H}_n . A parabolic Kazhdan-Lusztig basis on M_J was defined by Deodhar [D], by generalizing the notion of Kazhdan-Lusztig basis of \mathcal{H}_n introduced by Kazhdan and Lusztig [KL].

The notion of canonical basis for highest weight modules of U_v was introduced by Lusztig [L], which is a union of canonical bases for each weight subspace. In the case of highest weight module $V^{\otimes n}$, Frenkel, Khovanov and Kirillov [FKK] showed that the canonical basis of the weight subspace $V_\lambda^{\otimes n}$ coincides with the Kazhdan-Lusztig basis of M_J under the above isomorphism. Note that \mathcal{H}_n has a standard basis $\{T_\sigma \mid \sigma \in \mathfrak{S}_n\}$, and the Kazhdan-Lusztig basis of \mathcal{H}_n is characterized by the property that the transition matrix between this basis and the standard basis is of the unitriangular shape, and that it is fixed by a certain involution on \mathcal{H}_n , called the bar involution. In turn, $V^{\otimes n}$ has also a standard basis consisting of the tensor product of the given basis of V , and the canonical basis on $V^{\otimes n}$ is characterized by a certain involution ψ on it, together with some additional property related to the standard basis. The important step for proving the result in [FKK] is to show that these two involutions coincide with under the isomorphism $M_J \simeq V_\lambda^{\otimes n}$.

Let $W_{n,r}$ be the complex reflection group $\mathfrak{S}_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n$, and $\mathcal{H}_{n,r}$ the associated cyclotomic Hecke algebra, i.e., the Ariki-Koike algebra associated to $W_{n,r}$. In the case where $r = 1$, $\mathcal{H}_{n,r} \simeq \mathcal{H}_n$, and $\mathcal{H}_{n,r}$ is isomorphic to the Iwahori-Hecke algebra of type B_n if $r = 2$. $\mathcal{H}_{n,r}$ contains \mathcal{H}_n as a subalgebra, and in [SS] the action of \mathcal{H}_n on $V^{\otimes n}$ was extended to the action of $\mathcal{H}_{n,r}$. Each weight space $V^{\otimes n}$ is again $\mathcal{H}_{n,r}$ -stable. The aim of this paper is to extend the result of [FKK] to the case of certain induced $\mathcal{H}_{n,r}$ -modules. One of our main results is Theorem 2.4, which asserts that the bar involution of $\mathcal{H}_{n,r}$ is compatible with the involution ψ on $V^{\otimes n}$. By making use of this fact, one can show, in Theorem 4.3, that the weight subspace $V_\lambda^{\otimes n}$ is isomorphic to an $\mathcal{H}_{n,r}$ -module M_J induced from a “non-parabolic” subalgebra \mathcal{H}_J of $\mathcal{H}_{n,r}$, and that the canonical basis of $V_\lambda^{\otimes n}$ determines a basis of M_J fixed by the bar involution of $\mathcal{H}_{n,r}$. This may be regarded as a non-parabolic analogue of the result of [FKK].

However, if one focuses on the $\mathcal{H}_{n,r}$ -module M_J induced from the parabolic subalgebra \mathcal{H}_J of $\mathcal{H}_{n,r}$, for example $\mathcal{H}_{n,r}$ itself, the situation is much more complicated. There is no natural notion of standard basis nor Kazhdan-Lusztig basis of $\mathcal{H}_{n,r}$ for $r > 2$. Moreover, M_J turns out to be a direct sum of various weight subspaces $V_\lambda^{\otimes n}$. In order to treat these cases, we make use of the new generators of $\mathcal{H}_{n,r}$ introduced by [S]. By using the direct sum decomposition $M_J = \bigoplus_\lambda V_\lambda^{\otimes n}$, one can define two bases of M_J inherited from the standard basis and the canonical basis of $\bigoplus_\lambda V_\lambda^{\otimes n}$. As a special case, we can construct two bases of $\mathcal{H}_{n,r}$ in Theorem 4.7; the one has a property that the action

of generators of $\mathcal{H}_{n,r}$ on this basis is explicitly described, and the other has a property that it is fixed by the bar involution on $\mathcal{H}_{n,r}$, and the transition matrix between these two bases is described by various parabolic Kazhdan-Lusztig polynomials of type A .

We remark that even in the case where $r = 2$ (i.e., the case of Iwahori-Hecke algebras of type B_n), our basis does not coincide with the Kazhdan-Lusztig basis of $\mathcal{H}_{n,r}$. In section 5, we discuss the relationship between these two bases, with the standard basis and the Kazhdan-Lusztig basis of $\mathcal{H}_{n,r}$. In particular we show in Proposition 5.2 that the parabolic Kazhdan-Lusztig polynomials of type B_n can be determined uniquely by various parabolic Kazhdan-Lusztig polynomials of type A , together with the information on the transition matrix between the standard basis of $\mathcal{H}_{n,r}$ and the standard basis of $\bigoplus_{\lambda} V_{\lambda}^{\otimes n}$.

§1. Review on Ariki-Koike algebras

1.1. Let $K = \mathbb{Q}(v, u_1, \dots, u_r)$ be a field of rational functions in variables v, u_1, \dots, u_r . Let $W = W_{n,r}$ be the complex reflection group $\mathfrak{S}_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n$, and $\mathcal{H}_{n,r}$ the Ariki-Koike algebra associated to W . $\mathcal{H}_{n,r}$ is the associative algebra over K with generators a_1, \dots, a_n , and relations

$$\begin{aligned}
 (1.1.1) \quad & (a_1 - u_1)(a_1 - u_2) \cdots (a_1 - u_r) = 0, \\
 & (a_i - v)(a_i + v^{-1}) = 0 \quad (2 \leq i \leq n), \\
 & a_1 a_2 a_1 a_2 = a_2 a_1 a_2 a_1, \\
 & a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1} \quad (2 \leq i < n), \\
 & a_i a_j = a_j a_i \quad (|i - j| \geq 2).
 \end{aligned}$$

It is known that the subalgebra \mathcal{H}_n of $\mathcal{H}_{n,r}$ generated by a_2, \dots, a_n is isomorphic to the Hecke algebra associated to the symmetric group \mathfrak{S}_n with standard generators.

1.2. Let $U_v = U_v(\mathfrak{sl}_m)$ be the quantum group associated to the Lie algebra \mathfrak{sl}_m with generators E_i, F_i, K_i ($1 \leq i \leq m-1$) and standard relations. A priori, U_v is an associative algebra over $\mathbb{Q}(v)$, but for later discussion, we regard them as an algebra over K by an extension of scalars.

Let V be an m -dimensional vector space over K with basis e_1, \dots, e_m . The vector representation of U_v on V is defined by

$$\begin{aligned}
 E_i e_{i+1} &= e_i, & E_i e_j &= 0 \quad j \neq i+1, \\
 F_i e_i &= e_{i+1}, & F_i e_j &= 0 \quad j \neq i,
 \end{aligned}$$

$$K_i e_j = \begin{cases} v e_i & j = i, \\ v^{-1} e_{i+1} & j = i + 1, \\ e_j & j \neq i, i + 1. \end{cases}$$

It is known that U_v has a Hopf algebra structure with comultiplication $\Delta : U_v \rightarrow U_v \otimes U_v$ given by

$$\begin{aligned} \Delta(K_i) &= K_i \otimes K_i, \\ \Delta(E_i) &= E_i \otimes 1 + K_i \otimes E_i, \\ \Delta(F_i) &= F_i \otimes K_i^{-1} + 1 \otimes F_i. \end{aligned}$$

For a positive integer n , we consider the tensor space $V^{\otimes n}$ on which $U_v^{\otimes n}$ acts naturally. We define inductively an algebra homomorphism $\Delta^{(k)} : U_v \rightarrow U_v^{\otimes k}$, by starting from $\Delta^{(2)} = \Delta$ and by putting $\Delta^{(k)} = (\Delta^{(k-1)} \otimes \text{id}) \circ \Delta$ for each $k \geq 3$. By using $\Delta^{(n)}$, one can define an action of U_v on $V^{\otimes n}$.

1.3. In [Ji], Jimbo constructed an action of \mathcal{H}_n on $V^{\otimes n}$, commuting with the action of $U_v(\mathfrak{sl}_m)$. Let us fix integers m_1, \dots, m_r such that $\sum m_i = m$, and consider a Levi subalgebra $\mathfrak{g} = \mathfrak{sl}_{m_1} \oplus \dots \oplus \mathfrak{sl}_{m_r}$ of \mathfrak{sl}_m . The action of \mathcal{H}_n was extended by [SS] to the action of $\mathcal{H}_{n,r}$ on $V^{\otimes n}$ so that it commutes with the action of the subalgebra $U_v(\mathfrak{g})$ of $U_v(\mathfrak{sl}_m)$. We consider the decomposition $V = \bigoplus_i V_i$ with $\dim V_i = m_i$. We assume that a basis $\{e_j^{(k)}\}$ ($1 \leq j \leq m_k$) of V_k is chosen for $k = 1, \dots, r$, and that

$$e_1^{(1)}, \dots, e_{m_1}^{(1)}, e_1^{(2)}, \dots, e_{m_2}^{(2)}, \dots, e_1^{(r)}, \dots, e_{m_r}^{(r)}$$

gives the basis e_1, \dots, e_m of V in this order. The construction of the action of $\mathcal{H}_{n,r}$ on $V^{\otimes n}$ is given as follows. Let T be the element in $\text{End}(V \otimes V)$ defined by

$$(1.3.1) \quad T(e_i \otimes e_j) = \begin{cases} v e_j \otimes e_i & \text{if } i = j, \\ e_j \otimes e_i & \text{if } i > j, \\ e_j \otimes e_i + (v - v^{-1}) e_i \otimes e_j & \text{if } i < j. \end{cases}$$

Next we define a map $b : \{1, 2, \dots, m\} \rightarrow \mathbb{N}$ by $b(j) = k$ whenever $e_j \in V_k$. Let $wt : V \rightarrow V$ be a linear operator defined by $wt(e_j) = u_{b(j)} e_j$. Let us define linear operators, σ, S on $V^{\otimes 2}$ as follows.

$$\begin{aligned} \sigma(e_i \otimes e_j) &= e_j \otimes e_i, \\ S(e_i \otimes e_j) &= \begin{cases} T(e_i \otimes e_j) & \text{if } b(i) = b(j), \\ \sigma(e_i \otimes e_j) & \text{if } b(i) \neq b(j). \end{cases} \end{aligned}$$

Using these operators, we define operators $T_i, \sigma_i, S_i, \omega_j \in \text{End } V^{\otimes n}$, ($2 \leq i \leq n$), ($1 \leq j \leq n$), by the condition,

$$(1.3.2) \quad \begin{aligned} T_i &= \text{id}_V^{\otimes(i-2)} \otimes T \otimes \text{id}_V^{\otimes(n-i)}, \\ \sigma_i &= \text{id}_V^{\otimes(i-2)} \otimes \sigma \otimes \text{id}_V^{\otimes(n-i)}, \\ S_i &= \text{id}_V^{\otimes(i-2)} \otimes S \otimes \text{id}_V^{\otimes(n-i)}, \\ \omega_j &= \text{id}_V^{\otimes(j-1)} \otimes wt \otimes \text{id}_V^{\otimes(n-j)}. \end{aligned}$$

We now define an operator T_1 on $V^{\otimes n}$ by

$$(1.3.3) \quad T_1 = T_2^{-1} \cdots T_n^{-1} S_n \cdots S_2 \omega_1.$$

Then it is shown in [SS, Th.3.2] that $\tau : a_i \mapsto T_i$ ($1 \leq i \leq n$) gives rise to a representation of $\mathcal{H}_{n,r}$ on $V^{\otimes n}$.

Let $\bar{} : K \rightarrow K$ be the unique \mathbb{Q} -algebra involution such that $\bar{v} = v^{-1}$, $\bar{u}_i = u_i^{-1}$ for $i = 1, \dots, r$. We say that a map ϕ on a K vector space X is antilinear if $\phi(\lambda x) = \bar{\lambda} \phi(x)$ for $\lambda \in K, x \in X$. One can check by (1.1.1) that there exists a unique antilinear \mathbb{Q} -algebra automorphism $a \mapsto \bar{a}$ on $\mathcal{H}_{n,r}$ such that $\bar{a}_i = a_i^{-1}$ ($1 \leq i \leq n$). We call this map the bar involution on $\mathcal{H}_{n,r}$.

