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On a Conjecture of Fan Concerning Average Degrees and Long Cycles

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Abstract. In this paper, we prove the following result:
Let k be an integer with $k \geq 5$. Let C be a cycle in a graph G, and let H be a component of $G - C$. Suppose that C is locally longest with respect to H, and H is locally k-connected to C, $|V(H)| \geq k - 1$, $\delta(H) \geq \lfloor (k-1)/2 \rfloor$, and H is 3-connected. Let $r = (\sum_{x \in V(H)} \deg_G(x))/|V(H)|$. Then $l(C) \geq k(r + 2 - k)$, with equality only if r is an integer and either H is a complete graph of order $r + 1 - k$ and every vertex of H has the same k neighbours on C, or H is a complete graph of order $k - 1$ and every vertex of H has the same $r + 2 - k$ neighbours on C.

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§1. Introduction

In this paper, we consider only finite simple undirected graphs without loops or multiple edges. For a graph G, we let $V(G)$ and $E(G)$ denote the set of vertices and edges of G, respectively. For a vertex v of $V(G)$, we let $N_G(v)$ denote the set of vertices adjacent to v in G, and let $\deg_G(v) = |N_G(v)|$. The minimum degree of G is the minimun of $\deg_G(v)$ as v ranges over $V(G)$, and is denoted by $\delta(G)$. For a subset W of $V(G)$, the quantity $(\sum_{v \in W} \deg_G(v))/|W|$ is called the average degree of W in G. For $X \subseteq V(G)$, we let $G - X$ denote the subgraph of G obtained by deleting all vertices in X together with the edges incident with them; for a subgraph H of G, we write $G - H$ for $G - V(H)$.

Let $M = (v_0, v_1, \ldots, v_p)$ be a cycle or a path. Thus $V(M) = \{v_0, v_1, \ldots, v_p\}$ and $E(M) = \{v_0v_1, v_1v_2, \ldots, v_{p-1}v_p\}$ (if M is a cycle, then $p \geq 3$, $v_0, v_1, \ldots, v_{p-1}$ are distinct, and $v_p = v_0$; if M is a path, then $v_0, v_1, \ldots, v_p$ are all distinct). The length $l(M)$ of M is defined by $l(M) = p$, i.e., $l(M) = |E(M)|$.

When M is a cycle, for i, j with $i \le j < i + p$, we define the segment $M[v_i, v_j]$ of M by $M[v_i, v_j] = (v_i, v_{i+1}, \ldots, v_j)$ (indices are to be read modulo p); when M is a path, for i, j with $1 \le i \le j \le p$, we define the segment $M[v_i, v_j]$ of M by $M[v_i, v_j] = (v_i, v_{i+1}, \ldots, v_j)$. When M is a path, v_0 is called the initial vertex of M and v_p is called the terminal vertex of M. Now for $x, y \in V(G)$, a path in G having x as its initial vertex and y as its terminal vertex is called an (x, y)-path. For $x \in V(G)$ and $Y \subseteq V(G)$ with $x \notin Y$, an (x, y)-path with $y \in Y$ is called an (x, Y)-path. Two (x, Y)-paths are said to be *disjoint* if they have only the vertex x in common. Let H and M be two subgraphs of G with $V(H) \cap V(M) = \emptyset$. We say that H is *locally k-connected to M in G* if G contains k pairwise disjoint $(x, V(M))$-paths for every vertex $x \in V(H)$. Let now M be a cycle in G, and let H be a subgraph of $G - V(M)$. We say that M is *locally longest with respect to H in G* if we cannot obtain a cycle longer than M by replacing a segment $M[u, v]$ by a (u, v)-path of G through H.

In [2], Fan proved the following theorem(see also [3]) :

Theorem A [2, Theorem 2]. *Let k be an integer with $2 \le k \le 4$. Let C be a cycle in a graph G, and let H be a component of $G - C$. Suppose that C is locally longest with respect to H, and H is locally k-connected to C, and $|V(H)| \ge k - 1$. Let r denote the average degree of $V(H)$ in G. Then $l(C) \ge k(r + 2 - k)$, with equality only if r is an integer and either H is a complete graph of order $r + 1 - k$ and every vertex of H has the same k neighbours on C, or H is a complete graph of order $k - 1$ and every vertex of H has the same $r + 2 - k$ neighbours on C.*

He also conjectured that the same result holds for $k \ge 5$ as well.

Conjecture 1 [2, Conjecture 1]. *Let k be an integer with $k \ge 5$. Let C be a cycle in a graph G, and let H be a component of $G - C$. Suppose that C is locally longest with respect to H, and H is locally k-connected to C, and $|V(H)| \ge k - 1$. Let r denote the average degree of $V(H)$ in G. Then $l(C) \ge k(r + 2 - k)$, with equality only if r is an integer and either H is a complete graph of order $r + 1 - k$ and every vertex of H has the same k neighbours on C, or H is a complete graph of order $k - 1$ and every vertex of H has the same $r + 2 - k$ neighbours on C.*

However, Conjecture 1 does not hold. We here construct counter-examples. Let k, h be integers with $k \ge 5$ and $h \ge k - 1$. Let a be an integer with $a < \lfloor (k-1)/2 \rfloor$, and let H be a graph isomorphic to the complete bipartite graph $K_{a,h-a}$ with partite sets of cardinalities a and $h - a$. Let C be a cycle of length $2(a+1)t$ with $V(C) \cap V(H) = \emptyset$, and write $C = (c_1, c_2, \ldots, c_{2(a+1)t}, c_1)$, where $t \ge k$. Define a graph G by $V(G) = V(H) \cup V(C)$ and $E(G) = E(H) \cup E(C) \cup \{xy | x \in V(H), y \in \{c_{2(a+1)i} | i \in \{1, \cdots, t\}\}$. Then we can easily verify that C is locally longest with respect to H, and that H is locally k-connected to C. Let now $r = (\sum_{x \in V(H)} \deg_G(x))/h$. Then $r = (2a(h-a)+ht)/h$, and hence $k(r+2-k) - l(C) = (k-2(a+1))t + 2a(h-a)k/h - (k-2)k$. We also have $k - 2(a+1) > 0$ by the assumption that $a < \lfloor (k-1)/2 \rfloor$. Therefore if t is large enough, then $k(r+2-k) > l(C)$, which means that the conclusion of Conjecture 1 does not hold. Note that $\delta(H) = a$. Note also that this construction works for any integer a with $1 \le a < \lfloor (k-1)/2 \rfloor$. Having these observations in mind, we make the following new conjecture, which says that the conclusion of Conjecture 1 holds if we add the assumption that $\delta(H) \ge \lfloor (k-1)/2 \rfloor$.

Conjecture 2. *Let k be an integer with $k \ge 5$. Let C be a cycle in a graph G, and let H be a component of $G - C$. Suppose that C is locally longest with respect to H, and H is locally k-connected to C, $|V(H)| \ge k-1$ and $\delta(H) \ge \lfloor (k-1)/2 \rfloor$. Let r denote the average degree of $V(H)$ in G; i.e., $r = (\sum_{x \in V(H)} \deg_G(x))/|V(H)|$ (note that $\delta(H)$ denotes $\min\{\deg_H(x) | x \in V(H)\}$ but not $\min\{\deg_G(x) | x \in V(H)\}$). Then $l(C) \ge k(r+2-k)$, with equality only if r is an integer and either H is a complete graph of order $r+1-k$ and every vertex of H has the same k neighbours on C, or H is a complete graph of order $k-1$ and every vertex of H has the same $r+2-k$ neighbours on C.*

The purpose of this paper is to show that Conjecture 2 holds in the case where H is 3-connected.

Main Theorem. *Let k be an integer with $k \ge 5$. Let C be a cycle in a graph G, and let H be a component of $G - C$. Suppose that C is locally longest with respect to H, and H is locally k-connected to C, $|V(H)| \ge k-1$ and $\delta(H) \ge \lfloor (k-1)/2 \rfloor$. Suppose further that H is 3-connected. Let r denote the average degree of $V(H)$ in G; i.e., $r = (\sum_{x \in V(H)} \deg_G(x))/|V(H)|$. Then $l(C) \ge k(r+2-k)$, with equality only if r is an integer and either H is a complete graph of order $r+1-k$ and every vertex of H has the same k neighbours on C, or H is a complete graph of order $k-1$ and every vertex of H has the same $r+2-k$ neighbours on C.*

We here make some more definitions. Let G be a graph. For $X \subseteq V(G)$, we define $N_G(X) = \bigcup_{x \in V(G)} N_G(x)$. The subgraph of G induced by $X \subseteq V(G)$ is denoted by $\langle X \rangle_G$. A subgraph H of G is often identified with its vertex set $V(H)$; for example, $N_G(H)$ means $N_G(V(H))$ and, as is mentioned in the first paragraph, $G - H$ means $G - V(H)$. Also a vertex $x \in V(G)$ is identified with the set $\{x\}$; for example, $G - x$ means $G - \{x\}$.

Let x, y be distinct vertices of G. We define the *codistance $d^*_G(x,y)$ between x and y* to be the maximum length of an (x,y)-path in G. The *codiameter* $d^*(G)$ of G is defined by

$$d^*(G) = \min\{d^*_G(x,y) | x, y \in V(G), x \ne y\}.$$

We now list known results which we use in the proof of the Main Theorem.

Theorem B [1, Corollary 1]. *Let m be an integer with $m \geq 6$. Let u, v be two distinct vertices of a 3-connected graph G in which the degree sum of any pair of nonadjacent vertices of G is at least m. Then $d^*_G(u,v) \geq \min\{|V(G)|-1, m-2\}$. When $|V(G)| \geq m$, we have $d^*_G(u,v) = m-2$ if and only if m is even and one of the following holds*:

(i) *there exists $S \subseteq V(G)$ with $\{u,v\} \subseteq S$ and $|S| = m/2$ such that $E(G-S) = \emptyset$; or*

(ii) *$m/2 \geq 4$, and there exists $a \in V(G) - \{u,v\}$ such that for each component H of $G - \{u,v,a\}$, H is a complete graph of order $m/2-2$ and $N_G(x) - V(H) = \{u,v,a\}$ for every $x \in V(H)$.*

Theorem C [2, Theorem 1]. *Let x, y be two distinct vertices of a 2-connected graph G, and let r denote the average degree of $V(G) - \{x,y\}$ in G. Then $d^*_G(x,y) \geq r$. Futher, equality holds if and only if r is an integer, and for each component H of $G - \{x,y\}$, H is a complete graph of order $r-1$ and $N_G(x) - V(H) = \{u,v\}$ for every $x \in V(H)$.*

The following proposition is essentially proved in [2] in the course of the proof of Theorem 2 (see the second and the third paragraphs of the proof of [2, Theorem2]).

