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Hamiltonian cycles through a linear forest

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Abstract. Let G be a graph of order n. A graph is *linear forest* if every component is a path. Let S be a set of m edges of G that induces a linear forest. An edge $xy \in E(G)$ is called an *S-edge* if $xy \in S$. An *S-edge-length* of a cycle in G is defined as the number of S-edges that it contains. We prove that if the degree sum in G of every pair of nonadjacent vertices of G is at least $n + m$, then G contains hamiltonian cycles of every S-edge-length between 0 and $|S|$.

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§1. Introduction

In this paper, we consider only finite undirected graphs without loops or multiple edges. For standard graph-theoretic terminology not explained in this paper, we refer the reader to [3]. For $R \subseteq V(G)$ and a vertex $x \in V(G)$, we denote $N_R(x) = N_G(x) \cap R$. We denote the degree of a vertex x in G by $d_G(x)$. A path P connecting two vertices x and y is denoted by xPy, and is called an *x-y path*. The *distance* $d_G(x, y)$ is the length of a shortest x-y path in G; if there is no such path in G, we define $d_G(x, y) = \infty$. We write a cycle C with a given orientation by $\overrightarrow{C}$. For $x, y \in V(C)$, we denote by $x\overrightarrow{C}y$ a path from x to y on $\overrightarrow{C}$. The reverse sequence of $x\overrightarrow{C}y$ is denoted by $y\overleftarrow{C}x$. For $x \in V(C)$, we denote the successor of x on $\overrightarrow{C}$ by x^+. Let X be a subset of $V(C)$. The set X^+ (respectively, X^-) is the successors (predecessors, respectively) of the vertices of X in C and for $x, y \in C$, we define $C[x, y]$ ($C[x, y)$, $C(x, y)$, respectively) to be the subgraph of C from x to y (from x to y^-, from x^+ to y^-). A vertex v is called an *R-vertex* if $v \in R$. The *R-length* of a cycle in G is defined as the number of R-vertices that it contains. A graph on n vertices is called *pancyclic* if it contains cycles of every length l, $3 \le l \le n$. The graph G is said *R-pancyclable* if it contains cycles of all R-lengths from 3 to $|R|$. A *linear forest* is a graph each of whose component is a path. Let S be a set of edges of G that induces a linear forest. An edge $xy \in E(G)$ is called an *S-edge* if $xy \in S$. An *S-edge-length* of a cycle in G is defined as the number of S-edges that it contains.

Among many sufficient conditions for a graph to be hamiltonian, the following sufficient condition is well-known.

Theorem A (Ore [5]). *Let G be a graph of order $n \geq 3$. If $d_G(x)+d_G(y) \geq n$ for every pair of nonadjacent vertices x and y in G, then G is hamiltonian.*

Bondy [2] showed that the same condition as Theorem A implies the existence of cycles of every length between 3 and $|V(G)|$ (except for complete bipartite graphs).

Theorem B (Bondy [2]). *Let G be a graph of order n. If $d_G(x)+d_G(y) \geq n$ for every pair of nonadjacent vertices x and y in G, then G is either pancyclic or the complete bipartite graph $K_{n/2,n/2}$.*

About the cycles passing through some specified vertices, Bollobás and Brightwell [1] proved the following.

Theorem C (Bollobás and Brightwell [1]). *Let G be a graph on n vertices and R a subset of $V(G)$. If $|R| \geq 3$ and $d_G(x)+d_G(y) \geq n$ for every pair of nonadjacent vertices x and y in R, then G has a cycle that includes every vertex of R.*

Theorem C is generalized as follows, which shows the existence of a cycle through a specified number of vertices of a vertex set.

Theorem D (Favaron et al. [4] and Stacho [7]). *Let G be a graph of order n and R a subset of $V(G)$ such that $|R| \geq 3$. If $d_G(x)+d_G(y) \geq n$ for every pair of nonadjacent vertices x and y of R, then either G is R-pancyclable or else n is even, $R = V(G)$ and $G = K_{n/2,n/2}$ or $G[R] = K_{2,2} = C_4 = x_1x_2x_3x_4$ and the structure of G is as follows: $V(G)$ is partitioned into $S \cup V_1 \cup V_2 \cup V_3 \cup V_4$; for any i, $1 \leq i \leq 4$, $G[V_i]$ is any graph on $|V_i|$ vertices with $|V_i| \geq 0$, and each vertex x_i is adjacent to all the vertices of V_{i+1} and V_i where the index i is taken as modulo 4.*

On the other hand, on the existence of a cycle passing through a linear forest, the following theorem is known.