1.4. Recall that $\mathcal{H}_{n,r}$ has an alternative presentation given in [S, Th.3.7] as follows. (However, we remark that this presentation only admits a specialization of the type $\varphi : K \rightarrow K'$, where K' is a field such that $\varphi(\xi_i)$ are all distinct.) $\mathcal{H}_{n,r}$ is generated by $\{a_2, \dots, a_n, \xi_1, \dots, \xi_n\}$, subject to the following relations.

$$(1.4.1) \quad \begin{aligned} (a_i - v)(a_i + v^{-1}) &= 0 & (2 \leq i \leq n) \\ (\xi_i - u_1) \cdots (\xi_i - u_r) &= 0 & (1 \leq i \leq n) \\ a_i a_{i+1} a_i &= a_{i+1} a_i a_{i+1} & (2 \leq i \leq n) \\ a_i a_j &= a_j a_i & (|i - j| \geq 2) \\ \xi_i \xi_j &= \xi_j \xi_i & (1 \leq i, j \leq n), \end{aligned}$$

$$(1.4.2) \quad a_j \xi_j = \xi_{j-1} a_j + \Delta^{-2} \sum_{c_1 < c_2} (u_{c_2} - u_{c_1})(v - v^{-1}) F_{c_1}(\xi_{j-1}) F_{c_2}(\xi_j),$$

$$(1.4.3) \quad a_j \xi_{j-1} = \xi_j a_j - \Delta^{-2} \sum_{c_1 < c_2} (u_{c_2} - u_{c_1})(v - v^{-1}) F_{c_1}(\xi_{j-1}) F_{c_2}(\xi_j),$$

$$(1.4.4) \quad a_j \xi_k = \xi_k a_j \quad (k \neq j-1, j),$$

where $\Delta = \prod_{i > j} (u_i - u_j)$ is the Vandermonde determinant with respect to the parameters u_1, \dots, u_r , and the sum in (1.4.2) or (1.4.3) is taken for all integers $1 \leq c_1, c_2 \leq r$. For each integer $1 \leq c \leq r$, $F_c(X)$ is a certain polynomial in a

variable X with coefficients in $\mathbb{Z}[u_1, \dots, u_r]$, defined in [S, 3.3.2]. Note that the generators a_2, \dots, a_n above may be identified with the generators appeared in 1.1.

Under the representation $\tau : \mathcal{H}_{n,r} \rightarrow \text{End } V^{\otimes n}$, the generator ξ_i is mapped to ω_i for each i .

§2. Involutions associated to $\mathcal{H}_{n,r}$ and $U_v(\mathfrak{sl}_m)$

2.1 The operator T in (1.3.1) has its origin in the study of the universal R -matrix Θ attached to U_v . Let E be the orthogonal complement of $\sum \varepsilon_i$ in the Euclidean space \mathbb{R}^m with the standard basis $\varepsilon_1, \dots, \varepsilon_m$. The root system Φ of \mathfrak{sl}_m is given by the set $\{\varepsilon_i - \varepsilon_j \mid 1 \leq i \neq j \leq m\}$ with $\Phi^+ = \{\varepsilon_i - \varepsilon_j \mid i < j\}$. Thus the root lattice $\mathbb{Z}\Phi$ is given by the \mathbb{Z} -submodule of E consisting of integral vectors. Let (\cdot, \cdot) be the inner product of \mathbb{R}^m . The weight lattice Λ is given by the set of $\lambda \in E$ such that $(\lambda, \mu) \in \mathbb{Z}$ for any $\mu \in \Phi$. For $1 \leq i \leq m$, put $\bar{\varepsilon}_i = \varepsilon_i - \frac{1}{m} \sum_{j=1}^m \varepsilon_j \in \Lambda$. Then $\sum \bar{\varepsilon}_i = 0$, and $\bar{\varepsilon}_i$ is a weight with weight vector e_i . The weight lattice Λ is identified with the set $\mathbb{Z}^m / \mathbb{Z}(1, \dots, 1)$ by the correspondence $\lambda = \sum c_i \bar{\varepsilon}_i \leftrightarrow (c_1, \dots, c_m)$. For a U_v -module M , we denote by M_λ the weight subspace of M corresponding to $\lambda \in \Lambda$.

Let U_v^+ (resp. U_v^-) be the subalgebra of U_v generated by E_i, K_i (resp. F_i, K_i), respectively. For each $\mu \in \mathbb{Z}\Phi, \mu \geq 0$, we denote by U_μ^+, U_μ^- the weight subspace of U_v^\pm with respect to μ or $-\mu$, respectively. Then there exists an element $\Theta_\mu \in U_\mu^- \otimes U_\mu^+$ with $\Theta_0 = 1 \otimes 1$, for each μ , and $\Theta = \sum_{\mu \geq 0} \Theta_\mu$ (an element in a completion of $U_v \otimes U_v$, see [L, 4.1]) can be defined.

Let M and M' be finite dimensional U_v -modules. We fix an m -th root $v^{1/m}$ of v , and consider the extension field $K(v^{1/m})$ of K . (Accordingly, we regard U_v as the algebra over $K(v^{1/m})$ if needed). Following [Ja, 7.3, 7.9], we introduce a linear map $C' \in \text{End } M \otimes M'$ (\tilde{f} in the notation of [Ja]). We define a map $f : \Lambda \times \Lambda \rightarrow K(v^{1/m})^*$ by

$$f(\lambda, \mu) = (v^{1/m})^{-m(\lambda, \mu)}$$

for all $\lambda, \mu \in \Lambda$. Note that $(\lambda, \mu) \in \frac{1}{m}\mathbb{Z}$. In particular, we have

$$(2.1.1) \quad f(\bar{\varepsilon}_i, \bar{\varepsilon}_j) = v^{1/m - \delta_{ij}}.$$

Now C' is defined, for $\lambda, \mu \in \Lambda$, by

$$C'(x \otimes y) = f(\lambda, \mu)x \otimes y$$

for all $x \in M_\lambda$ and $y \in M_\mu$.

The element Θ induces a well-defined map $\Theta_{M, M'} \in \text{End } M \otimes M'$. It is known ([Ja, Th. 7.3]) that the map $\Theta_{M, M'} C' \sigma : M' \otimes M \rightarrow M \otimes M'$ gives

rise to an isomorphism of U_v -modules, where $\sigma : M' \otimes M \rightarrow M \otimes M'$ is the permutation of factors.

2.2 The bar involution on K can be extended obviously to an involution on $K(v^{1/n})$. The bar involution $-$ on U_v is an antilinear \mathbb{Q} -algebra automorphism on U_v defined on the generators by

$$\overline{E}_i = E_i, \quad \overline{F}_i = F_i, \quad \overline{K}_i = K_i^{-1}.$$

The bar involution is extended to $U_v \otimes U_v$ by $\overline{x \otimes y} = \overline{x} \otimes \overline{y}$. Let $\overline{\Theta} = - \circ \Theta \circ -$ be the bar conjugate of Θ . Then it is known by [L, 4.1] that $\Theta \overline{\Theta} = 1 \otimes 1$.

We consider the special case where $M = M' = V$, and write $\Theta_{V,V} \in \text{End}(V \otimes V)$ simply as Θ . Then as is well-known (cf. [FKK, Prop. 2.1]), we have

$$(2.2.1) \quad (\Theta C' \sigma)^{-1} = v^{-1/m} T.$$

More precisely, the action of C' and $\Theta = \sum \Theta_\mu$ on $V \otimes V$ are described as follows. Since $e_i \in V$ is a weight vector with weight $\bar{\varepsilon}_i \in \Lambda$, by the property of Θ_μ (cf. [Ja, Chap. 7]), we have

$$(2.2.2) \quad \Theta_\mu(e_i \otimes e_j) = \begin{cases} (v^{-1} - v)e_j \otimes e_i & \text{if } \mu = \varepsilon_i - \varepsilon_j \text{ with } i < j, \\ e_i \otimes e_j & \text{if } \mu = 0, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that

$$(2.2.3) \quad \Theta(e_i \otimes e_j) = \begin{cases} e_i \otimes e_j & \text{if } i \geq j, \\ e_i \otimes e_j - (v - v^{-1})e_j \otimes e_i & \text{if } i < j. \end{cases}$$

Put $C = v^{-1/m} C'$. Then by (2.1.1), we have

$$(2.2.4) \quad C(e_i \otimes e_j) = \begin{cases} e_i \otimes e_j & i \neq j, \\ v^{-1}e_i \otimes e_j & i = j. \end{cases}$$

We define an antilinear involution ψ on $V^{\otimes n}$ inductively as follows: First define ψ on V by

$$\psi\left(\sum c_i e_i\right) = \sum \bar{c}_i e_i.$$

Next let W_1, W_2 be tensor powers of V , and assume that the involutions ψ on W_1, W_2 are already defined. We define ψ on $W_1 \otimes W_2$ by

$$\psi(w_1 \otimes w_2) = \Theta(\psi(w_1) \otimes \psi(w_2)).$$

Then it is shown in [L, 4.2.4, 27.3.6] that ψ on $V^{\otimes n}$ does not depend on the decomposition $V^{\otimes n} = W_1 \otimes W_2$, and it is compatible with the U_v -module structure of $V^{\otimes n}$ in the following sense: $\psi(ux) = \bar{u}\psi(x)$ for $u \in U_v, x \in V^{\otimes n}$.

In [FKK], Frenkel, Khovanov and Kirillov studied the relationship between Kazhdan-Lusztig basis of \mathcal{H}_n and canonical basis of U_v by making use of the $\mathcal{H}_n \otimes U_v$ -module $V^{\otimes n}$. In particular, they showed

Proposition 2.3 ([FKK, Prop. 2.4]). *The bar involution of \mathcal{H}_n is compatible with the involution ψ on $V^{\otimes n}$, i.e., for any $a \in \mathcal{H}_n$, we have*

$$\psi \circ a = \bar{a} \circ \psi.$$

The main objective in this section is to extend this result to the case of $\mathcal{H}_{n,r}$. We shall show that

Theorem 2.4. *The bar involution on $\mathcal{H}_{n,r}$ is compatible with the involution ψ on $V^{\otimes n}$, i.e., for any $a \in \mathcal{H}_{n,r}$, we have*

$$(2.4.1) \quad \psi \circ a = \bar{a} \circ \psi.$$

2.5. The remainder of this section is devoted to the proof of Theorem 2.4. We denote by e_I with $I = (i_1, \dots, i_n)$ the vector $e_{i_1} \otimes \dots \otimes e_{i_n}$ of $V^{\otimes n}$. Hence $\{e_I \mid I \in [1, m]^n\}$ gives a basis of $V^{\otimes n}$. (Here $[1, m]$ means the set $\{1, 2, \dots, m\}$). The symmetric group \mathfrak{S}_n acts on $[1, m]^n$ by permuting the factors, compatible with the action on $V^{\otimes n}$, i.e. $\sigma(e_I) = e_{\sigma I}$ for $\sigma \in \mathfrak{S}_n$. If we denote by $m_I(i)$ the multiplicity of i occurring in $I = (i_1, \dots, i_n)$, then e_I is a weight vector of U_v -module $V^{\otimes n}$ with weight $\sum_i m_I(i) \bar{\epsilon}_i$.

We define an antilinear involution $-$ on $V^{\otimes n}$ by $\bar{x} = \sum_I \bar{c}_I e_I$ for $x = \sum_I c_I e_I$. Let Ψ_i be a linear map on $V^{\otimes n}$ defined by

$$\Psi_i = (\Delta^{(i-1)} \otimes 1) \Theta \otimes 1^{\otimes (n-i)}.$$

Then it follows from the definition that the involution ψ can be expressed as

$$(2.5.1) \quad \psi = \Psi_n \Psi_{n-1} \dots \Psi_2 \circ -.$$

In order to describe the involution ψ , first we shall concentrate on the map $\Psi_n = (\Delta^{(n-1)} \otimes 1) \Theta$. We prepare some notation. By $z \mapsto z_{ij}$, we denote the embedding $U_v^{\otimes 2} \rightarrow U_v^{\otimes n}$ subject to the i -th and j -th factors, i.e., for $z = a \otimes b \in U_v \otimes U_v$, we put

$$z_{ij} = x_1 \otimes x_2 \otimes \dots \otimes x_n$$

with $x_i = a, x_j = b$ and $x_k = 1$ for $k \neq i, j$.

For $\alpha, \beta \in \mathbb{Z}\Phi$ with $\alpha \geq \beta \geq 0$, we define $X_{\alpha, \beta}^i \in U_v^{\otimes n}$ by

$$(2.5.2) \quad X_{\alpha, \beta}^i = (1^{\otimes(n-i)} \otimes K_{\alpha}^{-1} \otimes 1^{\otimes(i-1)})(\Theta_{\alpha-\beta})_{n-i, n} \quad (2 \leq i \leq n-1),$$

where $K_{\alpha} = \prod K_i^{m_i}$ if $\alpha = \sum_i m_i(\varepsilon_i - \varepsilon_{i+1})$. The following lemma is a generalization of [J, Lemma 7.4]. The proof is reduced to the case $n = 3$ by making use of the relation $\Delta^{(n)} \otimes 1 = (\Delta \otimes 1^{\otimes(n-1)})(\Delta^{(n-1)} \otimes 1)$ (note that this relation is different from the defining relation for $\Delta^{(n)}$ in 1.2). The case $n = 3$ follows from Lemma 7.4 in [J].