Proposition D [2, Theorem 2]. *Let k be an integer with $k \geq 2$. Let C be a cycle in a graph G, and let H be a component of $G - C$. Suppose that C is locally longest with respect to H, and H is locally k-connected to C, and $|V(H)| \geq k-1$. Suppose further that H is nonseparable. Let r denote the average degree of $V(H)$ in G.*

Let $p = |N_G(H) \cap V(C)|$, and write $N_G(H) \cap V(C) = \{u_1, \ldots, u_p\}$ so that $u_1, \ldots, u_p$ occur along C in this order and, for each i, let $H_i = \langle V(H) \cup \{u_i, u_{i+1}\}\rangle_G$ (indices are to be read modulo p). Finally let $T = \{u_i | 1 \leq i \leq p, |N_G(\{u_i, u_{i+1}\}) \cap V(H)| \geq 2\}$, and suppose that either

$$\begin{aligned} |T| &\leq k-1 \quad \text{or} \\ \big|\{\, u_i \in T \,|\, d^*_{H_i}(u_i, u_{i+1}) < k \,\}\big| &\leq k. \end{aligned}$$

Then $l(C) \geq k(r+2-k)$, with equality only if r is an integer and either H is a complete graph with order $r+1-k$ and every vertex of H has the same k neighbours on C, or H is a complete graph of order $k-1$ and every vertex of H has the same $r+2-k$ neighbours on C.

We prove a proposition, Proposition 2.1, in Section 2, and derive corollaries of Proposition 2.1 in Section 3 and, using the corollaries in Section 3, we prove the Main Theorem in Section 4.

§2. Long paths in 3-connected graphs

For an integer $s \geq 1$, let K_s denote the complete graph of order s. In this section, we prove the following modification of Theorem B:

Proposition 2.1. *Let $d \geq 3$ be an integer, and let G be a 3-connected graph of order at least $2d+1$ with $\delta(G) \geq d$. Let $u, v \in V(G)$ with $u \neq v$, and suppose that $d^*_G(u,v) = 2d-1$. Then one of the following six statements holds (it is possible that some two of these six statements hold simultaneously) :*

(i) *there exists $S \subseteq V(G)$ with $|\{u,v\} \cap S| = 1$ and $|S| = d$ such that $E(G-S) = \emptyset$;*

(ii) *there exists $S \subseteq V(G)$ with $\{u,v\} \subset S$ and $|S| = d$ such that $|E(G-S)| = 1$;*

(iii) *$d \geq 4$, and there exists $a \in V(G) - \{u,v\}$ and there exists a component H_0 of $G - \{u,v,a\}$ such that $|V(H_0)| = d-1$ and $N_G(V(H_0)) - V(H_0) = \{u,v,a\}$, and such that for each component H of $G - \{u,v,a\}$ with $H \neq H_0$, we have $H \cong K_{d-2}$ and $N_G(x) - V(H) = \{u,v,a\}$ for every $x \in V(H)$;*

(iv) *there exist $a, b \in V(G) - \{u,v\}$ with $a \neq b$ and $ab \in E(G)$ such that for each component H of $G - \{u,v,a,b\}$, we have $H \cong K_{d-2}$, and either $N_G(x) - V(H) = \{u,v,a\}$ for every $x \in V(H)$ or $N_G(x) - V(H) = \{u,v,b\}$ for every $x \in V(H)$;*

(v) *$d = 5$, and there exists $S \subseteq V(G)$ with $\{u,v\} \subset S$ and $|S| = 4$ such that for each component H of $G-S$, we have $H \cong K_2$ and $N_G(x) - V(H) = S$ for every $x \in V(H)$; or*

(vi) *$d = 4$, and there exist $a, b \in V(G) - \{u,v\}$ with $a \neq b$ such that for each component H of $G - \{u,v,a,b\}$, either we have $H \cong K_1$ and $N_G(V(H)) = \{u,v,a,b\}$, or we have $H \cong K_2$ and $N_G(x) - V(H) = \{u,v,a\}$ for every $x \in V(H)$.*

Proof. Let P be a (u,v)-path of length $2d-1$, and write

$$P = (v_0, \ldots, v_{2d-1}),$$

where $v_0 = u$ and $v_{2d-1} = v$. Since $|V(G)| \geq 2d+1$, we have $|V(G-P)| \geq 1$. We present the rest of the proof by dividing it into two subsections.

2.1. Component of $G - P$

Throughout this subsection, we let H denote a component of $G - P$. We are mainly concerned with the structure of $\langle V(H) \cup V(P) \rangle_G$. The following lemma immediately follows from the maximality of the length of P :

Lemma 2.1. *Suppose that* $|V(H)| = 1$. *Then one of the following holds*:

(i) $N_G(H) \cap V(P) = \{v_{2i} | 0 \leq i \leq d-1\}$;
(ii) $N_G(H) \cap V(P) = \{v_{2i+1} | 0 \leq i \leq d-1\}$; *or*
(iii) *there exists* m *with* $0 \leq m \leq d-2$ *such that* $N_G(H) \cap V(P) = \{v_{2i} | 0 \leq i \leq m\} \cup \{v_{2i+1} | m+1 \leq i \leq d-1\}$.

Lemma 2.2. *Suppose that* $|V(H)| = 1$. *Then* $E_G(V(P) - N_G(H), V(P) - N_G(H)) - E(P) = \emptyset$.

Proof. Write $V(H) = \{a\}$. By way of contradiction, suppose that there exists $v_i v_{i'} \in E(G)$ with $0 \leq i < i' \leq 2d-1$ such that $v_i, v_{i'} \in V(P) - N_G(a)$ and $v_i v_{i'} \notin E(P)$. Note that if (i) of Lemma 2.1 holds, then

$$v_{i-1}, v_{i'-1} \in N_G(a) \cap V(P) \tag{2.1}$$

and, if (ii) of Lemma 2.1 holds, then

$$v_{i+1}, v_{i'+1} \in N_G(a) \cap V(P). \tag{2.2}$$

Further if (iii) of Lemma 2.1 holds, then from the fact that $v_0, v_{2d-1} \in N_G(a) \cap V(P)$ and $\{v_{2m+1}, v_{2m+2}\}$ is the only pair of consecutive vertices of P which does not intersect with $N_G(a) \cap V(P)$, we see that (2.1) or (2.2) holds. Thus in any case, (2.1) or (2.2) holds. By symmetry, we may assume that (2.1) holds. Then $(v_0, \ldots, v_{i-1}, a, v_{i'-1}, v_{i'-2}, \ldots, v_i, v_{i'}, \ldots, v_{2d-1})$ is a (v_0, v_{2d-1})-path of length $2d$, which contradicts the assumption that $d^*_G(v_0, v_{2d-1}) = 2d-1$. □

In the rest of this subsection, we consider the case where $|V(H)| \geq 2$.

Lemma 2.3. *Suppose that* $|V(H)| \geq 2$. *Then* H *is nonseparable.*

Proof. By way of contradiction, suppose that H is separable. Let B_1,B_2 be two distinct endblocks of H and, for each $1 \leq i \leq 2$, let b_i be the cut vertex of H such that $b_i \in V(B_i)$. Set $n' = |N_G(V(B_1 - b_1) \cup V(B_2 - b_2)) \cap V(P)|$, and write $N_G(V(B_1 - b_1) \cup V(B_2 - b_2)) \cap V(P) = \{x'_1, \ldots, x'_{n'}\}$, where $x'_1, \ldots, x'_{n'}$ occur in this order along P. Define

$$N' = \{(x'_\nu, x'_{\nu+1}) \,|\, 1 \leq \nu \leq n' - 1, N_G(\{x'_\nu, x'_{\nu+1}\}) \cap V(B_i - b_i) \neq \emptyset \text{ for each } i = 1, 2\}.$$

Clearly $|N'| \leq n' - 1$. Since G is 3-connected, $|N_G(B_i - b_i) \cap V(P)| \geq 2$ for each i, and hence $n' \geq 2$ and $|N'| \geq 1$. Our first aim is to prove $n' = 2$ and $|N'| = 1$. It follows from the maximality of the length of P that for each $1 \leq \nu \leq n' - 1$, we have $l(P[x'_\nu, x'_{\nu+1}]) \geq 2$ and, if $(x'_\nu, x'_{\nu+1}) \in N'$, then we further have $l(P[x'_\nu, x'_{\nu+1}]) \geq 4$ by the definition of N'. Hence $2d-1 = l(P) \geq \sum_{\nu=1}^{n'-1} l(P[x'_\nu, x'_{\nu+1}]) \geq 4 + 2(n' - 2) = 2n'$, which implies $n' \leq d - 1$.

Claim 2.1. *Let $(x'_\nu, x'_{\nu+1}) \in N'$. Then the following hold.*

(i) $l(P[x'_\nu, x'_{\nu+1}]) \geq 2(d - (n' - 1))$.

(ii) *If $|N'| < n' - 1$, then $l(P[x'_\nu, x'_{\nu+1}]) \geq 2(d - (n' - 1)) + 2$.*

Proof. We may assume $N_G(x'_\nu) \cap V(B_1 - b_1) \neq \emptyset$ and $N_G(x'_{\nu+1}) \cap V(B_2 - b_2) \neq \emptyset$. Let $B_{1,\nu}$ denote the graph obtained from $\langle V(B_1) \cup \{x'_\nu\}\rangle_G$ by joining x'_ν and b_1 (in the case where $x'_\nu b_1 \in E(G)$, this means that we simply let $B_{1,\nu} = \langle V(B_1) \cup \{x'_\nu\}\rangle_G$). Similarly let $B_{2,\nu+1}$ denote the graph obtained from $\langle V(B_2) \cup \{x'_{\nu+1}\}\rangle_G$ by joining $x'_{\nu+1}$ and b_2. Then $B_{1,\nu}$, $B_{2,\nu+1}$ are 2-connected. Note that $\deg_{B_{1,\nu}}(\alpha) \geq \delta(G) - |(N_G(V(B_1 - b_1) \cup V(B_2 - b_2)) \cap V(P)) - \{x'_\nu\}| \geq d - (n' - 1)$ for all $\alpha \in V(B_1 - b_1)$ and, similarly $\deg_{B_{2,\nu+1}}(\alpha) \geq d - (n' - 1)$ for all $\alpha \in V(B_2 - b_2)$. This, in particular, implies that the average degrees of $V(B_1 - b_1)$ in $B_{1,\nu}$ and $V(B_2 - b_2)$ in $B_{2,\nu+1}$ are at least $d - (n' - 1)$, and hence $d^*_{B_{1,\nu}}(x'_\nu, b_1) \geq d - (n' - 1)$ and $d^*_{B_{2,\nu+1}}(b_2, x'_{\nu+1}) \geq d - (n' - 1)$ by Theorem C. Since $l(P[x'_\nu, x'_{\nu+1}]) \geq d^*_{B_{1,\nu}}(x'_\nu, b_1) + d^*_{H-(B_1-b_1)-(B_2-b_2)}(b_1, b_2) + d^*_{B_{2,\nu+1}}(b_2, x'_{\nu+1}) \geq d^*_{B_{1,\nu}}(x'_\nu, b_1) + d^*_{B_{2,\nu+1}}(b_2, x'_{\nu+1})$ by the maximality of $l(P)$, this implies $l(P[x'_\nu, x'_{\nu+1}]) \geq 2(d - (n' - 1))$. Thus (i) is proved.