Theorem E (Pósa [6]). *Let m be a nonnegative integer, G a graph on n vertices, where $n \geq 3$, and S a set of m edges of G that induces a linear forest. If $d_G(x) + d_G(y) \geq n + m$ for every pair of nonadjacent vertices x and y, then G contains a hamiltonian cycle that includes every edge of S.*

In this paper, we prove the following theorem, which shows the existence of a hamiltonian cycle which contains a specified number of edges of a linear forest.

Theorem 1. *Let m be a nonnegative integer, G a graph on n vertices, where $n \geq 5$, and S a set of m edges of G that induces a linear forest. If $d_G(x) + d_G(y) \geq n+m$ for every pair of nonadjacent vertices x and y, then G contains hamiltonian cycles of all the S-edge-lengths from 0 to m.*

§2. Proof of Theorem 1

Let G be a graph on n vertices which satisfies the hypothesis. Let S be a set of m edges of G that induces a linear forest. By Theorem E, G contains a hamiltonian cycle H of G such that $S \subseteq E(H)$. We show that if G contains a hamiltonian cycle H of G such that $|E(H) \cap S| = l$, then there exists a hamiltonain cycle H' of G such that $|E(H') \cap S| = l - 1$. So we assume that G contains a hamiltonain cycle H of G such that $|E(H) \cap S| = l$. Set $H = x_1 x_2 \dots x_n x_1$ and consider the subscripts as modulo n. Let $Y = \{x_i | x_i x_{i+1} \in S\}$, $Z = \{x_i | x_i x_{i+1} \notin S\}$ and $q = |S \backslash E(H)|$. Note that $q = m - l$.

Lemma 1. *If there exist $x_i \in Y$ and $x_j \in Z$ such that $d_H(x_i, x_j) \geq 2$, $x_i x_j \in E(G) \backslash S$ and $x_{i+1} x_{j+1} \notin S$, then there exists a hamiltonian cycle H' such that $|E(H') \cap S| = l - 1$ and $x_i x_{i+1} \notin E(H')$.*

Proof of Lemma 1.

We assume that G contains $x_i x_j \in E(G)$ such that $x_i \in Y$, $x_j \in Z$ and $d_H(x_i, x_j) \geq 2$. If $x_{i+1} x_{j+1} \in E(G) \backslash S$, then G contains a hamiltonian cycle $H' = x_i x_j \overleftarrow{H} x_{i+1} x_{j+1} \overrightarrow{H} x_i$ such that $|E(H') \cap S| = l - 1$. So we assume that $x_{i+1} x_{j+1} \notin E(G)$. Then $d_G(x_{i+1}) + d_G(x_{j+1}) \geq n + m$. Let $G' = (V(G), E(G) \backslash \{S \backslash E(H)\})$. Let $p = \min\{q, 3\}$. Then $d_{G'}(x_{i+1}) + d_{G'}(x_{j+1}) \geq n + m - p$. Let $C_1 = V(H[x_{i+1}, x_j])$ and $C_2 = V(H[x_{j+1}, x_i])$. Let $X_1 = N^-_{G'}(x_{i+1}) \cap C_1$, $Y_1 = N_{G'}(x_{j+1}) \cap C_1$, $X_2 = N_{G'}(x_{i+1}) \cap C_2$ and $Y_2 = N^-_{G'}(x_{j+1}) \cap C_2$. By $q \geq p$, we have

$$\begin{aligned} |X_1 \cap Y_1| + |X_2 \cap Y_2| &= |X_1| + |Y_1| + |X_2| + |Y_2| - (|X_1 \cup Y_1| + |X_2 \cup Y_2|) \\ &\geq n + m - p - n \\ &= l + q - p \geq l. \end{aligned}$$

Since $x_i \notin \{X_1 \cap Y_1\} \cup \{X_2 \cap Y_2\}$, there exists a vertex $v \in \{X_1 \cap Y_1\} \cup \{X_2 \cap Y_2\}$ such that $v \notin Y$. If $v \in X_1 \cap Y_1$, then there exists a hamiltonian cycle $H' = x_{i+1}\overrightarrow{H}vx_{j+1}\overrightarrow{H}x_ix_j\overleftarrow{H}v^+x_{i+1}$ such that $|E(H') \cap S| = l-1$. If $v \in X_2 \cap Y_2$, then there exists a hamiltonian cycle $H' = x_{i+1}\overrightarrow{H}x_jx_i\overleftarrow{H}v^+x_{j+1}\overrightarrow{H}vx_{i+1}$ such that $|E(H') \cap S| = l-1$. Since G' is subgraph of G, G contains H'. □

Lemma 2. *If there exist $z_1, z_2, z_3 \in Z$ and $y \in Y$ such that $d_H(y, z_i) \geq 2$ and $yz_i \in E(G)$ for every i, $1 \leq i \leq 3$, then there exists a hamiltonian cycle H' such that $|E(H') \cap S| = l-1$.*

Proof of Lemma 2.