Lemma 2.6. *For all $\mu \in \mathbb{Z}\Phi$ with $\mu \geq 0$, we have*

$$(\Delta^{(n-1)} \otimes 1)\Theta_{\mu} = \sum (\Theta_{\mu-\nu_1})_{n-1, n} X_{\nu_1, \nu_2}^2 \cdots X_{\nu_{n-2}, \nu_{n-1}}^{n-1},$$

where the sum is taken over all the sequences $\mu \geq \nu_1 \geq \cdots \geq \nu_{n-1} = 0$ such that $\nu_i \in \mathbb{Z}\Phi$.

2.7. We shall describe $\Psi_n = (\Delta^{(n-1)} \otimes 1)\Theta$. For $I = (i_1, \dots, i_n)$, let (η_1, \dots, η_n) be the sequence defined by $\eta_k = \varepsilon_{i_k}$. Let us define a linear map $\Theta_{k, n}^{\#}$ on $V^{\otimes n}$, for $k = 1, \dots, n-1$, by

$$\Theta_{k, n}^{\#}(e_I) = \begin{cases} e_I & \text{if } i_k \geq i_n, \\ e_I + v^{-(\eta_k - \eta_n, \eta_{k+1} + \cdots + \eta_{n-1})}(v^{-1} - v)e_{I'} & \text{if } i_k < i_n, \end{cases}$$

where $I' = (k, n)I$. (In the case where $k = n-1$, we understand that the inner product in the second formula is equal to 0).

We have the following lemma.

Lemma 2.8. *As operators on $V^{\otimes n}$, we have*

$$(2.8.1) \quad (\Delta^{(n-1)} \otimes 1)\Theta = \Theta_{n-1, n}^{\#} \Theta_{n-2, n}^{\#} \cdots \Theta_{1, n}^{\#}.$$

Proof. First we compute $X_{\alpha, \beta}^{n-k}(e_I)$ for $I = (i_1, \dots, i_n)$. By (2.2.2) and (2.5.2), we see that

$$(2.8.2) \quad X_{\alpha, \beta}^{n-k}(e_I) = \begin{cases} v^{-(\alpha, \eta_{k+1})}(v^{-1} - v)e_{I'} & \text{if } \alpha - \beta = \eta_k - \eta_n > 0 \text{ and } i_k < i_n, \\ v^{-(\alpha, \eta_{k+1})}e_I & \text{if } \alpha = \beta, \\ 0 & \text{otherwise,} \end{cases}$$

with $I' = (k, n)I$.

Next we compute $(\Delta^{(n-1)} \otimes 1)\Theta(e_I)$. For a fixed $I = (i_1, \dots, i_n)$, let \mathcal{P}_I be the set of subsets $\mathbf{p} = \{p_1 < \cdots < p_k\}$ of $\{1, \dots, n-1\}$ such that $i_{p_k} < \cdots < i_{p_2} < i_{p_1} < i_n$. We put $k = |\mathbf{p}|$. For $\mathbf{p} \in \mathcal{P}_I$, let

$$I(\mathbf{p}) = (p_k, n) \cdots (p_2, n)(p_1, n)I = (n, p_1, p_2, \dots, p_k)I.$$

Then it follows from Lemma 2.6 and (2.8.2) that

$$(\Delta^{(n-1)} \otimes 1)\Theta(e_I) = \sum_{\mathbf{p} \in \mathcal{P}_I} v^{-c_{\mathbf{p}}}(v^{-1} - v)^{|\mathbf{p}|} e_{I(\mathbf{p})},$$

where

$$\begin{aligned} c_{\mathbf{p}} = & (\eta_{p_1} - \eta_n, \eta_{p_1+1} + \cdots + \eta_{p_2}) \\ & + (\eta_{p_2} - \eta_n, \eta_{p_2+1} + \cdots + \eta_{p_3}) \\ & + \quad \dots\dots\dots \\ & + (\eta_{p_k} - \eta_n, \eta_{p_k+1} + \cdots + \eta_{n-1}). \end{aligned}$$

On the other hand, the right hand side of (2.8.1) is easily computed. We have

$$\Theta_{n-1,n}^{\sharp} \Theta_{n-2,n}^{\sharp} \cdots \Theta_{1,n}^{\sharp}(e_I) = \sum_{\mathbf{p} \in \mathcal{P}_I} v^{-d_{\mathbf{p}}}(v^{-1} - v)^{|\mathbf{p}|} e_{I(\mathbf{p})}$$

with

$$\begin{aligned} d_{\mathbf{p}} = & (\eta_{p_1} - \eta_n, \eta_{p_1+1} + \cdots + \eta_{n-1}) \\ & + (\eta_{p_2} - \eta_{p_1}, \eta_{p_2+1} + \cdots + \eta_{n-1}) \\ & + \quad \dots\dots\dots \\ & + (\eta_{p_k} - \eta_{p_{k-1}}, \eta_{p_k+1} + \cdots + \eta_{n-1}). \end{aligned}$$

But then we have

$$d_{\mathbf{p}} = \sum_{j=1}^k (\eta_{p_j}, \eta_{p_j+1} + \cdots + \eta_{p_{j+1}}) - (\eta_n, \eta_{p_1+1} + \cdots + \eta_{n-1}),$$

where we use the convention that $p_{k+1} = n - 1$. This implies that $c_{\mathbf{p}} = d_{\mathbf{p}}$ for any $\mathbf{p} \in \mathcal{P}_I$, and the lemma follows. \square

2.9. For a fixed $1 \leq i, j \leq n$ with $i \neq j$, we define an embedding $\text{End } V^{\otimes 2} \rightarrow \text{End } V^{\otimes n}, x \mapsto x_{ij}$, in a similar way as in 2.5; x_{ij} denotes the transformation on $V^{\otimes n}$ which acts on i -th and j -th factors of $V^{\otimes n}$ via the map x , and acts trivially on other factors. Then it is easy to see for any $\sigma \in \mathfrak{S}_n$ that

$$(2.9.1) \quad \sigma x_{ij} \sigma^{-1} = x_{\sigma(i)\sigma(j)}.$$

In later discussions, we consider the operators $\Theta_{ij}, C_{ij}, T_{ij}, S_{ij}$ for $\Theta, C, T, S \in \text{End } V^{\otimes 2}$, respectively. In particular, we note that $T_{i-1,i} = T_i$ (resp. $S_{i-1,i} =$

S_i) for $i = 2, \dots, n$ in the notation of (1.3.2). We also note that $\Theta_{i-1,i} = \Theta_{i-1,i}^\sharp$ by (2.2.3) and 2.7. However, Θ_{ij}^\sharp does not mean the embedding in general.

Let $\overline{\Theta}_{ij}^\sharp$ be the linear transformation on $V^{\otimes n}$ defined by $\overline{\Theta}_{ij}^\sharp = - \circ \Theta_{ij}^\sharp \circ -$. Then $\overline{\Theta}_{ij}^\sharp$ coincides with the map defined in 2.7, but by replacing v by v^{-1} . The bar operation on $V^{\otimes 2}$ is compatible with the bar operation on Θ . It follows that $\overline{\Theta}_{ij} = - \circ \Theta_{ij} \circ -$. The following relations are easily verified.

$$(2.9.2) \quad \begin{aligned} - \circ \omega_i \circ - &= \omega_i^{-1}, \\ - \circ C_{ij} \circ - &= C_{ij}^{-1}, \\ - \circ T_{ij} \circ - &= T_{ji}^{-1}, \\ - \circ S_{ij} \circ - &= S_{ji}^{-1}. \end{aligned}$$

For a pair k, n such that $1 \leq k \leq n-1$, we put

$$D_{k,n} = \sigma_{k,n} C_{k,n} C_{k+1,n} \cdots C_{n-1,n},$$

where $\sigma_{k,n}$ denotes the cyclic permutation $(k, k+1, \dots, n)$. We have the following lemma.

Lemma 2.10. *For $1 \leq k \leq n-1$, we have*

$$(2.10.1) \quad \overline{\Theta}_{k,n}^\sharp \overline{\Theta}_{k,n-1}^\sharp \cdots \overline{\Theta}_{k,k+1}^\sharp = D_{k,n} T_n T_{n-1} \cdots T_{k+1}.$$

Proof. First consider the case where $k = n-1$. It follows from (2.2.1) that we have

$$(2.10.2) \quad T_n = (n-1, n) C_{n-1,n}^{-1} \overline{\Theta}_{n-1,n}^\sharp$$

since $\Theta \overline{\Theta} = 1$ and $\Theta_{n-1,n} = \Theta_{n-1,n}^\sharp$. Since $D_{n-1,n} = (n-1, n) C_{n-1,n} = C_{n-1,n}(n-1, n)$, we have $\overline{\Theta}_{n-1,n}^\sharp = D_{n-1,n} T_n$ as asserted.

Next we show that

$$(2.10.3) \quad \overline{\Theta}_{k,n}^\sharp D_{k,n-1} = D_{k,n} T_n \quad \text{for } 1 \leq k \leq n-2.$$

By (2.9.1) we have

$$\begin{aligned} &(n-1, n) C_{k,n} C_{k+1,n} \cdots C_{n-1,n} T_n \\ &= C_{k,n-1} C_{k+1,n-1} \cdots C_{n-2,n-1} C_{n-1,n} (n-1, n) T_n \\ &= C_{k,n-1} C_{k+1,n-1} \cdots C_{n-2,n-1} \overline{\Theta}_{n-1,n}^\sharp. \end{aligned}$$

The last formula follows from (2.10.2). In order to show (2.10.3), we have only to check that

$$\overline{\Theta}_{k,n}^\sharp \sigma_{k,n-1} C_{k,n-1} \cdots C_{n-2,n-1} = \sigma_{k,n-1} C_{k,n-1} \cdots C_{n-2,n-1} \overline{\Theta}_{n-1,n}^\sharp.$$

It is easy to evaluate the maps on both sides at e_I . For e_I with $I = (i_1, \dots, i_n)$, they have the common values

$$v^{-(\eta_k + \eta_{k+1} + \dots + \eta_{n-2}, \eta_{n-1})} e_{I'}$$

if $i_{n-1} \geq i_n$, and

$$v^{-(\eta_k + \eta_{k+1} + \dots + \eta_{n-2}, \eta_{n-1})} e_{I'} + v^{-(\eta_k + \eta_{k+1} + \dots + \eta_{n-2}, \eta_n)} (v - v^{-1}) e_{I''}$$

if $i_{n-1} < i_n$, with $I' = \sigma_{k,n-1} I$ and $I'' = \sigma_{k,n} I'$. This proves (2.10.3).

Now the lemma is immediate by substituting $\bar{\Theta}_{k,i}^\# = D_{k,i} T_i D_{k,i-1}^{-1}$ for $i \geq k+2$ by (2.10.3), and $\bar{\Theta}_{k,k+1}^\# = D_{k,k+1} T_{k+1}$ into the left hand side of (2.10.1). \square

By using Lemma 2.10, we can describe the involution ψ as follows.

Proposition 2.11. *Let $\sigma_0 = (1, n)(2, n-1) \dots$ be the longest length element in \mathfrak{S}_n , and put*

$$\hat{C} = \prod_{1 \leq i < j \leq n} C_{ij}.$$

(Note that the operators C_{ij} commute with each other). Then we have

$$\psi = {}^- \circ \sigma_0 \hat{C} T_2 (T_3 T_2) \dots (T_n T_{n-1} \dots T_2).$$

Proof. By (2.5.1) and Lemma 2.8, we have (cf. 2.9)

$$\begin{aligned} {}^- \circ \psi &= {}^- \circ (\Theta_{n-1,n}^\# \Theta_{n-2,n}^\# \dots \Theta_{1,n}^\#) (\Theta_{n-2,n-1}^\# \Theta_{n-3,n-1}^\# \dots \Theta_{1,n-1}^\#) \dots (\Theta_{12}^\#) \circ {}^- \\ &= (\bar{\Theta}_{n-1,n}^\# \bar{\Theta}_{n-2,n}^\# \dots \bar{\Theta}_{1,n}^\#) (\bar{\Theta}_{n-2,n-1}^\# \bar{\Theta}_{n-3,n-1}^\# \dots \bar{\Theta}_{1,n-1}^\#) \dots (\bar{\Theta}_{12}^\#). \end{aligned}$$

It is clear that $\bar{\Theta}_{ij}^\#$ and $\bar{\Theta}_{i'j'}^\#$ commute with each other when $\{i, j\} \cap \{i', j'\} = \emptyset$. Hence we have

$$\begin{aligned} {}^- \circ \psi &= (\bar{\Theta}_{n-1,n}^\#) (\bar{\Theta}_{n-2,n}^\# \bar{\Theta}_{n-2,n-1}^\#) \dots (\bar{\Theta}_{1,n}^\# \bar{\Theta}_{1,n-1}^\# \dots \bar{\Theta}_{12}^\#) \\ &= (D_{n-1,n} T_n) (D_{n-2,n} T_n T_{n-1}) \dots (D_{1,n} T_n T_{n-1} \dots T_2), \end{aligned}$$

where the second equality follows from Lemma 2.10. By definition of D_{ij} , and by using (2.9.1), the last formula is modified to

$$(2.11.1) \quad {}^- \circ \psi = \sigma_0 (C_{12} T_2) (C_{13} C_{23} T_3 T_2) \dots (C_{1,n} C_{2,n} \dots C_{n-1,n} T_n T_{n-1} \dots T_2).$$

Here we note that

(2.11.2) The product $C_{1,k}C_{2,k}\cdots C_{k-1,k}$ commutes with T_2, T_3, \dots, T_{k-1} .