To prove (ii), assume that $|N'| < n' - 1$, and choose $\lambda \in \{1, \ldots, n' - 1\} - \{\nu\}$ such that $(x'_\lambda, x'_{\lambda+1}) \notin N'$. Then we have $N_G(\{x'_\lambda, x'_{\lambda+1}\}) \cap V(B_1 - b_1) = \emptyset$ or $N_G(\{x'_\lambda, x'_{\lambda+1}\}) \cap V(B_2 - b_2) = \emptyset$. We may assume $N_G(\{x'_\lambda, x'_{\lambda+1}\}) \cap V(B_1 - b_1) = \emptyset$. Then all vertices of $B_1 - b_1$ have degree at least $d - (n' - 1) + 2$ in $B_{1,\nu}$, and hence $d^*_{B_{1,\nu}}(x_\nu, b_1) \geq d - (n' - 1) + 2$ by Theorem C. Consequently it again follows from the maximality of $l(P)$ that $l(P[x'_\nu, x'_{\nu+1}]) \geq d^*_{B_{1,\nu}}(x'_\nu, b_1) + d^*_{B_{2,\nu+1}}(b_2, x'_{\nu+1}) \geq 2(d - (n' - 1)) + 2$. □

We return to the proof of the lemma. Suppose that $|N'| \geq 2$, and take $(x'_\mu, x'_{\mu+1}), (x'_{\mu'}, x'_{\mu'+1}) \in N'$ with $\mu \neq \mu'$. Then since $n' \leq d-1$, $l(P[x'_\mu, x'_{\mu+1}]) + l(P[x'_{\mu'}, x'_{\mu'+1}]) \geq 4(d - (n' - 1)) \geq 2(d - (n' - 1)) + 4$ by Claim 2.1 (i).

Hence

$$\begin{aligned} l(P) &\geq l(P[x'_{\mu}, x'_{\mu+1}]) + l(P[x'_{\mu'}, x'_{\mu'+1}]) + \sum_{\substack{1 \leq \nu \leq n'-1, \\ \nu \neq \mu, \mu'}} l(P[x'_{\nu}, x'_{\nu+1}]) \\ &\geq 2(d-(n'-1)) + 4 + 2(n'-3) = 2d, \end{aligned}$$

which contradicts the assumption that $l(P) = 2d-1$. Thus $|N'| = 1$. Further, if $|N'| < n' - 1$, then it follows from Claim 2.1 (ii) that

$$\begin{aligned} l(P) &\geq \sum_{\nu=1}^{n'-1} l(P[x'_{\nu}, x'_{\nu+1}]) \\ &\geq 2(d-(n'-1)) + 2 + 2(n'-2) = 2d, \end{aligned}$$

a contradiction. Consequently $1 = |N'| = n' - 1$. Thus $n' = 2$.

Since G is 3-connected, this implies that there exist $y \in V(P) - \{x'_1, x'_2\}$ and $a \in V(H) - (V(B_1 - b_1) \cup V(B_2 - b_2))$ such that $ya \in E(G)$. Assume first that $y \notin V(P[x'_1, x'_2])$. We may assume $y \in V(P[x'_2, v])$. Then since $l(P[x'_2, y]) \geq 2$ by the maximality of $l(P)$, it follows from Claim 2.1 (i) that

$$\begin{aligned} l(P) &\geq l(P[x'_1, x'_2]) + l(P[x'_2, y]) \\ &\geq 2(d-1) + 2 = 2d, \end{aligned}$$

a contradiction. Assume now that $y \in V(P[x'_1, x'_2])$. As in the proof of Claim 2.1, we may assume $N_G(x'_1) \cap V(B_1 - b_1) \neq \emptyset$ and $N_G(x'_2) \cap V(B_2 - b_2) \neq \emptyset$. Define $B_{1,1}$ and $B_{2,2}$ as in Claim 2.1. Then arguing as in Claim 2.1, we obtain

$$\begin{aligned} l(P[x'_1, y]) &\geq d^*_{B_{1,1}}(x'_1, b_1) + d^*_{H-(B_1-b_1)-(B_2-b_2)}(b_1, a) + 1 \\ &\geq d^*_{B_{1,1}}(x'_1, b_1) + 1 \geq (d-1) + 1, \\ l(P[x'_2, y]) &\geq d^*_{B_{2,2}}(x'_2, b_2) + 1 \geq (d-1) + 1. \end{aligned}$$

Consequently

$$l(P) \geq l(P[x'_1, y]) + l(P[y, x'_2]) \geq 2((d-1)+1) = 2d,$$

which again contradicts the assumption that $l(P) = 2d - 1$. This completes the proof of the lemma. □

Now let $n = |N_G(H) \cap V(P)|$, and write $N_G(H) \cap V(P) = \{x_1, \ldots, x_n\}$, where $x_1, \ldots, x_n$ occur in this order along P. Define

$$N = \{(x_\nu, x_{\nu+1}) \,|\, 1 \leq \nu \leq n-1, |N_G(\{x_\nu, x_{\nu+1}\}) \cap V(H)| \geq 2\}.$$

Clearly $|N| \leq n - 1$.

Lemma 2.4. *Suppose that* $|V(H)| \geq 3$. *Then* $d \geq 5$, $H \cong K_{d-2}$, $N_G(x) \cap V(P) = N_G(H) \cap V(P)$ *for all* $x \in V(H)$, *and one of the following holds*:

(i) $N_G(H) \cap V(P) = \{v_0, v_{d-1}, v_{2d-1}\}$;
(ii) $N_G(H) \cap V(P) = \{v_0, v_d, v_{2d-1}\}$;
(iii) $N_G(H) \cap V(P) = \{v_0, v_{d-1}, v_{2d-2}\}$; *or*
(iv) $N_G(H) \cap V(P) = \{v_1, v_d, v_{2d-1}\}$.

Proof. By Lemma 2.3, H is nonseparable. Since $|V(H)| \geq 3$, this implies

(2.3) $$d^*_H(a, a') \geq 2 \text{ for all } a, a' \in V(H) \text{ with } a \neq a'.$$

It follows from the maximality of the length of P that for each $1 \leq \nu \leq n-1$, we have $l(P[x_\nu, x_{\nu+1}]) \geq 2$ and, if $(x_\nu, x_{\nu+1}) \in N$, then we further have $l(P[x_\nu, x_{\nu+1}]) \geq 4$ by (2.3). Since G is 3-connected, there exist three independent edges joining H and P, and hence $n \geq 3$ and $|N| \geq 2$. Consequently $2d-1 = l(P) \geq \sum_{\nu=1}^{n-1} l(P[x_\nu, x_{\nu+1}]) \geq 4 \cdot 2 + 2(n-3) = 2n+2$, which implies $n \leq d-2$.

Claim 2.2. *Let* $(x_\nu, x_{\nu+1}) \in N$. *Then the following hold.*

(i) $l(P[x_\nu, x_{\nu+1}]) \geq d-(n-2)$.

(ii) *If equality holds in* (i), *then* $H \cong K_{d-(n-2)-1}$, *and* $N_G(x) \cap V(P) = N_G(H) \cap V(P)$ *for all* $x \in V(H)$.

Proof. Let H_ν denote the graph obtained from $\langle V(H) \cup \{x_\nu, x_{\nu+1}\}\rangle_G$ by joining x_ν and $x_{\nu+1}$ (in the case where $x_\nu x_{\nu+1} \in E(G)$, this means that we simply let $H_\nu = \langle V(H) \cup \{x_\nu, x_{\nu+1}\}\rangle_G$). Then H_ν is 2-connected. Note that $\deg_{H_\nu}(\alpha) \geq \delta(G) - |(N_G(H) \cap V(P)) - \{x_\nu, x_{\nu+1}\})| \geq d-(n-2)$ for all $\alpha \in V(H)$. This, in particular, implies that the average degree of $V(H)$ in H_ν is at least $d-(n-2)$. Therefore $d^*_{H_\nu}(x_\nu, x_{\nu+1}) \geq d-(n-2)$ by Theorem C, and hence it follows from the maximality of the length of P that $l(P[x_\nu, x_{\nu+1}]) \geq d^*_{H_\nu}(x_\nu, x_{\nu+1}) \geq d-(n-2)$. Thus (i) is proved.

To prove (ii), suppose that $l(P[x_\nu, x_{\nu+1}]) = d-(n-2)$. Then $d^*_{H_\nu}(x_\nu, x_{\nu+1}) = d-(n-2)$. Hence $H \cong K_{d-(n-2)-1}$ by Theorem C and, since $\delta(G) \geq d$, this forces $N_G(x) \cap V(P) = N_G(H) \cap V(P)$ for each $x \in V(H)$. □

For convenience, define an integer ε by letting $\varepsilon = 0$ if $H \cong K_{d-(n-2)-1}$ and $N_G(x) \cap V(P) = N_G(H) \cap V(P)$ for all $x \in V(H)$, and letting $\varepsilon = 1$ otherwise. Then by Claim 2.2, $l(P[x_\nu, x_{\nu+1}]) \geq d-(n-2)+\varepsilon$ for all $(x_\nu, x_{\nu+1}) \in N$.