Assume that $z_1, z_2, z_3 \in Z$ and $y \in Y$ such that $d_H(y, z_i) \geq 2$ for every i, $1 \leq i \leq 3$. Since edges of S induce a linear forest, without loss of generality, we may assume $yz_1, yz_2 \notin S$ and $y^+z_1^+ \notin S$. By Lemma 1, G contains a hamiltonian cycle H' such that $|E(H') \cap S| = l-1$. □

Case 1. $m \leq n-4$.

If $q = 0$, since $|E(H) \backslash S| \geq 4$, there exist $z \in Z$ and $y \in Y$ such that $d_H(y, z) \geq 2$. If $yz, y^+z^+ \in E(G)$, then $H' = y^+\overrightarrow{H}zy\overleftarrow{H}z^+$ is a hamiltonian cycle such that $|E(H') \cap S| = l-1$. Hence we may consider only the case yz or $y^+z^+ \notin E(G)$. Concerning the reverse sequence of H in case of $y^+z^+ \notin E(G)$, we obtain that there exist $z \in Z$ and $y \in Y$ such that $yz \notin E(G)$. If $q > 1$, then $|E(H) \backslash S| \geq 5$ implies that, for any $y \in Y$, there exist $z_1, z_2, z_3 \in Z$ such that $d_H(y, z_i) \geq 2 (1 \leq i \leq 3)$. If $yz_1, yz_2, yz_3 \in E(G)$, by Lemma 2, G contains a hamiltonian cycle H' such that $|E(H') \cap S| = l-1$. Hence we may consider only the case where at least one of yz_1, yz_2 and yz_3 is not in $E(G)$. Therefore, in both cases $q = 0$ and $q \geq 1$, we may assume that there exists $y \in Y$ and $z \in Z$ such that $yz \notin E(G)$ and $d_H(y, z) \geq 2$. Clearly $|\{y^+, y^-\} \cap Z| \leq 2$. It follows from the facts $|Y| = l$ and $d_H(y, z) \geq 2$ that $|\{z^+, z^-\} \cap Y| \leq \min\{l-1, 2\}$. Hence

$$\begin{aligned} |\{y^+, y^-\} \cap Z| + |\{z^+, z^-\} \cap Y| &\leq 2 + \min\{l-1, 2\} \\ &\leq l+1. \end{aligned} \tag{1}$$

By $yz \notin E(G)$,

$$\begin{aligned} |N_Y(y)| + |N_Z(y)| + |N_Y(z)| + |N_Z(z)| &= d_G(y) + d_G(z) \\ &\geq n+m. \end{aligned}$$

By $|N_Y(y)| + |N_Z(z)| \leq n-2$,

$$|N_Z(y)| + |N_Y(z)| \geq m+2 \geq l+q+2. \tag{2}$$

From (1) and (2),

$$|N_Y(z)\backslash\{z^+, z^-\}| + |N_Z(y)\backslash\{y^+, y^-\}| \geq q + 1.$$

Hence G contains a set of edges E' of cardinality $q+1$ such that for any $uv \in E'$,

(i) $|\{u, v\} \cap \{y, z\}| = 1$,

(ii) $|\{u, v\} \cap Y| = 1, |\{u, v\} \cap Z| = 1$,

(iii) $d_H(u, v) \geq 2$ and

(iv) $uv \notin E(H)$.

Therefore, by pigeonhole principle, G contains $x_i \in Y$ and $x_j \in Z$ such that $d_H(x_i, x_j) \geq 2$, $x_ix_j \in E(G)\backslash S$ and $x_{i+1}x_{j+1} \notin S$. By Lemma 1, G contains a hamiltonian cycle H' such that $|E(H') \cap S| = l - 1$.

Case 2. $m \geq n - 3$.

By the degree condition, G is complete. If $l \leq n-5$, there exist $y \in Y$ and $z_1, z_2, z_3 \in Z$ such that $d_H(y, z_i) \geq 2$ for every i, $1 \leq i \leq 3$. By Lemma 2, G contains a hamiltonian cycle H' such that $|E(H') \cap S| = l - 1$. Hence we assume $n - 4 \leq l \leq n - 1$. If $q = 0$, immediately G contains a hamiltonian cycle H' such that $|E(H') \cap S| = l - 1$. Hence we may assume $q \geq 1$, then we have $n - 4 \leq l \leq n - 2$.