In fact, (2.11.2) is reduced to showing that $C_{a-1,k}C_{a,k}$ commutes with T_a , and this follows from the fact that $C_{a-1,k}C_{a,k}$ acts on the subspace of $V^{\otimes n}$ generated by e_I and $e_{(a-1,a)I}$ by a scalar multiplication $v^{-(\eta_{a-1}+\eta_a, \eta_k)}$.

Now, by using (2.11.2), (2.11.1) is further modified to

$${}^- \circ \psi = \sigma_0 \cdot \prod_{i < j} C_{ij} \cdot T_2(T_3T_2) \cdots (T_n \cdots T_2).$$

This proves the proposition. \square

2.12. We now proceed to the proof of Theorem 2.4. For the proof, it is enough to show (2.4.1) for the generators a_1, \dots, a_n . By Proposition 2.3, we know already that (2.4.1) holds for a_2, \dots, a_n . So, we have only to show it for a_1 , i.e., to show that

$$(2.12.1) \quad \psi T_1 = T_1^{-1} \psi.$$

We shall show (2.12.1). Let $\widehat{C} = \prod C_{ij}$ be as in Proposition 2.11. First we note that

$$(2.12.2) \quad \widehat{C} \text{ commutes with } T_{ij}, S_{ij}, \sigma \text{ for any } i \neq j \text{ and any } \sigma \in \mathfrak{S}_n.$$

In fact, let $V_I^{\otimes n}$ be the subspace of $V^{\otimes n}$ generated by $\{e_{\sigma I} \mid \sigma \in \mathfrak{S}_n\}$ for a fixed $I = (i_1, \dots, i_n)$. Then \widehat{C} acts on $V_I^{\otimes n}$ as a scalar multiplication by v^{-c} with $c = \sum_{i < j} (\eta_i, \eta_j)$. (2.12.2) follows from this.

By definition (1.3.3) and Proposition 2.11, we can write

$$(2.12.3) \quad \psi T_1 = Z \omega_1 \quad \text{with } Z = {}^- \circ \sigma_0 \widehat{C} T_2(T_3T_2) \cdots (T_{n-1} \cdots T_2) S_n S_{n-1} \cdots S_2.$$

We show that

$$(2.12.4) \quad Z = Z^{-1}.$$

In fact, by (2.9.2), (2.9.1) and (2.12.2), we have

$$\begin{aligned} Z^{-1} &= {}^- \circ (S_{21}S_{32} \cdots S_{n,n-1})(T_{21}T_{32} \cdots T_{n-1,n-2}) \cdots (T_{21}T_{32})T_{21} \widehat{C} \sigma_0 \\ &= {}^- \circ \sigma_0 \widehat{C} (S_n S_{n-1} \cdots S_2)(T_n T_{n-1} \cdots T_3) \cdots (T_n T_{n-1})T_n. \end{aligned}$$

It is known by [SS, Lemma 3.8] that

$$(2.12.5) \quad (S_n S_{n-1} \cdots S_2)T_j = T_{j-1}(S_n S_{n-1} \cdots S_2)$$

for $j = 3, \dots, n$. Therefore we have

$$Z^{-1} = {}^- \circ \sigma_0 \widehat{C}(T_{n-1} \cdots T_2) \cdots (T_{n-1} T_{n-2}) T_{n-1} (S_n \cdots S_2).$$

Now by using the relations

$$(T_{n-1} T_{n-2} \cdots T_{n-a}) T_i = T_{i-1} (T_{n-1} T_{n-2} \cdots T_{n-a})$$

for $i \geq n - a + 1$, which follows from the braid relations of \mathcal{H}_n , it is easy to see that

$$(T_{n-1} \cdots T_2) \cdots (T_{n-1} T_{n-2}) T_{n-1} = T_2 (T_3 T_2) \cdots (T_{n-1} \cdots T_2).$$

Hence (2.12.4) holds.

Since ψ is an involution, by using (2.12.4), we have

$$T_1^{-1} \psi = (\psi T_1)^{-1} = \omega_1^{-1} Z^{-1} = \omega_1^{-1} Z.$$

Hence to prove (2.12.1), it is enough to show that $\omega_1^{-1} Z = Z \omega_1$. Note that $\sigma_0 \omega_1 \sigma_0^{-1} = \omega_n$ and that ω_n commutes with \widehat{C} and T_2, \dots, T_{n-1} . Thus, by (2.9.2), we have

$$(2.12.6) \quad \omega_1^{-1} Z = {}^- \circ \sigma_0 \widehat{C} T_2 (T_3 T_2) \cdots (T_{n-1} \cdots T_2) \omega_n (S_n \cdots S_2).$$

Here we note the following formula.

$$(2.12.7) \quad \omega_i S_i = S_i \omega_{i-1} \quad \text{for } i = 2, \dots, n.$$

In fact, it is enough to see the formula for the case where $n = i = 2$, and $S_i = S$. Now ω_2 and ω_1 act as a (common) scalar multiplication on $e_j \otimes e_k$ and $e_k \otimes e_j$ if $b(j) = b(k)$. S permutes $e_j \otimes e_k$ and $e_k \otimes e_j$ if $b(j) \neq b(k)$. (2.12.7) follows easily from these facts.

Now by applying (2.12.7), we have $\omega_n (S_n \cdots S_2) = (S_n \cdots S_2) \omega_1$. Hence (2.12.6) implies that $\omega_1^{-1} Z = Z \omega_1$, and (2.12.1) holds. The theorem is proved.

2.13. By making use of Theorem 2.4, combined with Proposition 2.11, one can describe the bar involution for generators $\{a_2, \dots, a_n, \xi_1, \dots, \xi_n\}$ of $\mathcal{H}_{n,r}$ given in 1.4.

Proposition 2.14. *Let $\{a_2, \dots, a_n, \xi_1, \dots, \xi_n\}$ be the generators of $\mathcal{H}_{n,r}$ given in 1.4, and put $x = a_2(a_3 a_2) \cdots (a_n a_{n-1} \cdots a_2)$. Then we have*

$$\begin{aligned} \bar{a}_i &= a_i^{-1} & (2 \leq i \leq n), \\ \bar{\xi}_j &= x^{-1} \xi_{n-j+1}^{-1} x & (1 \leq j \leq n). \end{aligned}$$

Proof. It is enough to show the formula for ξ_j . By Theorem 2.4, we have

$$\bar{\omega}_j = \psi^{-1} \circ \omega_j \circ \psi.$$

Note that $^- \circ \omega_j \circ - = \omega_j^{-1}$, $\sigma_0 \omega_j \sigma_0 = \omega_{n-j+1}$, and that ω_j commutes with \widehat{C} . Then by Proposition 2.11, we see that

$$\psi^{-1} \circ \omega_j \circ \psi = X^{-1} \omega_{n-j+1}^{-1} X$$

with $X = T_2(T_3 T_2) \cdots (T_n T_{n-1} \cdots T_2)$. Since the representation τ is faithful if we choose $m_k \geq n$ for $k = 1, \dots, r$, and since $\tau(\xi_j) = \omega_j$, $\tau(x) = X$, this gives the required formula for ξ_j . \square

§3. Kazhdan-Lusztig basis and Canonical basis

3.1. Let W be a Weyl group with a set of generators S . We denote by \mathcal{H} the Hecke algebra associated to W . It is an associative algebra over $\mathbb{Q}(v)$ defined by generators a_s ($s \in S$) and relations

$$(3.1.1) \quad (a_s - v)(a_s + v^{-1}) = 0$$

together with usual braid relations. \mathcal{H} has a basis $\{a_w \mid w \in W\}$, where $a_w = a_{s_1} \cdots a_{s_q}$ for a reduced expression $w = s_1 \cdots s_q$.

For any subset J of S , we denote by W_J the parabolic subgroup in W generated by $s \in J$. Let W^J be the set of distinguished representatives in W/W_J . Hence W^J is the set of minimal elements w in wW_J with respect to the length $l(w)$ of W . Let \mathcal{H}_J be the subalgebra of \mathcal{H} generated by a_s with $s \in J$. Then \mathcal{H}_J is isomorphic to the Hecke algebra of W_J . Let φ be a homomorphism from \mathcal{H} to $\mathbb{Q}(v)$ defined by $a_s \mapsto v$ for any $s \in S$. We denote by φ_J the restriction of φ on \mathcal{H}_J . Let M_J be the induced \mathcal{H} -module $\text{Ind}_{\mathcal{H}_J}^{\mathcal{H}} \varphi_J$. Then by Deodhar [D], it is known that M_J has a basis $\{m_w \mid w \in W^J\}$ with the following properties,

$$(3.1.2) \quad a_s m_w = \begin{cases} m_{sw} + (v - v^{-1})m_w & \text{if } l(sw) < l(w), \\ m_{sw} & \text{if } l(sw) > l(w), sw \in W^J, \\ vm_w & \text{if } l(sw) > l(w), sw \notin W^J, \end{cases}$$

and $a_w m_1 = m_w$ for an identity element $1 \in W$ and $w \in W^J$. Note that $w \in W^J$ and $l(sw) < l(w)$ imply that $sw \in W^J$.

Let us define a bar involution on \mathcal{H} by $\bar{v} = v^{-1}$ and $\bar{a}_s = a_s^{-1}$ as in 1.3. We also define a bar involution on M_J by the condition that $\bar{\overline{m}}_e = m_e$ and

that $\overline{hm} = \overline{h}\overline{m}$ for $h \in \mathcal{H}$, $m \in M_J$. Let \leq be the partial order on W^J induced from the Bruhat order on W . The Kazhdan-Lusztig basis $\{C_w^J \mid w \in W^J\}$ of M_J was introduced by Kazhdan-Lusztig [KL] for $M_\emptyset \simeq \mathcal{H}$, and then extended by Deodhar [D] to the case M_J . They are characterized by the following two properties.

$$(3.1.3) \quad C_w^J \in m_w + \sum_{\substack{x \in W^J \\ x < w}} v^{-1} \mathbb{Z}[v^{-1}] m_x$$

$$(3.1.4) \quad \overline{C}_w^J = C_w^J.$$

The parabolic Kazhdan-Lusztig polynomial $P_{x,w}^J \in \mathbb{Z}[q]$ is defined, following Deodhar, in terms of the coefficient $p_{x,w}$ of m_x in the expression of C_w^J as follows.

$$p_{x,w} = q^{(l(w)-l(x))/2} \overline{P_{x,w}^J}(q) \quad \text{with } v = q^{-1/2}.$$

Note that in [KL], [D], \mathcal{H} is defined by the quadratic relation $(T_s - q)(T_s + 1)$ for an indeterminate q instead of (3.1.1). Then the relationship with our situation is given as follows; $v = q^{-1/2}$, and our a_s corresponds to $-vT_s$ in their setup. In particular, our C_σ^J corresponds to $(-1)^{l(\sigma)} C_\sigma^J$ under the notation of [KL], [D], and our m_w corresponds to $(-v)^{l(w)} m_w$ of [D].