Hence we obtain

$$\begin{aligned} l(P) &\geq \sum_{\nu=1}^{n-1} l(P[x_\nu, x_{\nu+1}]) \\ &= \sum_{\substack{1\leq\nu\leq n-1\\(x_\nu,x_{\nu+1})\in N}} l(P[x_\nu, x_{\nu+1}]) + \sum_{\substack{1\leq\nu\leq n-1\\(x_\nu,x_{\nu+1})\notin N}} l(P[x_\nu, x_{\nu+1}]) \\ &\geq (d-(n-2)+\varepsilon)|N| + 2(n-1-|N|) \\ &= (d-n)|N| + 2(n-1) + \varepsilon|N|; \end{aligned}$$

that is to say

$$l(P) \geq 2d + (d-n)(|N|-2) - 2 + \varepsilon|N|. \tag{2.4}$$

Recall that $n \leq d-2$ and $|N| \geq 2$. Hence if $|N| \geq 3$ or $\varepsilon = 1$, then it follows from (2.4) that $l(P) \geq 2d$, a contradiction. Thus $|N| = 2$ and $\varepsilon = 0$. Hence by the definition of ε, $H \cong K_{d-(n-2)-1}$ and $N_G(x) \cap V(P) = N_G(H) \cap V(P)$ for all $x \in V(H)$. By the definition of N, this implies $|N| = n-1$. Thus $2 = |N| = n-1$, which implies $n = 3$. Consequently $H \cong K_{d-(n-2)-1} = K_{d-2}$ and, by Claim 2.2 (i), $l(P[x_\nu, x_{\nu+1}]) \geq d-(n-2) = d-1$ for each $\nu \in \{1, 2\}$. Since $l(P) = 2d-1$, this implies that one of (i) through (iv) holds. □

Lemma 2.5. *Suppose that $|V(H)| = 2$. Then $H \cong K_2$, $N_G(x) \cap V(P) = N_G(H) \cap V(P)$ for each $x \in V(H)$ and, further, one of the following holds.*

(I) $d = 4$ *and*

- (i) $N_G(H) \cap V(P) = \{v_0, v_3, v_7\}$;
- (ii) $N_G(H) \cap V(P) = \{v_0, v_4, v_7\}$;
- (iii) $N_G(H) \cap V(P) = \{v_0, v_3, v_6\}$; *or*
- (iv) $N_G(H) \cap V(P) = \{v_1, v_4, v_7\}$.

(II) $d = 5$ *and* $N_G(H) \cap V(P) = \{v_0, v_3, v_6, v_9\}$.

Proof. Since $|V(H)| = 2$, $|N_G(x) \cap V(P)| \geq d-1$ for each $x \in V(H)$, and hence $n \geq d-1$. It follows from the maximality of the length of P that for each $1 \leq \nu \leq n-1$, we have $l(P[x_\nu, x_{\nu+1}]) \geq 2$ and, if $(x_\nu, x_{\nu+1}) \in N$, then we further have $l(P[x_\nu, x_{\nu+1}]) \geq 3$. Thus $2d-1 = l(P) \geq 2(n-1)$, which implies $n \leq d$. By the definition of N, this implies $|N| = n-1$. Consequently

$$l(P[x_\nu, x_{\nu+1}]) \geq 3 \text{ for each } 1 \leq \nu \leq n-1, \tag{2.5}$$

and hence

$$2d-1 = l(P) \geq 3(n-1) \tag{2.6}$$

Since $d \geq 3$ and $n \geq d-1$, (2.6) implies $n = d-1$. Therefore $N_G(x) \cap V(P) = N_G(H) \cap V(P)$ for each $x \in V(H)$. Since $n = |N_G(H) \cap V(P)| \geq 3$ by the assumption that G is 3-connected, we also have $d \geq 4$. Further since $2d-1 \geq 3(d-2)$ by (2.6), $d = 4$ or 5. Since $l(P) = 2d-1$, it now follows from (2.5) that (I) or (II) holds according as $d = 4$ or 5. □

Next we show that (iii), (iv) of Lemma 2.4 and (iii), (iv) of Lemma 2.5 (I) do not actually occur (Lemma 2.7). For this purpose, we first prove the following lemma.

Lemma 2.6. *Suppose that* $|V(H)| \geq 2$, *and* (iii) *of Lemma* 2.4 *or* (I) (iii) *of Lemma* 2.5 *holds. Then* $N_G(\{v_1, \ldots, v_{d-2}\}) - \{v_1, \ldots, v_{d-2}\} = N_G(\{v_d, \ldots, v_{2d-3}\}) - \{v_d, \ldots, v_{2d-3}\} = \{v_0, v_{d-1}, v_{2d-2}\}$.

Proof. By Lemmas 2.4 and 2.5, $d \geq 4$. Since $H \cong K_{d-2}$, H contains a path Q of length $d-3$. Let P' denote the (v_0, v_{2d-1})-path of length $2d-1$ obtained from P by replacing $P[v_1, v_{d-2}]$ by Q. Set $X = \{v_1, \ldots, v_{d-2}\}$, and let H' denote the component of $G - P'$ such that $X \subset V(H')$. Since $|V(H')| \geq |X| = d-2$, we obtain $H' \cong K_{d-2}$ by applying Lemmas 2.4 and 2.5 to P' and H'. In particular, $X = V(H')$. Since H' is a component of $G - P'$, this implies $N_G(X) \cap V(P') = N_G(X) - X$. Since $v_0, v_{d-1} \in N_G(X)$, it again follows from Lemmas 2.4 and 2.5 that

$$N_G(x) - X = N_G(X) - X \text{ for all } x \in X, \tag{2.7}$$

and $N_G(X) - X = \{v_0, v_{d-1}, v_{2d-2}\}$ or $\{v_0, v_{d-1}, v_{2d-1}\}$. But if $N_G(X) - X = \{v_0, v_{d-1}, v_{2d-1}\}$, then $v_1 v_{2d-1} \in E(G)$ by (2.7), and hence $(v_0, Q, v_{2d-2}, v_{2d-3}, \ldots, v_1, v_{2d-1})$ is a (v_0, v_{2d-1})-path with length $3d-3 > 2d$, which contradicts the assumption that $d^*_G(v_0, v_{2d-1}) = 2d-1$. Thus $N_G(X) - X = \{v_0, v_{d-1}, v_{2d-2}\}$.

Let now $Y = \{v_d, \ldots, v_{2d-3}\}$. Then $v_{d-1}, v_{2d-2} \in N_G(Y)$. Hence applying Lemmas 2.4 and 2.5 to the path obtained from P by replacing $P[v_d, v_{2d-3}]$ by Q, we obtain ($\langle Y \rangle_G \cong K_{d-2}$ and) $N_G(Y) - Y = \{v_0, v_{d-1}, v_{2d-2}\}$. □

Lemma 2.7. *Suppose that* $|V(H)| \geq 2$. *Then one of the following holds.*

(I) $d \geq 4$, $H \cong K_{d-2}$, $N_G(x) \cap V(P) = N_G(H) \cap V(P)$ *for all* $x \in V(H)$, *and*

 (i) $N_G(H) \cap V(P) = \{v_0, v_{d-1}, v_{2d-1}\}$ *or*

 (ii) $N_G(H) \cap V(P) = \{v_0, v_d, v_{2d-1}\}$; *or*

(II) $d = 5$, $H \cong K_2$, $N_G(x) \cap V(P) = N_G(H) \cap V(P)$ *for all* $x \in V(H)$, *and* $N_G(H) \cap V(P) = \{v_0, v_3, v_6, v_9\}$.

Proof. In view of Lemmas 2.4 and 2.5, it suffices to show that $N_G(H) \cap V(P) \neq \{v_0, v_{d-1}, v_{2d-2}\}$, $\{v_1, v_d, v_{2d-1}\}$. Suppose that $N_G(H) \cap V(P) = \{v_0, v_{d-1}, v_{2d-2}\}$ or $\{v_1, v_d, v_{2d-1}\}$. By symmetry, we may assume that $N_G(H) \cap V(P) = \{v_0, v_{d-1}, v_{2d-2}\}$. Then by Lemmas 2.4 and 2.5, $d \geq 4$, $H \cong K_{d-2}$ and

$$N_G(x) \cap V(P) = \{v_0, v_{d-1}, v_{2d-2}\} \text{ for all } x \in V(H). \tag{2.8}$$

By Lemma 2.6, $N_G(v_{2d-1}) \cap V(P) \subseteq \{v_0, v_{d-1}, v_{2d-2}\}$. Since $\delta(G) \geq d \geq 4$, this together with (2.8) implies that there exists $z \in V(G - P - H)$ such that $v_{2d-1}z \in E(G)$. Let $H^{'}$ denote the component of $G - P$ with $z \in V(H^{'})$. Then by Lemma 2.6,

$$N_G(H^{'}) \cap V(P) \subseteq \{v_0, v_{d-1}, v_{2d-2}, v_{2d-1}\}. \tag{2.9}$$

In view of Lemma 2.1, (2.9) implies $|V(H^{'})| \geq 2$. Since $v_{2d-1} \in N_G(H^{'}) \cap V(P)$, it now follows from (2.9) and Lemmas 2.4 and 2.5 that $H^{'} \cong K_{d-2}$, and

$$N_G(x) \cap V(P) = \{v_0, v_{d-1}, v_{2d-1}\} \text{ for all } x \in V(H^{'}). \tag{2.10}$$

Since $H \cong K_{d-2}$ and $H^{'} \cong K_{d-2}$, H and $H^{'}$ contain paths Q and $Q^{'}$ of length $d-3$, respectively. But then by (2.8) and (2.10), $(v_0, Q, v_{2d-2}, v_{2d-3}, \ldots, v_d, v_{d-1}, Q^{'}, v_{2d-1})$ is a (v_0, v_{2d-1})-path with length $3d-3$, which contradicts the assumption that $d_G^*(v_0, v_{2d-1}) = 2d - 1$. This completes the proof of the lemma. □

Lemma 2.8. (i) *Suppose that* (I) (i) *of Lemma* 2.7 *holds, and let* $X = \{v_1, \ldots, v_{d-2}\}$. *Then* $\langle X \rangle_G \cong K_{d-2}$ *and* $N_G(x) - X = \{v_0, v_{d-1}, v_{2d-1}\}$ *for all* $x \in X$.

(ii) *Suppose that* (I) (ii) *of Lemma* 2.7 *holds, and let* $X^{'} = \{v_{d+1}, \ldots, v_{2d-2}\}$. *Then* $\langle X \rangle_G \cong K_{d-2}$ *and* $N_G(x) - X = \{v_0, v_d, v_{2d-1}\}$ *for all* $x \in X$.