Subcase 2.1. $l = n - 2$.

In this case we have $q = 1$. Let $z_1, z_2 \in Z$. Since $n \geq 5$, there exist $y_1, y_2 \in Y$ such that $d_H(y_i, z_i) \geq 2$ ($i = 1$ and 2). It follows from $q = 1$ that $y_iz_i, y_i^+z_i^+ \notin S$ for $i = 1$ or 2, hence $y_i\overleftarrow{H}z_i^+y_i^+\overrightarrow{H}z_iy_i$ is a required cycle.

Subcase 2.2. $l = n - 3$.

By $l = n - 3$, we have $q \leq 2$. If $n = 5$, then we may assume $H = x_1x_2x_3x_4x_5x_1$. If $Y = \{x_1, x_2\}$ and $Z = \{x_3, x_4, x_5\}$, since edges of S induce a linear forest, we have $x_1x_3, x_2x_4 \notin S$. Hence $H' = x_1x_3x_2x_4x_5x_1$ is a required cycle. If $Y = \{x_1, x_3\}$, then $Z = \{x_2, x_4, x_5\}$. First we suppose $x_1x_4 \in S$. If $x_2x_5, x_3x_5 \in S$, then the edges of S do not induce linear forest. Hence, without loss of generality, we may assume $x_2x_5 \notin S$. Since the edges of S induce a linear forest, we have $x_1x_3 \notin S$. Then $H' = x_1x_4x_5x_2x_3x_1$ is a required cycle. Next, we suppose $x_1x_4 \notin S$. If $x_3x_5 \notin S$, then $H' = x_1x_2x_3x_5x_4x_1$ is a required cycle. So we assume $x_3x_5 \in S$. If $x_2x_5 \notin S$, then $H' = x_1x_4x_3x_2x_5x_1$ is a required cycle. If $x_2x_5 \in S$, then $x_1x_3 \notin S$. Thus $H' = x_1x_3x_2x_5x_4x_1$ is a required cycle. We can prove the other case in $n = 5$ by the same argument as above, so we assume $n \geq 6$. Let $y_1, y_2, y_3 \in Y$, then there exist $z_1, z_2, z_3 \in Z$,$z_i \neq z_j (i \neq j, 1 \leq i \leq 3, 1 \leq j \leq 3)$ such that $d_H(y_i, z_i) \geq 2, 1 \leq i \leq 3$. Since $q \leq 2$, $y_iz_i \notin S$ and $y_i^+z_i^+ \notin S$ for some i with $1 \leq i \leq 3$. Hence $y_i\overleftarrow{H}z_i^+y_i^+\overrightarrow{H}z_iy_i$ is a required cycle.

Subcase 2.3. $l = n - 4$.

If $n = 5$, without loss of generality we may assume $Y = \{x_1\}$ and $Z = \{x_2, x_3, x_4, x_5\}$. If $x_1x_3, x_2x_4 \notin S$, then $H' = x_1x_3x_2x_4x_5x_1$ is a required cycle. If $x_2x_4 \in S$, since the edges of S induce a linear forest, we have $x_1x_4, x_2x_5 \notin S$. Hence $H' = x_1x_4x_3x_2x_5x_1$ is a required cycle. If $x_1x_3 \in S$ and $x_2x_4 \notin S$, then we have $x_1x_4 \notin S$. If $x_2x_5 \notin S$, then $H' = x_1x_4x_3x_2x_5x_1$ is a required cycle. If $x_2x_5 \in S$, then we obtain $x_3x_5 \notin S$. Thus $H' = x_1x_4x_2x_3x_5x_1$ is a required cycle. Hence we may assume $n \geq 6$. Let $y_1, y_2 \in Y$, then there exist $z_1, z_1', z_2, z_2' \in Z$ such that $d_H(z_i, y_i) \geq 2$ and $d_H(z_i', y_i') \geq 2$ for $i = 1, 2$. Since $q \leq 3$, $\{y_iz_i, y_i^+z_i^+\} \cap S = \phi$ or $\{y_iz_i', y_i^+z_i'^+\} \cap S = \phi$ holds for $i = 1$ or 2. Without loss of generality, we may assume that $\{y_iz_i, y_i^+z_i^+\} \cap S = \phi$. Then $y_i\overleftarrow{H}z_i^+y_i^+\overrightarrow{H}z_iy_i$ is a required cycle. $\square$

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