3.2. For $x, w \in W^J$ such that $x < w$, we denote by $\mu(x, w)$ the coefficient of $q^{(l(w)-l(x)-1)/2}$ in $P_{x,w}^J(q)$. Note that $\deg P_{x,w}^J \leq \frac{1}{2}(l(w)-l(x)-1)$. Let $s \in S$ be such that $l(sw) < l(w)$ for $w \in W^J$. Then C_w^J is determined inductively, with respect to the Bruhat order, by

$$(3.2.1) \quad C_w^J = (a_s + v^{-1})C_{sw}^J - \sum_{\substack{y \in W^J, y \leq sw \\ sy < y \text{ or } sy \notin W^J}} (-1)^{l(w)-l(y)} \mu(y, sw) C_y^J$$

Now the action of a_s on C_w^J is given as follows; for $s \in S$ and $w \in W^J$,

$$(3.2.2) \quad a_s C_w^J = \begin{cases} -v^{-1} C_w^J + C_{sw}^J - \sum_{\substack{y < w \\ sy \leq y \text{ or } sy \notin W^J}} (-1)^{l(w)-l(y)} \mu(y, w) C_y^J, \\ v C_w^J, \end{cases}$$

where the first equality occurs when $sw > w$ and $sw \in W^J$, and the second occurs when $sw < w$ or $sw \notin W^J$. In fact, (3.2.2) can be shown as in a similar way in [KL, 2.3] once we know that

$$(3.2.3) \quad a_s C_w^J = v C_w^J \quad \text{if } sw > w \text{ and } sw \notin W^J.$$

We show (3.2.3). By comparing the coefficients of m_w on both sides, we see that (3.2.3) is equivalent to the identities,

$$(3.2.4) \quad P_{sx,w}^J = P_{x,w}^J \quad \text{if } sx \in W^J.$$

(Note that $sx \in W^J$ if $sx < x$ and $x \in W^J$). Since we are in a setting $u = -1$ in the notation of [D], we have by [D, Prop. 3.4],

$$(3.2.5) \quad P_{x,w}^J = P_{xw_J,ww_J}$$

for $x, w \in W^J$, where w_J is the longest element in W^J , and $P_{x,y}$ is the original Kazhdan-Lusztig polynomial for W . By our assumption, $sw > w$ and $sw \notin W^J$. Then there exists $s' \in W_J$ such that $sw = ws'$ (e.g., [H, Lemma 7.2]), and we have $sww_J < ww_J$. It follows that $a_s C_{ww_J} = v C_{ww_J}$ by [KL, 2.3] (C_w is a Kazhdan-Lusztig basis for \mathcal{H}). This induces similar identities for P_{xw_J,ww_J} and P_{sxw_J,ww_J} as in (3.2.4). Now (3.2.4) follows from these identities in view of (3.2.5).

3.3. We assume that $W \simeq \mathfrak{S}_n$, and consider the $U_v \otimes \mathcal{H}_n$ -module $V^{\otimes n}$. The weights of U_v on $V^{\otimes n}$ are given by $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{Z}_{\geq 0}^m$ with $\sum \lambda_i = n$. The weight subspace $V_\lambda^{\otimes n}$ has a basis $\{e_I\}$, with $I = (i_1, \dots, i_n)$ such that $\#\{a \mid i_a = k\} = \lambda_k$. The involution ψ on $V^{\otimes n}$ stabilizes the subspace $V_\lambda^{\otimes n}$. Moreover $V_\lambda^{\otimes n}$ is an \mathcal{H}_n -submodule of $V^{\otimes n}$ generated by a single element e_{I_λ} , where

$$(3.3.1) \quad I_\lambda = (\underbrace{m, \dots, m}_{\lambda_m \text{-times}}, \dots, \underbrace{1, \dots, 1}_{\lambda_1 \text{-times}}).$$

Let $\mathfrak{S}_\lambda \simeq \mathfrak{S}_{\lambda_m} \times \dots \times \mathfrak{S}_{\lambda_1}$ be the stabilizer of I_λ in \mathfrak{S}_n . Then \mathfrak{S}_λ is a parabolic subgroup W_J of \mathfrak{S}_n , and we denote by \mathcal{H}_λ the parabolic subalgebra \mathcal{H}_J corresponding to \mathfrak{S}_λ . It is easy to see that \mathcal{H}_n -module $V_\lambda^{\otimes n}$ is isomorphic to $M_J = \text{Ind}_{\mathcal{H}_\lambda}^{\mathcal{H}_n} \varphi$.

Recall that $\{e_I \mid I \in [1, m]^n\}$ is the basis of $V^{\otimes n}$, which we call the standard basis of $V^{\otimes n}$. The canonical basis $\{b_I \mid I \in [1, m]^n\}$ of U_v -module $V^{\otimes n}$ is characterized by the following two properties ([L, Chap. 27]).

$$(3.3.2) \quad b_I \in e_I + \sum_{I'} v^{-1} \mathbb{Z}[v^{-1}] e_{I'},$$

$$\psi(b_I) = b_I,$$

where the sum in the first formula is taken over all I' having the same weight as I .

It is shown in [FKK] that the map $f : m_\sigma \mapsto e_{\sigma(I_\lambda)}$ gives an isomorphism $M_J \simeq V_\lambda^{\otimes n}$, which transfers the bar involution on M_J to the involution ψ on $V_\lambda^{\otimes n}$. We identify M_J with $V_\lambda^{\otimes n}$.

We define a partial order $I < I'$ on $[1, m]^n$ as the transitive closure of the relation

$$(\dots, a, \dots, b, \dots) < (\dots, b, \dots, a, \dots) \quad \text{if } a > b.$$

Then we have the following.

Lemma 3.4. *Let $\sigma, \tau \in \mathfrak{S}_n^J$, and assume that $\sigma < \tau$. Then we have $\sigma(I_\lambda) < \tau(I_\lambda)$.*

Proof. The proof is reduced to the case where $\tau = \sigma s$ with a (not necessarily simple) reflection $s \in \mathfrak{S}_n$. So we assume that s is a transposition (p, q) with $1 \leq p < q \leq n$. Then it is easy to check that $\sigma^{-1}(p) < \sigma^{-1}(q)$ if $l(\sigma^{-1}) < l(s\sigma^{-1})$. If we write $I_\lambda = (i_1, \dots, i_n)$, we have

$$\sigma(I_\lambda) = (\dots, i_{\sigma^{-1}(p)}, \dots, i_{\sigma^{-1}(q)}, \dots), \quad \sigma s(I_\lambda) = (\dots, i_{\sigma^{-1}(q)}, \dots, i_{\sigma^{-1}(p)}, \dots).$$

Now by (3.3.1) and by our assumption, we have $i_{\sigma^{-1}(q)} \leq i_{\sigma^{-1}(p)}$. The lemma follows from this. \square

The following special case is worth mentioning.

Lemma 3.5 ([FKK, Lemma 2.1]). *Let $\sigma \in W^J$ and $s \in S$ be a transposition $(i, i+1)$. Let a and b be i -th and $(i+1)$ -th entries of $\sigma(I_\lambda)$, respectively. Then we have*

$$\begin{aligned} \text{if } a > b, & \quad \text{then } s\sigma > \sigma \text{ and } s\sigma \in W^J, \\ \text{if } a = b, & \quad \text{then } s\sigma > \sigma \text{ and } s\sigma \notin W^J, \\ \text{if } a < b, & \quad \text{then } s\sigma < \sigma. \end{aligned}$$

The following result shows that the Kazhdan-Lusztig basis is obtained as a special case of the canonical basis of $V^{\otimes n}$.

Theorem 3.6 ([FKK, Th. 2.5]). *Assume that $W \simeq \mathfrak{S}_n$. Then, under the identification $M_J \simeq V_\lambda^{\otimes n}$, we have for each $\sigma \in W^J$,*

$$C_\sigma^J = b_{\sigma(I_\lambda)}.$$

Combining Theorem 3.6 with Lemma 3.4, we have the following refinement of (3.3.2).

Corollary 3.7. *Under the above notation, we have*

$$b_I \in e_I + \sum_{I' < I} v^{-1} \mathbb{Z}[v^{-1}] e_{I'}.$$

The following corollary is also immediate from (3.2.2), Lemma 3.5 and Theorem 3.6.

Corollary 3.8. *Let $s = (a, a+1)$ be a transposition. Then for $I = (i_1, \dots, i_n)$, we have*

$$T_s b_I = v b_I \quad \text{if } i_a \leq i_{a+1}.$$

§4. $\mathcal{H}_{n,r}$ -submodules of $V^{\otimes n}$

4.1. We now return to the setup in section 1, and assume that $W = W_{n,r}$. We consider the $\mathcal{H}_{n,r}$ -module $V^{\otimes n}$ with the graded vector space $V = \bigoplus_{i=1}^r V_i$ as before. We prepare some notation in addition to 3.1. Let $S = \{s_1, \dots, s_n\}$ be the set of generators of $W_{n,r}$, where $t_1 = s_1$ has order r , and s_2, \dots, s_n are generators of \mathfrak{S}_n corresponding to transposition $(1, 2), \dots, (n-1, n)$. We define $t_i \in W_{n,r}$ by $t_i = s_i \cdots s_2 t_1 s_2 \cdots s_i$ for $i = 2, \dots, n$. Then t_1, \dots, t_n gives rise to a set of generators of the group $(\mathbb{Z}/r\mathbb{Z})^n$.

The basis vector of $V^{\otimes n}$ is given by e_I with $I = (i_1, \dots, i_n)$ as before. By 1.3, e_I can also be written as $e_{j_1}^{(\varepsilon_1)} \otimes \cdots \otimes e_{j_n}^{(\varepsilon_n)}$. In this case, we write I as $I = (j_1^{(\varepsilon_1)}, \dots, j_n^{(\varepsilon_n)})$. The weight λ of U_v on $V^{\otimes n}$ is expressed as $\lambda = (\lambda_1, \dots, \lambda_m)$ as in 3.3. In our situation, I determines a multi-composition $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(r)})$, with $\lambda^{(k)} = (\lambda_1^{(k)}, \dots, \lambda_{m_k}^{(k)}) \in \mathbb{Z}_{\geq 0}^{m_k}$, such that $\sum_{j,k} \lambda_j^{(k)} = n$; the correspondence is given by $\lambda_j^{(k)} = \#\{a \mid j_a = j, \varepsilon_a = k\}$. If one ignores the superscripts of $\lambda_j^{(k)}$, $\boldsymbol{\lambda}$ reduces to λ . We call $\boldsymbol{\lambda}$ the weight of e_I . We denote by $V_{\boldsymbol{\lambda}}^{\otimes n}$ the subspace of $V^{\otimes n}$ generated by e_I whose weight is $\boldsymbol{\lambda}$. It is easy to check that the action of $\mathcal{H}_{n,r}$ on $V^{\otimes n}$ stabilizes the subspace $V_{\boldsymbol{\lambda}}^{\otimes n}$. For the weight $\boldsymbol{\lambda}$, put

$$(4.1.1) \quad e_{\lambda^{(k)}} = \underbrace{e_{m_k}^{(k)} \otimes \cdots \otimes e_{m_k}^{(k)}}_{\lambda_{m_k}^{(k)} \text{-times}} \otimes \cdots \otimes \underbrace{e_1^{(k)} \otimes \cdots \otimes e_1^{(k)}}_{\lambda_1^{(k)} \text{-times}}$$

and define a vector $e_{\boldsymbol{\lambda}} \in V_{\boldsymbol{\lambda}}^{\otimes n}$ by

$$(4.1.2) \quad e_{\boldsymbol{\lambda}} = e_{\lambda^{(r)}} \otimes e_{\lambda^{(r-1)}} \otimes \cdots \otimes e_{\lambda^{(1)}}.$$

The stabilizer of $e_{\boldsymbol{\lambda}}$ in \mathfrak{S}_n is isomorphic to

$$\mathfrak{S}_{\boldsymbol{\lambda}} = \mathfrak{S}_{\lambda^{(r)}} \times \mathfrak{S}_{\lambda^{(r-1)}} \times \cdots \times \mathfrak{S}_{\lambda^{(1)}}$$

with $\mathfrak{S}_{\lambda^{(k)}} = \mathfrak{S}_{\lambda_{m_k}^{(k)}} \times \cdots \times \mathfrak{S}_{\lambda_1^{(k)}}$. We define a subgroup W_{λ} of $W_{n,r}$ by $W_{\lambda} = \mathfrak{S}_{\lambda} \ltimes (\mathbb{Z}/r\mathbb{Z})^n$, i.e.,

$$W_{\lambda} \simeq W_{\lambda^{(r)}} \times W_{\lambda^{(r-1)}} \times \cdots \times W_{\lambda^{(1)}}$$

with $W_{\lambda^{(k)}} = W_{\lambda_{m_k}^{(k)},r} \times \cdots \times W_{\lambda_1^{(k)},r}$. Let \mathcal{H}_{λ} be the Ariki-Koike algebra associated to W_{λ} , i.e.,

$$(4.1.3) \quad \mathcal{H}_{\lambda} = \mathcal{H}_{\lambda^{(r)}} \otimes \mathcal{H}_{\lambda^{(r-1)}} \otimes \cdots \otimes \mathcal{H}_{\lambda^{(1)}}$$

with $\mathcal{H}_{\lambda^{(k)}} = \mathcal{H}_{\lambda_{m_k}^{(k)},r} \otimes \cdots \otimes \mathcal{H}_{\lambda_1^{(k)},r}$. One can regard \mathcal{H}_{λ} as a subalgebra of $\mathcal{H}_{n,r}$ in a natural way, by making use of generators $\{\xi_1, \dots, \xi_n\}$ as discussed in [S, 4.2]. (For example, if $a + b = n$, then $\mathcal{H}_{a,r} \otimes \mathcal{H}_{b,r} \hookrightarrow \mathcal{H}_{n,r}$, where $\mathcal{H}_{a,r}$ (resp. $\mathcal{H}_{b,r}$) is the subalgebra of $\mathcal{H}_{n,r}$ generated by $s_2, \dots, s_a, \xi_1, \dots, \xi_a$, (resp. $s_{a+2}, \dots, s_n, \xi_{a+1}, \dots, \xi_n$), respectively.)