Proof. Let X be as in (i). We argue as in Lemma 2.6. Let $P^{'}$ denote the path obtained from P by replacing $P[v_1, v_{d-2}]$ by a path of length $d-3$ in H. Applying Lemma 2.7 to $P^{'}$, we see that $\langle X \rangle_G$ is a component of $G - P^{'}$, $\langle X \rangle_G \cong K_{d-2}$, and $N_G(x) - X = N_G(X) - X$ for all $x \in X$. Since $v_0, v_{d-1} \in N_G(X)$, we also have $N_G(X) - X = \{v_0, v_{d-1}, v_{2d-1}\}$ by Lemma 2.7. Thus (i) is proved, and (ii) can be verified in a similar way. □

Lemma 2.9. *Suppose that* (II) *of Lemma* 2.7 *holds. For* j *with* $0 \leq j \leq 2$, *let* $X_j = \{v_{3j+1}, v_{3j+2}\}$. *Then for each* $0 \leq j \leq 2$, *we have* $\langle X_j \rangle_G \cong K_2$ *and* $N_G(x) - X_j = \{v_0, v_3, v_6, v_9\}$ *for every* $x \in X_j$.

Proof. Let $0 \le j \le 2$. Let $P^{'}$ denote the path obtained from P by replacing $P[v_{3j+1}, v_{3j+2}]$ by H, and let $H^{'}$ denote the component of $G-P$ containing X_j. Since $v_{3j}, v_{3j+3} \in N_G(H^{'})$, (I) of Lemma 2.7 cannot hold for $P^{'}$ and $H^{'}$, and hence (II) of Lemma 2.7 holds. In particular, $X_j = V(H^{'})$, and hence $N_G(X_j) \cap V(P^{'}) = N_G(X_j) - X_j$. Consequently it follows from Lemma 2.7 (II) that $N_G(x) - X_j = \{v_0, v_3, v_6, v_9\}$ for every $x \in X_j$. □

2.2. Proof of the proposition

We now prove three lemmas concerning the structure of $\langle V(H_1) \cup V(H_2) \cup V(P)\rangle_G$, where H_1 and H_2 are components of $G-P$.

Lemma 2.10. *Let H_1, H_2 be components of $G-P$ with $|V(H_1)| = |V(H_2)| = 1$. Then one of the following holds*:

(i) $N_G(H_1) \cap V(P) = N_G(H_2) \cap V(P)$; *or*

(ii) $d = 3$, *and* $N_G(H_1) \cap V(P) = \{v_0, v_p, v_5\}$ *and* $N_G(H_2) \cap V(P) = \{v_0, v_q, v_5\}$, *where* $\{p, q\} = \{2, 3\}$.

Proof. Write $V(H_1) = \{a_1\}$ and $V(H_2) = \{a_2\}$. We may assume

$$N_G(a_1) \cap V(P) \ne N_G(a_2) \cap V(P). \tag{2.11}$$

Claim 2.3. *Both H_1 and H_2 satisfy* (iii) *of Lemma* 2.1.

Proof. Suppose that H_1 satisfies (i) or (ii) of Lemma 2.1. By symmetry, we may assume H_1 satisfies Lemma 2.1 (ii). Then by (2.11) and Lemma 2.1, $v_0 \in N_G(a_2)$, and either $v_2 \in N_G(a_2)$ or $v_3 \in N_G(a_2)$. If $v_2 \in N_G(a_2)$, $(v_0, a_2, v_2, v_1, a_1, v_3, v_4, \dots, v_{2d-1})$ is a (v_0, v_{2d-1})-path with length $2d+1$; if $v_3 \in N_G(a_2)$, $(v_0, a_2, v_3, v_2, v_1, a_1, v_5, \dots, v_{2d-1})$ is a (v_0, v_{2d-1})-path with length $2d$. In either case, we get a contradiction to the assumption that $d^*_G(v_0, v_{2d-1}) = 2d-1$. □

By (2.11) and Claim 2.3, there exist $m, m^{'}$ with $m \ne m^{'}$ and $0 \le m, m^{'} \le d-2$ such that $N_G(a_1) \cap V(P) = \{v_{2i} | 0 \le i \le m\} \cup \{v_{2i+1} | m+1 \le i \le d-1\}$ and $N_G(a_2) \cap V(P) = \{v_{2i} | 0 \le i \le m^{'}\} \cup \{v_{2i+1} | m^{'}+1 \le i \le d-1\}$. We may assume $0 \le m < m^{'} \le d-2$. If $m+1 < m^{'}$, $(v_0, \dots, v_{2m}, a_1, v_{2(m+1)+1}, v_{2(m+1)}, a_2, v_{2(m+1)+2}, \dots, v_{2d-1})$ is a (v_0, v_{2d-1})-path with length $2d$, a contradiction. Thus $m+1 = m^{'}$. Suppose that $d \ge 4$. Then we have $m \ge 1$ or $m^{'} \le d-3$. By symmetry, we may assumme $m \ge 1$. But then $(v_0, \dots, v_{2(m-1)}, a_2, v_{2(m+1)}, v_{2m+1}, v_{2m}, a_1, v_{2(m+1)+1}, \dots, v_{2d-1})$ is a (v_0, v_{2d-1}) -path with length $2d$, a contradiction. Thus $d = 3$, and hence (ii) holds. □

Lemma 2.11. *Let H_1, H_2 be components of $G - P$ with $|V(H_1)| = 1$ and $|V(H_2)| \geq 2$. Then $d = 4$, and one of the following holds*:

(i) $N_G(H_1) \cap V(P) = \{v_0, v_3, v_5, v_7\}$, $N_G(H_2) \cap V(P) = \{v_0, v_3, v_7\}$; *or*

(ii) $N_G(H_1) \cap V(P) = \{v_0, v_2, v_4, v_7\}$, $N_G(H_2) \cap V(P) = \{v_0, v_4, v_7\}$.

Proof. By Lemma 2.7, we have $d \geq 4$. By symmetry, we may assume that H_2 satisfies (I) (i) or (II) of Lemma 2.7. Then it follows from (i) of Lemma 2.8 and Lemma 2.9 that $N_G(\{v_1, v_2\}) \subseteq V(P)$, which implies $\{v_1, v_2\} \cap N_G(H_1) = \emptyset$. By Lemma 2.1, this implies $N_G(H_1) \cap V(P) = \{v_0\} \cup \{v_{2i+1} | 1 \leq i \leq d-1\}$. In particular,

$$v_3 \in N_G(H_1) \tag{2.12}$$

and

$$v_5 \in N_G(H_1). \tag{2.13}$$

In view of Lemma 2.9, (2.13) implies that H_2 cannot satisfy (II) of Lemma 2.7. Thus H_2 satisfies (I) (i) of Lemma 2.7. Consequently it follows from (2.12) and Lemma 2.8 (i) that $d = 4$, and hence (i) holds. □

Lemma 2.12. *Let H_1, H_2 be components of $G - P$ with $|V(H_1)| \geq 2$ and $|V(H_2)| \geq 2$. Then one of the following holds*:

(i) $N_G(H_1) \cap V(P) = N_G(H_2) \cap V(P)$; *or*

(ii) $N_G(H_1) \cap V(P) = \{v_0, v_{d-1}, v_{2d-1}\}$, $N_G(H_2) \cap V(P) = \{v_0, v_d, v_{2d-1}\}$.

Proof. We may assume

$$N_G(H_1) \cap V(P) \neq N_G(H_2) \cap V(P). \tag{2.14}$$

If $d \neq 5$, the desired conclusion immediately follows from Lemma 2.7. Thus we may assume $d = 5$. It suffices to show that neither H_1 nor H_2 satisfies (II) of Lemma 2.7. Suppose that H_1 satisfies (II) of Lemma 2.7. Then $v_3 \in N_G(H_1)$. By (2.14), H_2 satisfies (I) of Lemma 2.7. We may assume H_2 satisfies (I) (i) of Lemma 2.7. But then it follows from (i) of Lemma 2.8 that $N_G(v_3) \subset V(P)$, which contradicts the earlier assertion that $v_3 \in N_G(H_1)$. □

We are now in a position to complete the proof of Proposition 2.1.

Case 1 $|V(H)| = 1$ for every component H of $G - P$.

We first consider the case where $N_G(H_1) \cap V(P) = N_G(H_2) \cap V(P)$ for any two components H_1, H_2 of $G - P$. If (i) of Lemma 2.1 holds for every component H of $G - P$, or if (ii) of Lemma 2.1 holds for every component

H of $G-P$, then by Lemma 2.2, (i) of Proposition 2.1 holds with $S = N_G(G-P) \cap V(P)$. If (iii) of Lemma 2.1 holds for every component H of $G-P$, then by Lemma 2.2, (ii) of Proposition 2.1 holds. We now consider the case where there exist components H_1, H_2 of $G-P$ such that $N_G(H_1) \cap V(P) \neq N_G(H_2) \cap V(P)$. In this case, it follows from Lemma 2.10 that $d = 3$, and $N_G(H) \cap V(P) = \{v_0, v_2, v_5\}$ or $\{v_0, v_3, v_5\}$ for each component H of $G-P$. Since $v_1v_4, v_1v_3, v_2v_4 \notin E(G)$ by Lemma 2.2, this implies that (iv) holds with $\{a, b\} = \{v_2, v_3\}$.

Case 2 Suppose that there exist components H_1 and H_2 of $G-P$ such that $|V(H_1)| = 1$ and $|V(H_2)| \geq 2$.

By Lemma 2.11, $d = 4$. By symmetry, we may assume that (i) of Lemma 2.11 holds. Then by Lemma 2.11, (iii) of Lemma 2.1 holds with $m = 0$ for each component H of $G-P$ with $|V(H)| = 1$, and (I) (i) of Lemma 2.7 holds for each component H of $G-P$ with $|V(H)| \geq 2$. This in particular implies that $N_G(v_4), N_G(v_6) \subseteq V(P)$. Hence $N_G(v_4), N_G(v_6) \subseteq \{v_0, v_3, v_5, v_7\}$ by Lemma 2.2. Since $\delta(G) \geq d = 4$, this forces $N_G(v_4) = N_G(v_6) = \{v_0, v_3, v_5, v_7\}$. Also $N_G(v_1) - \{v_1, v_2\} = N_G(v_2) - \{v_1, v_2\} = \{v_0, v_3, v_7\}$ by Lemma 2.8 (i). Consequently (vi) holds with $a = v_3$ and $b = v_5$.

Case 3 $|V(H)| \geq 2$ for every component H of $G-P$.