We can define a linear character $\varphi_n^{(k)} : \mathcal{H}_{n,r} \rightarrow K$ by

$$\begin{aligned} \varphi_n^{(k)}(a_i) &= v & (2 \leq i \leq n), \\ \varphi_n^{(k)}(\xi_j) &= u_k & (1 \leq j \leq n) \end{aligned}$$

(cf. [S, (3.3.3), 5.2]), and define $\varphi_{\lambda^{(k)}} : \mathcal{H}_{\lambda^{(k)}} \rightarrow K$ by $\varphi_{\lambda^{(k)}} = \varphi_{\lambda_{m_k}^{(k)}}^{(k)} \otimes \cdots \otimes \varphi_{\lambda_1^{(k)}}^{(k)}$.

Then we define a linear character $\varphi_{\lambda} : \mathcal{H}_{\lambda} \rightarrow K$ by

$$(4.1.4) \quad \varphi_{\lambda} = \varphi_{\lambda^{(r)}} \otimes \varphi_{\lambda^{(r-1)}} \otimes \cdots \otimes \varphi_{\lambda^{(1)}}$$

according to the embedding into $\mathcal{H}_{n,r}$ given in (4.1.3). Put $V_{\lambda}^{\otimes n} = M_{\lambda}$. Then we have the following result.

Proposition 4.2. *Let the notations be as above.*

- (i) M_{λ} is generated by e_{λ} as $\mathcal{H}_{n,r}$ -module, and we have

$$M_{\lambda} = \mathcal{H}_{n,r} e_{\lambda} \simeq \text{Ind}_{\mathcal{H}_{\lambda}}^{\mathcal{H}_{n,r}} \varphi_{\lambda}$$

as $\mathcal{H}_{n,r}$ -modules.

- (ii) M_{λ} has a basis $\{e_{\sigma}\}$ indexed by the set \mathfrak{S}_n^J (here we regard \mathfrak{S}_{λ} as a parabolic subgroup $(\mathfrak{S}_n)_J$ of \mathfrak{S}_n). The action of $\mathcal{H}_{n,r}$ on this basis is given as follows:

$$a_s e_{\sigma} = \begin{cases} e_{s\sigma} + (v - v^{-1})e_{\sigma} & \text{if } l(s\sigma) < l(\sigma), \\ e_{s\sigma} & \text{if } l(s\sigma) > l(\sigma), s\sigma \in \mathfrak{S}_n^J, \\ ve_{\sigma} & \text{if } l(s\sigma) > l(\sigma), s\sigma \notin \mathfrak{S}_n^J, \end{cases}$$

$$\xi_j e_{\sigma} = u_{\varepsilon(j,\sigma)} e_{\sigma},$$

where $\varepsilon(j, \sigma) \in \{1, \dots, r\}$ is given as follows; write $e_{\lambda} = e_I$ as in 4.1, and put $\varepsilon(j, \sigma) = \varepsilon_j$ for $\sigma(I) = (j_1^{(\varepsilon_1)}, \dots, j_n^{(\varepsilon_n)})$.

- (iii) *There exists an involution $\bar{} : M_{\lambda} \rightarrow M_{\lambda}$ satisfying the property that $\overline{hm} = \bar{h}\bar{m}$ for $h \in \mathcal{H}_{n,r}$, $m \in M_{\lambda}$, and that $\bar{e}_{\sigma} = e_{\sigma}$ for $\sigma = 1$.*

Proof. Let λ be the weight of e_{λ} as U_v -module. Then M_{λ} coincides with $V_{\lambda}^{\otimes n}$ and e_{λ} is nothing but $e_{I_{\lambda}}$ given in (3.3.1). Then by [FKK, Prop. 2.1], $V_{\lambda}^{\otimes n}$ is generated by $e_{I_{\lambda}}$ as \mathcal{H}_n -module, and is isomorphic to M_J as in 3.3, for a parabolic subgroup $\mathfrak{S}_{\lambda} = \mathfrak{S}_n^J$. In particular, we see that $M_{\lambda} = \mathcal{H}_{n,r}e_{\lambda}$, and that

$$\dim V_{\lambda}^{\otimes n} = |\mathfrak{S}_n^J| = \dim \text{Ind}_{\mathcal{H}_{\lambda}}^{\mathcal{H}_{n,r}} \varphi_{\lambda}.$$

Since it is easy to see that Ke_{λ} is a one-dimensional \mathcal{H}_{λ} -module affording φ_{λ} , the first assertion follows.

Now we define a basis $\{e_{\sigma} \mid \sigma \in \mathfrak{S}_n^J\}$ in M_{λ} by using the basis $\{m_{\sigma}\}$ in M_J . Then we have $e_{\sigma} = m_{\sigma} = e_{\sigma(I_{\lambda})}$ by [FKK]. The first three formula in (ii) now follows from (3.1.2). The last formula in (ii) follows by considering the action of ω_j on $e_{\sigma(I_{\lambda})} \in V^{\otimes n}$.

The involution ψ on $V^{\otimes n}$ stabilizes the subspace $V_{\lambda}^{\otimes n}$. We define the bar involution $\bar{}$ on $M_{\lambda} = V_{\lambda}^{\otimes n}$ in terms of ψ . Then we have $\overline{hm} = \bar{h}\bar{m}$ by Theorem 2.4. Since $\psi(e_{\lambda}) = e_{\lambda}$, we have $\bar{e}_1 = e_1$. The proposition is proved. \square

The following result is an analogue to the case of $\mathcal{H}_{n,r}$ of the result of Frenkel, Khovanov and Kirillov (cf. Theorem 3.6) concerning the Kazhdan-Lusztig basis of \mathcal{H}_n and canonical basis of U_q , and also of the parabolic Kazhdan-Lusztig basis of Deodhar (cf. 3.1). But note that \mathcal{H}_{λ} is no longer a parabolic subalgebra of $\mathcal{H}_{n,r}$.

Theorem 4.3. *Let $M_{\lambda} \simeq \text{Ind}_{\mathcal{H}_{\lambda}}^{\mathcal{H}_{n,r}} \varphi_{\lambda}$ be the induced $\mathcal{H}_{n,r}$ -module. Then there exists a unique basis $\{b_{\sigma} \mid \sigma \in \mathfrak{S}_n^J\}$ in M_{λ} satisfying the following properties.*

$$b_{\sigma} \in e_{\sigma} + \sum_{\substack{\tau \in \mathfrak{S}_n^J \\ \tau < \sigma}} v^{-1} \mathbb{Z}[v^{-1}] e_{\tau},$$

$$\bar{b}_{\sigma} = b_{\sigma}.$$

The coefficient $p_{\tau,\sigma}$ of e_{τ} in the expression of b_{σ} is given by the parabolic Kazhdan-Lusztig polynomial for the case of $\mathfrak{S}_n^J \subset \mathfrak{S}_n$ just as in 3.1.

Proof. By Theorem 3.6, canonical basis $\{b_{\sigma(I_{\lambda})} \mid \sigma \in \mathfrak{S}_n^J\}$ gives rise to a basis of $V_{\lambda}^{\otimes n}$, which corresponds to the parabolic Kazhdan-Lusztig basis $\{C_{\sigma}^J\}$ in M_J . Hence, if we define the basis $\{b_{\sigma}\}$ in $M_{\lambda} = V_{\lambda}^{\otimes n}$ in terms of $\{b_{\sigma(I_{\lambda})}\}$, the assertions in the theorem follow from 3.1 and Proposition 4.2. \square

4.4. We now pass to a more general situation. Take an integer $t \geq 0$ such that $t \leq m_k$ for $k = 1, \dots, r$. Let $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)})$ be an r -tuple of compositions as in 4.1, but here we assume that $\lambda^{(k)} = \emptyset$ for $k \neq r$, and that $\lambda^{(r)} = (\lambda_{t+1}^{(r)}, \dots, \lambda_{m_r}^{(r)}) \in \mathbb{Z}_{\geq 0}^{m_r-t}$. We consider a pair $(\lambda; \mathbf{c})$, with $\mathbf{c} = (c_1, \dots, c_t) \in \mathbb{Z}_{>0}^t$ a composition such that $\sum_{j,k} \lambda_j^{(k)} + \sum_i c_i = n$. We put $c = \sum c_i$. We denote by $M_{\lambda, \mathbf{c}}$ the subspace of $V^{\otimes n}$ generated by e_I with $I = (j_1^{(\varepsilon_1)}, \dots, j_n^{(\varepsilon_n)})$ such that $\lambda_j^{(r)} = \#\{a \mid j_a = j, \varepsilon_a = r\}$ and that $c_i = \#\{a \mid j_a = i\}$. Then $M_{\lambda, \mathbf{c}}$ is a direct sum of various weight spaces $V_{\nu}^{\otimes n}$, and so has a structure of $\mathcal{H}_{n,r}$ -module. The decomposition of $M_{\lambda, \mathbf{c}}$ into $V_{\nu}^{\otimes n}$ is described more precisely as follows. Let ν be a pair $(\lambda; \mu)$, where λ is as above, and $\mu = (\mu^{(1)}, \dots, \mu^{(r)})$ is an r -tuples of compositions $\mu^{(k)} = (\mu_1^{(k)}, \dots, \mu_t^{(k)}) \in \mathbb{Z}_{\geq 0}^t$ such that $\sum_{k=1}^r \mu_i^{(k)} = c_i$ for $1 \leq i \leq t$. We denote by $\mathcal{P}_{\lambda, \mathbf{c}}$ the set of such $\nu = (\lambda; \mu)$. Note that ν can be written, by rearranging the entries, as $(\nu^{(1)}, \dots, \nu^{(r)})$ with $\nu^{(r)} = (\mu_1^{(r)}, \dots, \mu_t^{(r)}, \lambda_{t+1}^{(r)}, \dots, \lambda_{m_r}^{(r)})$, and $\nu^{(k)} = (\mu_1^{(k)}, \dots, \mu_t^{(k)}, 0, \dots, 0)$ for $k \neq r$. Hence it determines an $\mathcal{H}_{n,r}$ -subspace $V_{\nu}^{\otimes n}$. It is easy to check that

$$M_{\lambda, \mathbf{c}} = \bigoplus_{\nu \in \mathcal{P}_{\lambda, \mathbf{c}}} V_{\nu}^{\otimes n}.$$

We shall investigate the $\mathcal{H}_{n,r}$ -module structure of $M_{\lambda, \mathbf{c}}$. For each $\nu \in \mathcal{P}_{\lambda, \mathbf{c}}$, we define $e^{\nu} \in V_{\nu}^{\otimes n}$ by $e^{\nu} = e_{\lambda} \otimes e^{\mu}$, where $e_{\lambda} = e_{\lambda^{(r)}}$ is defined just as in (4.1.1), by restricting the factors in between $e_{m_r}^{(r)}$ and $e_{t+1}^{(r)}$. $e^{\mu} \in V^{\otimes c}$ is defined by $e^{\mu} = E_1 \otimes E_2 \otimes \dots \otimes E_t$, with

$$(4.4.1) \quad E_i = (e_{t-i+1}^{(1)})^{\mu_i^{(1)}} \otimes \dots \otimes (e_{t-i+1}^{(r)})^{\mu_i^{(r)}} \in V^{\otimes c_i}.$$

Now e^{ν} can be written as $e^{\nu} = e_I$ for some I , and we denote by b^{ν} the canonical basis $b_I \in V_{\nu}^{\otimes n}$ corresponding to e_I . We define $m_{\lambda, \mathbf{c}} \in M_{\lambda, \mathbf{c}}$ by

$$(4.4.2) \quad m_{\lambda, \mathbf{c}} = \sum_{\nu \in \mathcal{P}_{\lambda, \mathbf{c}}} b^{\nu}.$$

We define a subalgebra $\mathcal{H}_{\lambda, \mathbf{c}}$ of $\mathcal{H}_{n,r}$ by $\mathcal{H}_{\lambda, \mathbf{c}} = \mathcal{H}_{\lambda} \otimes \mathcal{H}_{\mathbf{c}}$, where $\mathcal{H}_{\lambda} = \mathcal{H}_{\lambda^{(r)}}$ is defined as in (4.1.3), by modifying the definition of $\mathcal{H}_{\lambda^{(r)}}$ appropriately, and $\mathcal{H}_{\mathbf{c}}$ is defined by

$$(4.4.3) \quad \mathcal{H}_{\mathbf{c}} = \mathcal{H}_{c_1} \otimes \dots \otimes \mathcal{H}_{c_t}.$$

(Remember that \mathcal{H}_i is the Iwahori-Hecke algebra of type A_{i-1}). We define a linear character $\varphi_{\lambda, \mathbf{c}}$ of $\mathcal{H}_{\lambda, \mathbf{c}}$ by $\varphi_{\lambda, \mathbf{c}} = \varphi_{\lambda} \otimes \varphi_{\mathbf{c}}$, where $\varphi_{\lambda} = \varphi_{\lambda^{(r)}}$ is given as in (4.1.4). $\varphi_{\mathbf{c}}$ is given by $\varphi_{\mathbf{c}} = \varphi_{c_1} \otimes \dots \otimes \varphi_{c_t}$, where φ_n is the linear character of \mathcal{H}_n defined by $\varphi_n(a_j) = v$ for all generators a_j .