We first consider the case where $N_G(H_1) \cap V(P) = N_G(H_2) \cap V(P)$ for any two components H_1, H_2 of $G-P$. Assume for the moment that (I) (i) of Lemma 2.7 holds for every component H of $G-P$. Set $Y = \{v_d, v_{d+1}, \ldots, v_{2d-2}\}$. Then $N_G(Y) \subseteq V(P)$. Since $N_G(\{v_1, v_2, \ldots, v_{d-2}\}) \cap Y = \emptyset$ by Lemma 2.8 (i), this implies $N_G(Y) - Y \subseteq \{v_0, v_{d-1}, v_{2d-1}\}$, and hence $N_G(Y) - Y = \{v_0, v_{d-1}, v_{2d-1}\}$ by the assumption that G is 3-connected. Therefore it follows from Lemma 2.8 (i) that (iii) holds with $a = v_{d-1}$ and $H_0 = \langle Y \rangle_G$. Similarly if (I) (ii) of Lemma 2.7 holds for every component H of $G-P$, then by Lemma 2.8 (ii), (iii) holds with $a = v_d$ and $H_0 = \langle \{v_1, v_2, \ldots, v_{d-1}\} \rangle_G$. Also if (II) of Lemma 2.7 holds for every component H of $G-P$, then by Lemma 2.9, (v) holds with $S = \{v_0, v_3, v_6, v_9\}$. We now consider the case where there exist components H_1, H_2 of $G-P$ such that $N_G(H_1) \cap V(P) \neq N_G(H_2) \cap V(P)$. In this case, it follows from Lemma 2.12 that (I) (i) or (I) (ii) of Lemma 2.7 holds for each component H of $G-P$. Therefore it follows from Lemma 2.8 that (iv) holds with $\{a, b\} = \{v_{d-1}, v_d\}$. This completes the proof of the proposition.

§3. Codistance in 3-connected graphs

As corollaries of Proposition 2.1, we now derive results concerning the distribution of pairs of vertices with small codistance.

Corollary 3.1. *Let $d \geq 3$ be an integer, and let G be a 3-connected graph with $|V(G)| \geq 2d$ and $\delta(G) \geq d$. Suppose that $d^*(G) \leq 2d-1$. Suppose further that*

$$\text{if } |V(G)| = 2d, \text{ then } d^*(G) \leq 2d-2. \tag{3.1}$$

Let A_1, A_2 be subsets of $V(G)$ with $A_1 \neq \emptyset$, $A_2 \neq \emptyset$ and $|A_1 \cup A_2| \geq 2$. Suppose that $d^*_G(a_1, a_2) \leq 2d-1$ for all $a_1 \in A_1$ and $a_2 \in A_2$ with $a_1 \neq a_2$. Then one of the following holds:

(i) $|A_1| + |A_2| \leq |V(G)|$; *or*

(ii) *there exists $S \subseteq V(G)$ with $|S| = d$ such that $E(G-S) = \emptyset$.*

Proof. By Theorem B, $d^*(G) \geq 2d-2$, and hence $d^*(G) = 2d-2$ or $2d-1$. Take $u, v \in V(G)$ with $u \neq v$ such that $d^*_G(u,v) = d^*(G)$. If $d^*_G(u,v) = 2d-2$, then (i) or (ii) of Theorem B holds with $m = 2d$. If $d^*_G(u,v) = 2d-1$, then $|V(G)| \geq 2d+1$ by (3.1), and hence one of (i) through (vi) of Proposition 2.1 holds. If (i) of Theorem B or (i) of Proposition 2.1 holds, then (ii) holds. Thus we may assume (ii) of Theorem B or one of (ii) through (vi) of Proposition 2.1 holds.

Case 1. (ii) of Theorem B holds.

Since $|V(G)| \geq 2d$ and $|V(H)| = d-2$ for each component H of $G - \{u, v, a\}$,

$$G - \{u, v, a\} \text{ contains at least three components.} \tag{3.2}$$

Since $d \geq 4$,

$$3d - 4 \geq 2d. \tag{3.3}$$

Claim 3.1. *Let $x \in V(G) - \{u, v, a\}$ and $y \in \{u, v, a\}$. Then $d^*_G(x, y) \geq 2d$.*

Proof. Let H_1 be the component of $G - \{u, v, a\}$ with $x \in V(H_1)$. We may assume $y = v$ (note that the roles of u, v, a are symmetric in (ii) of Theorem B). By (3.2), there exist two components H_2, H_3 of $G - \{u, v, a\}$ with $x \notin V(H_2) \cup V(H_3)$. For each $1 \leq i \leq 3$, $H_i \cong K_{d-2}$, and hence H_i contains a path Q_i of length $d-3$; in particular, we can choose Q_1 so that x is the initial vertex of Q_1. Then (Q_1, u, Q_2, a, Q_3, v) is an (x, y)-path of length $3d-4$, and hence $d^*_G(x, y) \geq 2d$ by (3.3). □

Claim 3.2. *Let $x, y \in V(G) - \{u, v, a\}$ with $x \neq y$. Then $d^*_G(x, y) \geq 2d$.*

Proof. Let H_1, H_2 be the components of $G - \{u, v, a\}$ such that $x \in V(H_1)$ and $y \in V(H_2)$. We divide the proof into two cases according as $H_1 = H_2$ or $H_1 \neq H_2$.

Case a. $H_1 = H_2$.

By (3.2), there exist two components H_3, H_4 of $G - \{u, v, a\}$ with $\{x, y\} \cap (V(H_3) \cup V(H_4)) = \emptyset$. For each $3 \leq i \leq 4$, let Q_i be a path of length $d - 3$ in H_i. Then $(x, u, Q_3, a, Q_4, v, y)$ is an (x, y)-path of length $2d$.

Case b. $H_1 \neq H_2$.

Again by (3.2), there exists a component H_5 of $G - \{u, v, a\}$ with $\{x, y\} \cap V(H_5) = \emptyset$. Let Q_5 be a path of length $d - 3$ in H_5, and let Q_2 be a path of length $d - 3$ in H_2 with terminal vertex y. Take $z \in V(H_1)$ with $z \neq x$ (note that $d - 2 \geq 2$). Then $(x, u, z, a, Q_5, v, Q_2)$ is an (x, y)-path of length $2d$. □

It follows from Claims 3.1 and 3.2 that $A_1, A_2 \subseteq \{u, v, a\}$, and hence $|A_1| + |A_2| \leq 6 \leq 2d \leq |V(G)|$.

Case 2. (ii) of Proposition 2.1 holds.

Since $|V(G)| \geq 2d+1$, we have $|V(G)-S| \geq d+1$. Write $E(G-S) = \{z_1 z_2\}$.

Claim 3.3. *Let $x \in V(G) - S$ and $y \in S$. Then $d^*_G(x, y) \geq 2d$.*

Proof. Since G is 3-connected, $G - y$ is 2-connected, and hence there exist $y_1, y_2 \in S - \{y\}$ with $y_1 \neq y_2$ such that $z_1 y_1, z_2 y_2 \in E(G)$. Assume first that $x \notin \{z_1, z_2\}$. Since each vertex in $V(G) - S - \{z_1, z_2\}$ is agjacent to all vertices in S, $G - \{x, z_1, z_2, y_1\}$ contains a (y_2, y)-path P of length $2d - 4$, and we have $xy_1 \in E(G)$. Thus (x, y_1, z_1, z_2, P) is an (x, y)-path of length $2d$. Assume now that $x \in \{z_1, z_2\}$. We may assume $x = z_1$. Then $G - \{x, z_2\}$ contains a (y_2, y)-path Q of length $2d - 2$. Thus (x, z_2, Q) is an (x, y)-path of length $2d$. □

Claim 3.4. *Let $x, y \in V(G) - S$ with $x \neq y$. Then $d^*_G(x, y) \geq 2d$.*

Proof. Since $\delta(G) \geq d \geq 3$, there exists $y_0 \in S$ such that $z_1 y_0, z_2 y_0 \in E(G)$. Since G is 3-connected, there exist $y_1, y_2 \in S - \{y_0\}$ with $y_1 \neq y_2$ such that $z_1 y_1, z_2 y_2 \in E(G)$. First assume $\{x, y\} \cap \{z_1, z_2\} = \emptyset$. Then $G - \{x, z_1, z_2, y_0, y_1\}$ contains a (y_2, y)-path P of length $2d - 5$. Thus $(x, y_1, z_1, y_0, z_2, P)$ is an (x, y)-path of length $2d$. Next assume $|\{x, y\} \cap \{z_1, z_2\}| = 1$. We may assume $x = z_1$. Then $G - \{x, z_2, y_0\}$ contains a (y_2, y)-path Q of length $2d - 3$. Thus $(x, y_0, z_2, Q)$is an (x, y)-path of length $2d$. Finally assume $\{x, y\} = \{z_1, z_2\}$. We may assume $x = z_1$ and $y = z_2$. Then $G - \{x, y\}$ contains a (y_1, y_2)-path R of length $2d - 2$. Thus (x, R, y) is an (x, y)-path of length $2d$. □

It follows from Claims 3.3 and 3.4 that $A_1, A_2 \subseteq S$, and hence $|A_1|+|A_2| \leq 2d < |V(G)|$.

Case 3. (iii) of Proposition 2.1 holds.
Since $\delta(G) \geq d$,

$$\deg_{H_0}(w) \geq d-3 \text{ for all } w \in V(H_0), \tag{3.4}$$

and

$$|N_G(w) \cap \{u,v,a\}| \geq 2 \text{ for all } w \in V(H_0). \tag{3.5}$$

Claim 3.5. *Let $w_1, w_2 \in V(H_0)$ with $w_1 \neq w_2$. Then H_0 contains a (w_1,w_2)-path with length at least $d-3$.*

Proof. In view of (3.4), it is easy to verify the claim for $d=4$. Thus suppose that $d \geq 5$. Then by (3.4), H_0 is 2-connected. Hence again by (3.4), the desired conclusion follows from Theorem C. □

Now in view of Claim 3.5 and (3.5), we can argue as in Case 1 to obtain $A_1, A_2 \subseteq \{u,v,a\}$, and hence $|A_1|+|A_2| \leq 6 \leq 2d \leq |V(G)|$.

Case 4. (iv) of Proposition 2.1 holds.
If no component H of $G-\{u,v,a,b\}$ satisfies $N_G(H)-V(H)=\{u,v,b\}$, then $d^*_G(u,v)=2d-2$, which contradicts the assumption that we are in Case (iv) of Proposition 2.1. Further if there exists precisely one component, say $H_0^{'}$, of $G-\{u,v,a,b\}$ such that $N_G(H_0^{'})-V(H_0^{'})=\{u,v,b\}$, then in the case where $d \geq 4$, (iii) of Proposition 2.1 holds with $H_0=\langle V(H_0^{'}) \cup \{b\}\rangle_G$ and, in the case where $d=3$, (ii) of Proposition 2.1 holds with $S=\{u,v,a\}$. Thus we may assume

$$\begin{aligned}&\text{there exist at least two components } H \text{ of } G-\{u,v,a,b\}\\ &\text{such that } N_G(H)-V(H)=\{u,v,b\}.\end{aligned} \tag{3.6}$$

Similarly we may assume

$$\begin{aligned}&\text{there exist at least two components } H \text{ of } G-\{u,v,a,b\}\\ &\text{such that } N_G(H)-V(H)=\{u,v,a\}.\end{aligned} \tag{3.7}$$

Note that (3.6) and (3.7) imply that $|V(G)| \geq 4d-4$. We divide the proof into two cases according as $d=3$ or $d \geq 4$.