Under these notations, we have the following result.

Proposition 4.5. $M_{\lambda, \mathbf{c}}$ is generated by $m_{\lambda, \mathbf{c}}$ as $\mathcal{H}_{n, r}$ -module, and we have

$$M_{\lambda, \mathbf{c}} = \mathcal{H}_{n, r} m_{\lambda, \mathbf{c}} \simeq \text{Ind}_{\mathcal{H}_{\lambda, \mathbf{c}}}^{\mathcal{H}_{n, r}} \varphi_{\lambda, \mathbf{c}}, \quad \psi(m_{\lambda, \mathbf{c}}) = m_{\lambda, \mathbf{c}}.$$

Proof. It is clear that $m_{\lambda, \mathbf{c}}$ is fixed by ψ . We show the first two equalities. First we note that

$$(4.5.1) \quad hm_{\lambda, \mathbf{c}} = \varphi_{\lambda, \mathbf{c}}(h)m_{\lambda, \mathbf{c}} \quad \text{for } h \in \mathcal{H}_{\lambda, \mathbf{c}}.$$

In fact to show (4.5.1), it is enough to see, for each $\nu \in \mathcal{P}_{\lambda, \mathbf{c}}$, that

$$(4.5.2) \quad hb^\nu = \begin{cases} \varphi_\lambda(h)b^\nu & \text{if } h \in \mathcal{H}_\lambda, \\ \varphi_{\mathbf{c}}(h)b^\nu & \text{if } h \in \mathcal{H}_{\mathbf{c}}. \end{cases}$$

We show (4.5.2). In view of (4.4.1) and Corollary 3.8, we see that $a_j b^\nu = v b^\nu$ for all generators $a_j \in \mathcal{H}_{c_i}$. This implies the second equality in (4.5.2). Next we consider the first equality. By (modified form of) (4.1.1), (4.1.2), together with (4.4.1), we see that $e^\nu = e_I$, where I is of the form $I = (i_1, \dots, i_n)$ with $i_1 \geq i_2 \geq \dots \geq i_{n-c}$, and with $i_{n-c} > i_k$ for all $c+1 \leq k \leq n$. Then by Corollary 3.7, b^ν is written as a linear combination of $e_{I'}$, where $e_{I'}$ is of the form $e_\lambda \otimes e^{\mu'}$, for some $e^{\mu'} \in V^{\otimes c}$. Then as in the case of Proposition 4.2, one can check that $he_{I'} = \varphi_\lambda(h)e_{I'}$ for $h \in \mathcal{H}_\lambda$. The first equality follows from this, and so (4.5.2) holds.

Next we show that

$$(4.5.3) \quad M_{\lambda, \mathbf{c}} = \mathcal{H}_{n, r} m_{\lambda, \mathbf{c}}.$$

Let ζ be a primitive r -th root of unity. By the specialization $v \mapsto 1, u_i \mapsto \zeta^i$, $\mathcal{H}_{n, r}$ turns out to be the group algebra $\mathbb{C}W_{n, r}$. (Note that in order to apply the specialization argument, one has to replace $\mathcal{H}_{n, r}$ by its “integral form” defined over a subring $R_1 = \mathbb{Z}[v, v^{-1}, u_1, \dots, u_r, \Delta^{-1}]$ of K as in [S, 3.6]. Accordingly one needs to replace V by its R_1 -lattice with basis e_i . All the ingredients up to now make sense for this setup, and we use them freely without referring R_1 in the discussion below.)

Let $\bar{V} = \bigoplus \bar{V}_i$ be the \mathbb{C} -vector space with $\dim \bar{V}_i = m_i$. We denote by $\{\bar{e}_j^{(i)}\}$ the basis of \bar{V}_i . Then the $\mathcal{H}_{n, r}$ -module $V^{\otimes n}$ is specialized to the $\mathbb{C}W_{n, r}$ -module $\bar{V}^{\otimes n}$. Let t_i be as in 4.1. Then the action of t_i on $\bar{V}^{\otimes n}$ is given by $t_i e_I = \zeta^{\varepsilon_i} e_I$ for $I = (j_1^{(\varepsilon_1)}, \dots, j_n^{(\varepsilon_n)})$, which is the specialization of ξ_i on $V^{\otimes n}$. The previous construction for $M_{\lambda, \mathbf{c}} = \bigoplus_\nu V_\nu^{\otimes n}$ makes sense, and by the specialization we have a $W_{n, r}$ -module $\bar{M}_{\lambda, \mathbf{c}} = \bigoplus_\nu \bar{V}_\nu^{\otimes n}$. Let $\bar{e}^\nu, \bar{b}^\nu, \bar{m}_{\lambda, \mathbf{c}}$ be the elements in $\bar{M}_{\lambda, \mathbf{c}}$ obtained from $e^\nu, b^\nu, m_{\lambda, \mathbf{c}}$ by the specialization.

To show (4.5.3), it is enough to see that

$$(4.5.4) \quad \bar{M}_{\lambda, \mathbf{c}} = \mathbb{C}W_{n, r} \bar{m}_{\lambda, \mathbf{c}}.$$

We show (4.5.4). We prepare some notation. For $I = (j_1^{(\varepsilon_1)}, \dots, j_n^{(\varepsilon_n)})$, we call $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$ the signature of e_I , and $\mathbf{j} = (j_1, \dots, j_n)$ the foot of e_I . Put $\mathcal{U} = \mathbb{C}W_{n,r}\overline{m}_{\lambda,\mathbf{c}}$. Now $\overline{m}_{\lambda,\mathbf{c}}$ can be written as $\overline{m}_{\lambda,\mathbf{c}} = \sum_{\varepsilon \in [1,r]^n} \overline{m}(\varepsilon)$, where $\overline{m}(\varepsilon)$ is a linear combination of vectors \bar{e}_I whose signature is ε . Note that t_1, \dots, t_n are generators of the subgroup $(\mathbb{Z}/r\mathbb{Z})^n$ of $W_{n,r}$, and $e_{j_1}^{(\varepsilon_1)} \otimes \dots \otimes e_{j_n}^{(\varepsilon_n)}$ generates a one dimensional representation φ_ε of $(\mathbb{Z}/r\mathbb{Z})^n$ given by $t_i \mapsto \zeta^{\varepsilon_i}$. It follows that each $\overline{m}(\varepsilon)$ belongs to \mathcal{U} .

Let us consider the partial order $<$ on $[1, m]^n$ defined in 3.3. Let $F(\varepsilon)$ be the set of vectors in $\overline{M}_{\lambda,\mathbf{c}}$ consisting of \bar{e}_I with signature ε , together with the vectors $e_{I'}$ obtained from those e_I by permuting the factors. Clearly $\bigcup_{\varepsilon} F(\varepsilon)$ gives rise to a basis of $\overline{M}_{\lambda,\mathbf{c}}$. We show, by backward induction on the partial order of the set of signatures, that $F(\varepsilon) \subset \mathcal{U}$. Take $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$ and assume that $F(\varepsilon') \subset \mathcal{U}$ holds for any $\varepsilon' > \varepsilon$. Now $\overline{m}(\varepsilon)$ can be written as

$$(4.5.5) \quad \overline{m}(\varepsilon) \in \bar{e}_I + \sum_{I'} \mathbb{C}\bar{e}_{I'},$$

where the foot \mathbf{j} of I is given by

$$(4.5.6) \quad \mathbf{j} = (\underbrace{m_r, \dots, m_r}_{\lambda_{m_r}^{(r)}\text{-times}}, \dots, \underbrace{t+1, \dots, t+1}_{\lambda_{t+1}^{(r)}\text{-times}}, \underbrace{t, \dots, t}_{c_1\text{-times}}, \dots, \underbrace{1, \dots, 1}_{c_t\text{-times}}),$$

and $e_{I'}$ is a summand of some $b^\nu = b_{I''}$, not equal to e^ν . Thus $e_{I'}$ is obtained from $e_{I''}$ by permuting the factors. Note that $e_{I''}$ has the same foot as (4.5.6), and we have $I' < I''$ by Corollary 3.7. Let ε'' be the signature of I'' . As in the proof of (4.5.2), one can write $e_{I''} = e_\lambda \otimes e^\mu$ and $e_{I'} = e_\lambda \otimes e^{\mu'}$. Since $(t, \dots, t, \dots, 1, \dots, 1)$ is ordered decreasingly, the condition that $I' < I''$ implies that $\varepsilon < \varepsilon''$. It follows, by induction, that $e_{I''} \in \mathcal{U}$. By operating \mathfrak{S}_n , we see that $e_{I'} \in \mathcal{U}$ also. This implies that e_I and all its permutations of factors belong to \mathcal{U} . Hence we have $F(\varepsilon) \subset \mathcal{U}$. Thus (4.5.4), and so (4.5.3) holds.

It is easy to see that $\dim M_{\lambda,\mathbf{c}} = \dim \overline{M}_{\lambda,\mathbf{c}} = |W_{n,r}|/|W_{\lambda,\mathbf{c}}|$, where $W_{\lambda,\mathbf{c}}$ is the subgroup of $W_{n,r}$ corresponding to the subalgebra $\mathcal{H}_{\lambda,\mathbf{c}}$ of $\mathcal{H}_{n,r}$. Then (4.5.1) and (4.5.3) implies that $M_{\lambda,\mathbf{c}} \simeq \text{Ind}_{\mathcal{H}_{\lambda,\mathbf{c}}}^{\mathcal{H}_{n,r}} \varphi_{\lambda,\mathbf{c}}$. The proposition is proved. \square

4.6. The space $M_{\lambda,\mathbf{c}}$ can be decomposed into a direct sum of weight spaces $V_\nu^{\otimes n}$. Hence in view of Proposition 4.2 and Theorem 4.3, $M_{\lambda,\mathbf{c}}$ have bases inherited from the basis $\{e_I\}$ and $\{b_I\}$ of various $V_\nu^{\otimes n}$. In particular, a bar involution on $M_{\lambda,\mathbf{c}}$ can be defined, and one obtains a basis invariant under the bar involution.

Here we consider the special case where $M_{\lambda,\mathbf{c}}$ is isomorphic to the regular representation of $\mathcal{H}_{n,r}$. Hence we assume that $\lambda = \emptyset$, and $\mathbf{c} = (1^n)$. So

$\mathcal{H}_{\lambda, \mathbf{c}} \simeq K$ and $\varphi_{\lambda, \mathbf{c}} = 1_K$. $\mathcal{P}_{\lambda, \mathbf{c}}$ is in bijection with the set $[1, r]^n$, and the vector $e^\nu \in V_\nu^{\otimes n}$ corresponding to $\nu \in \mathcal{P}_{\lambda, \mathbf{c}}$ in 4.4 is given by

$$(4.6.1) \quad e^\nu = e_n^{(\varepsilon_1)} \otimes e_{n-1}^{(\varepsilon_2)} \otimes \cdots \otimes e_1^{(\varepsilon_n)}$$

under the correspondence $\nu \leftrightarrow \varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in [1, r]^n$. The basis of $V_\nu^{\otimes n}$ is obtained by permuting the factors of e^ν . If we write $e^\nu = e_I$ and $e_{\sigma(I)} = e_{\sigma, \varepsilon}$, then $\{e_{\sigma, \varepsilon} \mid \sigma \in \mathfrak{S}_n\}$ forms a basis of $V_\nu^{\otimes n}$, and

$$\{e_{\sigma, \varepsilon} \mid \sigma \in \mathfrak{S}_n, \varepsilon \in [1, r]^n\}$$

gives rise to a basis of $M_{\lambda, \mathbf{c}}$. We define a partial order on the set $\{(\sigma, \varepsilon) \mid \sigma \in \mathfrak{S}_n\}$ (for a fixed ε) as follows. Let $\tau_0 \in \mathfrak{S}_n$ be an element such that $e^\nu = e_{\tau_0(I)}$, where $I = (i_1, \dots, i_n)$ with $i_1 \geq \cdots \geq i_n$. Then we put $(\tau, \varepsilon) < (\sigma, \varepsilon)$ if $\tau\tau_0 < \sigma\tau_0$.