Case a. $d \geq 4$.
Since $ab \in E(G)$, we can adapt to this case the construction of desired paths in Case 1 by replacing the segment (a) of length 0 by the path (a,b) or (b,a) of length 1 or the path (b) of length 0. Consequently, arguing as in Case

1, we obtain $A_1, A_2 \subseteq \{u, v, a, b\}$ and hence $|A_1| + |A_2| \leq 8 < |V(G)|$ (thus we do not need (3.6), (3.7) in this case).

Case b. $d = 3$.

Note that for each component H of $G - \{u, v, a, b\}$, $|V(H)| = 1$. Write

$$\begin{aligned} N_G(a) - \{u, v, a, b\} &= \{\alpha_1, \dots, \alpha_\lambda\}, \\ N_G(b) - \{u, v, a, b\} &= \{\beta_1, \dots, \beta_\mu\}. \end{aligned}$$

We have $\lambda \geq 2$ and $\mu \geq 2$ by (3.6) and (3.7), respectively. Note also that for each $1 \leq i \leq \lambda$ and $1 \leq j \leq \mu$, $N_G(\alpha_i) = \{u, v, a\}$ and $N_G(\beta_j) = \{u, v, b\}$ by (iv) of Proposition 2.1.

Claim 3.6. *Let $x \in V(G) - \{u, v, a, b\}$ and $y \in \{u, v, a, b\}$. Then $d^*_G(x, y) \geq 6$.*

Proof. By symmetry, we may assume $x = \alpha_1$. Then

$$(\alpha_1, a, \alpha_2, v, \beta_1, b, \beta_2, u), (\alpha_1, a, \alpha_2, u, \beta_1, b, \beta_2, v),$$
$$(\alpha_1, u, \beta_1, b, \beta_2, v, \alpha_2, a) \text{ or } (\alpha_1, a, \alpha_2, u, \beta_1, v, \beta_2, b)$$

is an (x, y)-path of length 7 according as $y = u, v, a$ or b. □

Claim 3.7. *Let $x, y \in V(G) - \{u, v, a, b\}$ with $x \neq y$. Then $d^*_G(x, y) \geq 6$.*

Proof. By symmetry, we may assume that either $x = \alpha_1$ and $y = \beta_1$, or $x = \alpha_1$ and $y = \alpha_2$. If $x = \alpha_1$ and $y = \beta_1$, $(\alpha_1, a, \alpha_2, u, \beta_2, v, \beta_1)$ is an (x, y)-path of length 6. If $x = \alpha_1$ and $y = \alpha_2$, $(\alpha_1, u, \beta_1, b, \beta_2, v, \alpha_2)$ is an (x, y)-path of length 6 □

It follows from Claims 3.6 and 3.7 that $A_1, A_2 \subseteq \{u, v, a, b\}$, and hence $|A_1| + |A_2| \leq 8 = 4d - 4 \leq |V(G)|$.

Case 5. (v) of Proposition 2.1 holds.

Aguing as in the proof of Case 1, we see that $A_1, A_2 \subseteq S$, and hence $|A_1| + |A_2| \leq 8 < 2d < |V(G)|$.

Case 6. (vi) of Proposition 2.1 holds.

Let $F_1, \dots, F_\lambda$ be the components of $G - \{u, v, a, b\}$ having cardinality 1, and let $H_1, \dots, H_\mu$ be the components of $G - \{u, v, a, b\}$ having cardinality 2. For each $1 \leq i \leq \lambda$, write $V(F_i) = \{z_i\}$. If $\lambda \leq 1$ or $\mu = 0$, then $d^*_G(u, v) = 6 = 2d - 2$, a contradiction. Thus we have $\lambda \geq 2$ and $\mu \geq 1$. Further if $\lambda = 2$, then (iii) of Proposition 2.1 holds with $H_0 = \langle \{z_1, z_2, b\} \rangle_G$. Thus we may assume $\lambda \geq 3$.

Claim 3.8. *Let $x \in V(G) - \{u, v, a, b\}$ and $y \in V(G)$ with $x \neq y$. Then $d^*_G(x, y) \geq 8$.*

Proof. By symmetry, we may assume that either $x \in V(H_1)$ and $y \in V(H_2)$, or $x, y \in V(H_1) \cup \{z_1, z_2, u, v, a, b\}$. If $x \in V(H_1)$ and $y \in V(H_2)$, then $(x, u, z_1, a, z_2, b, z_3, v, y)$ is an (x, y)-path of length 8. If $x, y \in V(H_1) \cup \{z_1, z_2, u, v, a, b\}$, then since $\langle V(H_1) \cup \{z_1, z_2, z_3, u, v, a, b\}\rangle_G$ satisfies (ii) of Proposition 2.1, the desired conclusion follows from Claims 3.3 and 3.4. □

It follows from Claim 3.8 that $A_1, A_2 \subseteq \{u, v, a, b\}$, and hence $|A_1| + |A_2| \leq 8 < |V(G)|$. This completes the proof of Corollary 3.1. □

Corollary 3.2. *Let d, G, A_1,A_2 be as in Corollary* 3.1. *Then one of the following holds*:

(i) $|A_1| + |A_2| \leq |V(G)| + d$; *or*

(ii) $|V(G)| = 2d$, *and there exists $S \subseteq V(G)$ with $|S| = d$ such that $E(G - S) = \emptyset$.*

Proof. By Corollary 3.1, (i) or (ii) of Corollary 3.1 holds. If (i) of Corollary 3.1 holds, then clearly (i) holds. Thus we may assume that (ii) of Corollary 3.1 holds. Note that

(3.8) if $|V(G)| \geq 2d + 1$,
then $d^*_G(x, y) \geq 2d$ for all $x, y \in V(G) - S$ with $x \neq y$.

Now if $|V(G)| = 2d$, then (ii) holds. Thus we may assume $|V(G)| \geq 2d + 1$. Then it follows from (3.8) that we have $A_1 \subseteq S$ or $A_2 \subseteq S$, and hence $|A_1| + |A_2| \leq |V(G)| + d$, as desired. □

Corollary 3.3. *Let d, G, A_1,A_2 be as in Corollary* 3.1, *and suppose that $d^*_G(a_1, a_2) \leq 2d - 2$ for all $a_1 \in A_1$ and $a_2 \in A_2$ with $a_1 \neq a_2$. Then $|A_1| + |A_2| \leq |V(G)|$.*

Proof. As in the proof of Corollary 3.2, we may assume that (ii) of Corollary 3.1 holds. Then

(3.9) $d^*_G(x, y) \geq 2d - 1$ for all $x \in S$ and all $y \in V(G) - S$.

If $|V(G)| \geq 2d + 1$, then it follows from (3.8) and (3.9) that $A_1, A_2 \subseteq S$, and hence $|A_1| + |A_2| \leq 2d < |V(G)|$. Thus we may assume $|V(G)| = 2d$. Then $|V(G) - S| = |S| = d$. Further it follows from (3.9) that we have $A_1, A_2 \subseteq S$ or $A_1, A_2 \subseteq V(G) - S$. Consequently $|A_1| + |A_2| \leq 2d = |V(G)|$, as desired. □

§4. Proof of Main Theorem

Let G, C, H, k, r be as in the Main Theorem. Set $p = |N_G(H) \cap V(C)|$, and write $N_G(H) \cap V(C) = \{u_1, u_2, \cdots, u_p\}$, where $u_1, \ldots, u_p$ are in this order along C (indices are to be read modulo p). For each $1 \le i \le p$, let H_i denote the graph obtained from $\langle V(H) \cup \{u_i, u_{i+1}\}\rangle_G$ by joining u_i and u_{i+1} (in the case where $u_i u_{i+1} \in E(G)$, this means that we simply let $H_i = \langle V(H) \cup \{u_i, u_{i+1}\}\rangle_G$). Define

$$\begin{aligned} T &= \{u_i \,|\, 1 \le i \le p, |N_G(\{u_i, u_{i+1}\}) \cap V(H)| \ge 2\}, \\ \overline{T} &= (N_G(H) \cap V(C)) - T, \\ T_1 &= \{u_i \in T \,|\, d^*_{H_i}(u_i, u_{i+1}) < k\}, \\ T_2 &= \{u_i \in T \,|\, d^*_{H_i}(u_i, u_{i+1}) \ge k\}. \end{aligned}$$

Set $h = |V(H)|$, $s = |\overline{T}|$, $t_1 = |T_1|$, $t_2 = |T_2|$. The following claims immediately follow from the definition of H_i and $\overline{T}$.

Claim 4.1. *Let $1 \le i \le p$, and take $a \in N_G(u_i) \cap V(H)$ and $b \in N_G(u_{i+1}) \cap V(H)$. Then*

$$d^*_{H_i}(u_i, u_{i+1}) \ge \begin{cases} 2 & \text{if } a = b, \\ d^*_H(a, b) + 2 & \text{otherwise.} \end{cases}$$

Claim 4.2. *Let $u_i \in \overline{T}$ $(1 \le i \le p)$. Then*

$$|N_G(u_i) \cap V(H)| = |N_G(u_{i+1}) \cap V(H)| = 1.$$

The following claim follows from the assumption that C is locally longest with respect to H in G.

Claim 4.3. *Let $1 \le i \le p$. Then*

$$l(C[u_i, u_{i+1}]) \ge d^*_{H_i}(u_i, u_{i+1}).$$

Now if $t_1 \le k$, then the desired conclusion follows from Proposition D. Thus we may assume that

$$t_1 > k. \tag{4.1}$$

Suppose that $\delta(H) \ge k/2$. Then since H is 3-connected and $h - 1 \ge k - 2$, we see that $d^*_H(a, b) \ge k - 2$ for all $a, b \in V(H)$ with $a \ne b$ by applying Theorem B with $m = k$. Hence by Claim 4.1, $d^*_{H_i}(u_i, u_{i+1}) \ge k$ for each $u_i \in T$.