We write $m_0 = m_{\lambda, \mathbf{c}} = \sum b^\nu$ as in (4.4.2). Then the map $h \mapsto hm_0$ gives an isomorphism $\mathcal{H}_{n, r} \simeq M_{\lambda, \mathbf{c}}$. We denote by the same symbol the basis of $\mathcal{H}_{n, r}$ obtained from the basis $\{e_{\sigma, \varepsilon}\}$ of $M_{\lambda, \mathbf{c}}$. Since m_0 is ψ -invariant, it follows from Theorem 2.4 that the bar involution on $\mathcal{H}_{n, r}$ can be identified, under the above isomorphism, with the involution ψ on $M_{\lambda, \mathbf{c}}$. Let $\{b_{\sigma, \varepsilon}\}$ be the basis of $\mathcal{H}_{n, r}$ obtained by transferring the canonical basis of $M_{\lambda, \mathbf{c}}$ attached to $\{e_{\sigma, \varepsilon}\} \subset M_{\lambda, \mathbf{c}}$. Then the following result is immediate from Theorem 4.3.

Theorem 4.7. *There exists a unique basis $\{b_{\sigma, \varepsilon} \mid \sigma \in \mathfrak{S}_n, \varepsilon \in [1, r]^n\}$ of $\mathcal{H}_{n, r}$ satisfying the following properties.*

$$b_{\sigma, \varepsilon} \in e_{\sigma, \varepsilon} + \sum_{\substack{\tau \in \mathfrak{S}_n \\ (\tau, \varepsilon) < (\sigma, \varepsilon)}} v^{-1} \mathbb{Z}[v^{-1}] e_{\tau, \varepsilon},$$

$$\bar{b}_{\sigma, \varepsilon} = b_{\sigma, \varepsilon}.$$

The coefficient $p_{(\tau, \varepsilon), (\sigma, \varepsilon)}$ of $e_{\tau, \varepsilon}$ in the expression of $b_{\sigma, \varepsilon}$ is described by the parabolic Kazhdan-Lusztig polynomials of type A for the weight space $V_\nu^{\otimes n}$ under the correspondence $\nu \leftrightarrow \varepsilon$ in (4.6.1).

§5. The case of Iwahori-Hecke algebras of type B_n

5.1. We consider the case where $W = W_{n, 2}$ is the Weyl group of type B_n . We specify the parameters of $\mathcal{H}_{n, 2}$ by putting $u_1 = -v^{-1}, u_2 = v$, so that $\mathcal{H}_{n, 2}$ is the Hecke algebra \mathcal{H} of W as given in 3.1. We discuss the relationship between Kazhdan-Lusztig basis of \mathcal{H} and the previous basis.

Let us consider the subalgebra $\mathcal{H}_{\lambda, \mathbf{c}}$ of \mathcal{H} as in 4.4, and assume that $\mathcal{H}_{\lambda, \mathbf{c}}$ is the subalgebra \mathcal{H}_J associated to a parabolic subgroup W_J of W . We also assume that the linear character $\varphi_{\lambda, \mathbf{c}} : \mathcal{H}_J \rightarrow K$ in 4.4 is of the form φ_J in 3.1 (hence $\lambda = (\lambda^{(1)}; \lambda^{(2)}) = (-; k)$ for some $k \geq 0$). Then the \mathcal{H} -submodule $M_{\lambda, \mathbf{c}} = \bigoplus_{\nu} V_{\nu}^{\otimes n}$ of $V^{\otimes n}$ can be identified with M_J in 3.1, where $m_{\lambda, \mathbf{c}} \in M_{\lambda, \mathbf{c}}$ corresponds to $m_e \in M_J$. By Theorem 2.4 and Proposition 4.5, the bar involution on M_J given in 3.1 coincides with the involution ψ on $M_{\lambda, \mathbf{c}}$. We shall compare various bases on M_J . Put $m_{\sigma} = T_{\sigma} m_e$ for $\sigma \in W^J$. Then $\mathcal{M} = \{m_{\sigma} \mid \sigma \in W^J\}$ gives a basis of M_J . Let $\mathcal{C} = \{C_{\sigma}^J \mid \sigma \in W^J\}$ be the Kazhdan-Lusztig basis of M_J given in 3.1. We also put $\mathcal{E} = \{e_I \mid I \in \mathcal{I}_J\}$ and $\mathcal{B} = \{b_I \mid I \in \mathcal{I}_J\}$ the bases of M_J arising from the standard basis and the canonical basis of $\bigoplus_{\nu} V_{\nu}^{\otimes n}$. For a two bases $X = (x_i)$ and $Y = (y_j)$ of M_J indexed by the set $\mathcal{I}_J \simeq W^J$, we denote by $M(X, Y) = (a_{ij})$ the transition matrix from X to Y given by

$$y_i = \sum_{j \in \mathcal{I}} a_{ji} x_j.$$

Put

$$P_B = M(\mathcal{M}, \mathcal{C}), \quad P_A = M(\mathcal{E}, \mathcal{B}), \quad X = M(\mathcal{B}, \mathcal{C}), \quad Y = M(\mathcal{E}, \mathcal{M}).$$

We define a total order on W^J which is compatible with the converse of the Bruhat order on W^J , and consider the matrix $P_B = (p_{\tau, \sigma})_{\tau, \sigma \in W^J}$ with respect to this order. Then P_B is a lower unitriangular matrix. Moreover, $p_{\tau, \sigma} \in v^{-1}\mathbb{Z}[v^{-1}]$, and $p_{\tau, \sigma}$ represents the parabolic Kazhdan-Lusztig polynomials of type B_n associated to W_J up to a power of v . If we fix a total order on the set \mathcal{I}_J compatible with the weight decomposition $M_J = \bigoplus_{\nu} V_{\nu}^{\otimes n}$, the matrix P_A is a block-wise diagonal matrix, and diagonal blocks correspond to the weights on M_J . The diagonal block P_A^{ν} corresponding to the weight ν is the matrix of the parabolic Kazhdan-Lusztig polynomials of type A associated to the parabolic subgroup \mathfrak{S}_{ν} (up to powers of v), where \mathfrak{S}_{ν} is the stabilizer of e^{ν} in \mathfrak{S}_n .

We have the following.

Proposition 5.2. *The matrices P_A, P_B, X, Y satisfy the following relation.*

$$(5.2.1) \quad P_B = Y^{-1} P_A X.$$

Moreover, the matrices P_B and X are determined uniquely by P_A and Y . In other words, the parabolic Kazhdan-Lusztig polynomials of type B_n can be determined by various parabolic Kazhdan-Lusztig polynomials of type A and by the matrix Y .

Proof. It is clear that P_A, P_B, X, Y satisfy (5.2.1). We show that P_A and Y determine P_B and X uniquely. Write the equation (5.2.1) as

$$(5.2.2) \quad P_B X^{-1} = Y^{-1} P_A$$

and consider (5.2.2) as the matrix equation with unknown matrices P_B and X . We fix a bijection $W^J \simeq \mathcal{I}_J$, and write the matrices as $P_B = (p_{ij}), X^{-1} = (x_{ij})$ with $i, j \in \mathcal{I}_J$ along the order inherited from the order on W^J . Here $p_{ij} \in v^{-1}\mathbb{Z}[v^{-1}]$ and $x_{ij} \in \mathbb{Q}(v)$ such that $\bar{x}_{ij} = x_{ij}$. We determine the matrices P_B and X^{-1} row wisely. Suppose that the first $(i-1)$ -rows of P_B and X^{-1} are determined. Since P_B is lower unitriangular, one can write

$$(5.2.3) \quad \sum_{j=1}^{i-1} p_{ij} \mathbf{x}_j + \mathbf{x}_i = \boldsymbol{\alpha}_i,$$

where \mathbf{x}_j (resp. $\boldsymbol{\alpha}_j$) denotes the j -th row of X^{-1} (resp. $Y^{-1}P_A$), respectively. By applying the bar involution on (5.2.3), and by subtracting each other, one has

$$(5.2.4) \quad \sum_{j=1}^{i-1} (p_{ij} - \bar{p}_{ij}) \mathbf{x}_j = \boldsymbol{\alpha}_i - \bar{\boldsymbol{\alpha}}_i.$$

Here $\mathbf{x}_1, \dots, \mathbf{x}_{i-1}, \boldsymbol{\alpha}_i - \bar{\boldsymbol{\alpha}}_i$ are known vectors. Since $\mathbf{x}_1, \dots, \mathbf{x}_{i-1}$ are linearly independent, (5.2.4) determines $d_{ij} = p_{ij} - \bar{p}_{ij}$ uniquely. But since $p_{ij} \in v^{-1}\mathbb{Z}[v^{-1}]$, d_{ij} determines p_{ij} uniquely. Thus the i -th row of P_B is determined. By substituting p_{ij} into (5.2.3), the i -th row \mathbf{x}_i is also determined. Thus the matrices P_B and X^{-1} are determined. \square

5.3. The bases $\{e_{\sigma, \varepsilon}\}$ and $\{b_{\sigma, \varepsilon}\}$ appeared in Theorem 4.7 are nothing but the bases \mathcal{E} and \mathcal{B} , respectively. In order to relate these bases to the Kazhdan-Lusztig basis, it is essential to know about the matrix X since the matrix Y is more or less simpler than X . It would be an interesting problem to study the matrix X . One might expect that X has a relatively simple form compared to the matrix P_B . We give below a simple example of the matrix X , i.e., the relation between parabolic Kazhdan-Lusztig basis and the canonical basis.

Assume that W is the Weyl group of type B_n , and let W_J be the parabolic subgroup of type B_{n-1} . We put $J = \{t_1, s_2, \dots, s_{n-1}\}$. Then the distinguished representatives W^J are given as

$$W^J = \{s_i \cdots s_n \mid 2 \leq i \leq n+1\} \cup \{s_i \cdots s_2 t_1 s_2 \cdots s_n \mid 1 \leq i \leq n\}$$

under the convention that $s_1 = s_{n+1} = 1$. Assume that $V = V_1 \oplus V_2$ with $\dim V_1 = 1, \dim V_2 = 2$. We fix bases $e_1^{(1)}$ of V_1 and $e_1^{(2)}, e_2^{(2)}$ of V_2 , respectively.

We also write $e_3 = e_2^{(2)}, e_2 = e_1^{(2)}, e_1 = e_1^{(1)}$. Let us consider $M_{\lambda, \mathbf{c}}$ as in 4.4, where $\lambda = (\lambda^{(1)}, \lambda^{(2)}) = (-; n-1)$ and $\mathbf{c} = (1)$ (i.e., $t = 1$). Then $M_{\lambda, \mathbf{c}}$ is isomorphic to the induced representation $\text{Ind}_{\mathcal{H}_J}^{\mathcal{H}} \varphi_J$, where \mathcal{H}_J is the parabolic subalgebra of $\mathcal{H} = \mathcal{H}_{n,2}$ of type B_{n-1} and φ_J is as in 3.1. It can be decomposed into the direct sum of weight spaces $M_{\lambda, \mathbf{c}} = V_{\nu}^{\otimes n} \oplus V_{\nu'}^{\otimes n}$, where $\nu = (0, 1, n-1), \nu' = (1, 0, n-1)$ as weights for U_v . We define, for $1 \leq i \leq n$, I_i, I'_i by

$$I_i = (\underbrace{2^{(2)}, \dots, 2^{(2)}}_{i-1\text{-times}}, 2^{(1)}, 2^{(2)}, \dots, 2^{(2)}),$$

$$I'_i = (\underbrace{2^{(2)}, \dots, 2^{(2)}}_{i-1\text{-times}}, 1^{(1)}, 2^{(2)}, \dots, 2^{(2)}).$$

Then $b^\nu = b_{I_n} = e_{I_n}$ and $b^{\nu'} = b_{I'_n} = e_{I'_n}$, and we have $m_{\lambda, \mathbf{c}} = b_{I_n} + b_{I'_n}$. The Kazhdan-Lusztig basis C_σ^J for $\sigma \in W^J$ can be expressed in terms of canonical basis, as

$$C_\sigma^J = \begin{cases} b_{I_{i-1}} + b_{I'_{i-1}} & \text{if } \sigma = s_i \cdots s_n, \\ (v^i + v^{-i})b_{I_1} - b_{I_{i+1}} + b_{I'_{i+1}} & \text{if } \sigma = s_i \cdots s_2 t s_2 \cdots s_n. \end{cases}$$

This determines the matrix X completely.

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