This implies that $T_1 = \emptyset$, which contradicts (4.1). Thus $\delta(G) \leq \frac{k-1}{2}$. Since $\delta(H) \geq \lfloor (k-1)/2 \rfloor$ by assumption, this implies

$$\delta(H) = \left\lfloor \frac{k-1}{2} \right\rfloor . \tag{4.2}$$

Again since H is 3-connected, $\delta(H) \geq 3$, and hence

$$k \geq 7 \tag{4.3}$$

by (4.2). Let $E_G(C, H)$ denote the set of those edges of G which join a vertex of C and a vertex of H.

Claim 4.4. (I) *One of the following holds*:

(i) $|E_G(C, H)| \leq \frac{h}{2} t_1 + h t_2 + s$; *or*

(ii) *k is even and there exists $S \subseteq V(H)$ with $|S| = \delta(H)$ such that $E(H - S) = \emptyset$.*

(II) *If G satisfies* (I) (ii)*, then* $|E_G(C, H)| \leq \dfrac{h + \delta(H)}{2} t_1 + h t_2 + s.$

Proof. We first show that either

$$|N_G(u_i) \cap V(H)| + |N_G(u_{i+1}) \cap V(H)| \leq h \tag{4.4}$$

for every $u_i \in T_1$ or (I) (ii) holds, and that if G satisfies (I) (ii), then

$$|N_G(u_i) \cap V(H)| + |N_G(u_{i+1}) \cap V(H)| \leq h + \delta(G) \tag{4.5}$$

for every $u_i \in T$. Let $u_i \in T_1$. Then by the definition of $T (\supseteq T_1)$, there exist $a \in N_G(u_i) \cap V(H)$ and $b \in N_G(u_{i+1}) \cap V(H)$ with $a \neq b$, and hence $N_G(u_i) \cap V(H)$ and $N_G(u_{i+1}) \cap V(H)$ are nonempty subsets of H satisfiying $|(N_G(u_i) \cap V(H)) \cup (N_G(u_{i+1}) \cap V(H))| \geq 2$. Since $d^*_{H_i}(u_i, u_{i+1}) < k$ by the definition of T_1, it follows from Claim 4.1 that

$$d^*_H(a, b) \leq d^*_{H_i}(u_i, u_{i+1}) - 2 \leq k - 3 \tag{4.6}$$

for all $a \in N_G(u_i) \cap V(H)$ and $b \in N_G(u_{i+1}) \cap V(H)$ with $a \neq b$. Also recall that $h \geq k - 1$. We may assume (4.4) does not hold. We aim at showing that (I) (ii) and (4.5) hold. If k is odd, then it follows from (4.6) and (4.2) that $d^*_G(a, b) \leq 2\delta(H) - 2$ for all $a \in N_G(u_i) \cap V(H)$ and $b \in N_G(u_{i+1}) \cap V(H)$ with $a \neq b$, and we also have $h \geq 2\delta(H)$, and hence by Corollary 3.3, we get a contradiction to the assumption that (4.4) does not hold. Thus k is even. Again it follows from (4.6) and (4.2) that $d^*_G(a, b) \leq 2\delta(H) - 1$ for all $a \in N_G(u_i) \cap V(H)$ and $b \in N_G(u_{i+1}) \cap V(H)$ with $a \neq b$, and we also have $h \geq 2\delta(H) + 1$, and hence by Corollary 3.1, (I) (ii) holds. Further since $h \geq 2\delta(H) + 1$, it follows from Corollary 3.2 that (4.5) holds.

Now for each $u_i \in T_2$, we clearly have

$$|N_G(u_i) \cap V(H)| + |N_G(u_{i+1}) \cap V(H)| \leq |V(H)| + |V(H)| \leq 2h \tag{4.7}$$

and, for each $u_i \in \overline{T}$

$$|N_G(u_i) \cap V(H)| + |N_G(u_{i+1}) \cap V(H)| = 2 \tag{4.8}$$

by Claim 4.2. Define

$$\theta = \begin{cases} h + \delta(H) & \text{if } G \text{ satisfies (I) (ii);} \\ h & \text{otherwise.} \end{cases}$$

Then it follows from (4.7), (4.8), and (4.4) or (4.5) that

$$\begin{aligned} |E_G(C,H)| &= \sum_{u \in N_G(H) \cap V(C)} |N_G(u) \cap V(H)| \\ &= \sum_{u_i \in N_G(H) \cap V(C)} \frac{|N_G(u_i) \cap V(H)| + |N_G(u_{i+1}) \cap V(H)|}{2} \\ &= \sum_{u_i \in T_1} \frac{|N_G(u_i) \cap V(H)| + |N_G(u_{i+1}) \cap V(H)|}{2} \\ &+ \sum_{u_i \in T_2} \frac{|N_G(u_i) \cap V(H)| + |N_G(u_{i+1}) \cap V(H)|}{2} \\ &+ \sum_{u_i \in \overline{T}} \frac{|N_G(u_i) \cap V(H)| + |N_G(u_{i+1}) \cap V(H)|}{2} \\ &\leq \frac{\theta}{2} t_1 + h t_2 + s. \qquad \square \end{aligned}$$

Claim 4.5.

$$l(C) \geq (d^*(H) + 2)t_1 + kt_2 + 2s. \tag{4.9}$$

Proof. Note that for each $u_i \in T$, there exist $a \in N_G(u_i) \cap V(H)$ and $b \in N_G(u_{i+1}) \cap V(H)$ with $a \neq b$ by definition, and hence $d^*_{H_i}(u_i, u_{i+1}) \geq d^*_H(a, b) + 2$ by Claim 4.1. Thus $d^*_{H_i}(u_i, u_{i+1}) \geq d^*(H) + 2$ for all $u_i \in T_1 (\subseteq T)$. Again by definition, $d^*_{H_i}(u_i, u_{i+1}) \geq k$ for all $u_i \in T_2$. Further by Claim 4.1,

$d^*_{H_i}(u_i, u_{i+1}) \geq 2$ for all $u_i \in \bar{T}$. Consequently it follows from Claim 4.3 that

$$\begin{aligned} l(C) &= \sum_{u_i \in T} l(C[u_i, u_{i+1}]) + \sum_{u_i \in \bar{T}} l(C[u_i, u_{i+1}]) \\ &\geq \sum_{u_i \in T} d^*_{H_i}(u_i, u_{i+1}) + \sum_{u_i \in \bar{T}} d^*_{H_i}(u_i, u_{i+1}) \\ &\geq \left(\sum_{u_i \in T_1} d^*_{H_i}(u_i, u_{i+1}) + \sum_{u_i \in T_2} d^*_{H_i}(u_i, u_{i+1}) \right) + 2s \\ &\geq (d^*(H) + 2)t_1 + kt_2 + 2s. \end{aligned}$$

□

We are now in a position to complete the proof of the Main Theorem. Define θ as in Claim 4.4. Write

$$t_1 = t_4 + k, \tag{4.10}$$

and

$$(\theta/2)t_1 + ht_2 + s = t_3 + |E_G(C, H)|.$$

Then

$$t_2 = (1/h)t_3 - (\theta/2h)t_4 - (1/h)s + |E_G(C, H)|/h - k\theta/2h. \tag{4.11}$$

Also t_3 and t_4 are nonnegative by (4.1) and Claim 4.4. Substituting (4.10) and (4.11) for t_1 and t_2 in (4.9), we obtain

$$\begin{aligned} l(C) &\geq \frac{k}{h}t_3 + \left\{ (d^*(H) + 2) - \frac{k\theta}{2h} \right\} t_4 + \left(2 - \frac{k}{h} \right) s \\ &+ k\,(d^*(H) + 2) + k \left\{ \frac{1}{h}|E_G(C, H)| - \frac{k\theta}{2h} \right\}. \end{aligned} \tag{4.12}$$

In the rest of the proof, we consider each term of (4.12). Since t_3 is nonnegative,

$$\frac{k}{h}t_3 \geq 0. \tag{4.13}$$

Since $h \geq k - 1$, $2 - k/h \geq 0$, and hence

$$\left(2 - \frac{k}{h} \right) s \geq 0. \tag{4.14}$$

We now consider the second term of (4.12).

Claim 4.6. (i) $k-2 \geq \dfrac{k}{2}$.

(ii) *If* $k \geq 8$, *then* $k-2 \geq \dfrac{k(h+\delta(H))}{2h}$.

Proof. Since $k \geq 5$, (i) clearly holds. Since $h \geq k-1$, it follows from (4.2) that $\delta(H) \leq h/2$, and hence $(h+\delta(H))/2h \leq 3/4$, which implies (ii). □

Recall that t_4 is nonnegative. Also note that if $\theta = h + \delta(H)$, then k is even by the definition of θ, and hence $k \geq 8$ by (4.3). Thus the following claim shows that the second term of (4.12) is nonnegative.

Claim 4.7. (i) $d^*(H) + 2 - \dfrac{k}{2} \geq 0$.

(ii) *If* $k \geq 8$, *then* $d^*(H) + 2 - \dfrac{k(h+\delta(H))}{2h} \geq 0$.

Proof. Since H is 3-connected and $h \geq k-1$, and since $2\delta(H) = 2\lfloor (k-1)/2 \rfloor \geq k-2$ by (4.2), we obtain $d^*(H) \geq k-4$ by applying Theorem B with $m = k-2$. Hence the desired inequalities follow from Claim 4.6. □

Finally we consider the sum of the fourth and the fifth terms of (4.12).

Claim 4.8. (i) $k\left(d^*(H)+2\right) + k\left(\dfrac{1}{h}|E_G(C,H)| - \dfrac{k}{2}\right) > k(r+2-k)$.

(ii) *If* $k \geq 8$, *then*

$$k\left(d^*(H)+2\right) + k\left(\frac{1}{h}|E_G(C,H)| - \frac{k(h+\delta(H))}{2h}\right) > k(r+2-k).$$

Proof. Take $x, y \in V(H)$ with $x \neq y$ such that $d^*_H(x,y) = d^*(H)$. Let H' be the graph obtained from H by adding two new vertices u, v with $u \neq v$ and $u, v \notin V(H)$, and three new edges xu, uv, vy. Since H is 3-connected, H' is 2-connected. Also the average degree of $V(H)$ in H' is $(\sum_{w \in V(H)} \deg_{H'}(w))/h = (rh - |E_G(C,H)| + 2)/h$. Therefore it follows from Theorem C that $d^*_H(x,y) + 2 = d^*_{H'}(u,v) \geq (rh - |E_G(C,H)| + 2)/h$, and hence

$$(d^*(H)+2) + |E_G(C,H)|/h > r. \tag{4.15}$$

Combining (4.15) and Claim 4.6, we obtain the desired inequalities. □

By (4.12), (4.13), (4.14) and Claims 4.7 and 4.8, we obtain $l(C) > k(r+2-k)$. This completes the proof of the Main Theorem.

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