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An inverse problem for the elastic equation in plane-stratified media

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Abstract. Assume that two media are laying in a half-space and the interface wall is parallel to the boundary of the half-space. We can directly observe the data near the boundary of the half-space, but we cannot directly observe inside the half-space. In this situation, we try to identify these unknown things by creating artificial explosions and observing on the boundary the waves generated by the explosions. Here, the waves are described by the elastic equation.

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§1. Introduction

Our problem originates from a simplified model of an experiment conducted by geophysicists. We cannot directly observe the structure inside the earth. Then, for example, we perform the following experiment in order to guess it: We create an artificial explosion at a certain point near the earth's surface. Waves generated by the explosion travel in the earth. We observe the waves on the earth's surface, and determine the structure inside the earth from the observation data.

We consider this problem, in particular, in the case when the earth consists of some layers. This problem has been studied by Bartoloni-Lodovici-Zirilli [2], Fatone-Maponi-Pignotti-Zirilli [3], Hansen [4], and Nagayasu [6] for instance. They deal with the wave equation as an equation which describes the behavior of waves. However, in the model which we consider, the media through which waves travel are the earth, and the waves which travel inside the earth are described by the elastic equation rather than the wave equation. Therefore, in this paper, we consider the problem by dealing with the elastic equation as the equation which describes the waves through the media. We begin with

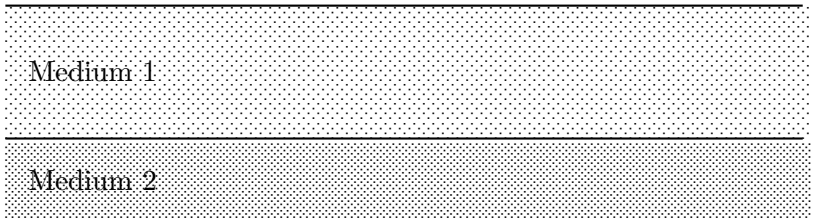

Figure 1: The situation which we consider.

treating the simplified situation as follows, and we hope that the result for it will suggest the results for more general situation.

Assume that two media, Medium 1 and Medium 2, are laying in a half-space, and the interface wall is parallel to the boundary of the half-space (see Figure 1). We assume that the speeds of the (primary and shear) waves and the density of the medium in Medium 1 are known, but the width of Medium 1, the speeds of the waves and the density of the medium in Medium 2 are unknown. Under this situation, we try to identify these unknown data by using the known data or the data which can be observed near the boundary.

In Nagayasu [6], from the known data we determined the width of Medium 1, the speed of the waves through Medium 2 and the density of Medium 2 if the waves which travel through the media are described by the wave equation. Since we deal with the elastic equation in this paper, we must determine the speeds of two kinds of the waves, namely, the primary and shear waves through Medium 2. However we may expect to be able to determine these speeds because we can observe these two kinds of the waves also. The objective of this paper is to justify this consideration.

Now, we introduce the notations and formulate this problem. Let us write $x' = (x_0, x_1, x_2)$, $x'' = (x_1, x_2, x_3)$ and $x''' = (x_1, x_2)$ for the coordinate $x = (x_0, x_1, x_2, x_3)$ in $\mathbb{R}^4$. The variable x_0 plays the role of the time and x'' the physical space. We introduce x' for short notation when we apply the Fourier-Laplace transformations with respect to (x_0, x_1, x_2).

Let $h > 0$, and $\Omega_1 := \{x'' \in \mathbb{R}^3 : 0 < x_3 < h\}$, $\Omega_2 := \{x'' \in \mathbb{R}^3 : x_3 > h\}$. The constant h describes the width of Medium 1, and Ω_k Medium k for $k = 1, 2$. We set $D_{x_j} := (1/i)(\partial/\partial x_j)$, $\nabla_{x''} = (D_{x_1}, D_{x_2}, D_{x_3})$, $\Delta_{x''} = D_{x_1}^2 + D_{x_2}^2 + D_{x_3}^2$. Let c_{p_k}, c_{s_k}, and ρ_k be positive real numbers and set

$$P_k(D_x)u := -D_{x_0}^2 u + (c_{p_k}^2 - c_{s_k}^2)\nabla_{x''}(\nabla_{x''} \cdot u) + c_{s_k}^2 \Delta_{x''} u,$$

$$B_k(D_x) := i\rho_k \begin{bmatrix} c_{s_k}^2 D_{x_3} & 0 & c_{s_k}^2 D_{x_1} \\ 0 & c_{s_k}^2 D_{x_3} & c_{s_k}^2 D_{x_2} \\ (c_{p_k}^2 - 2c_{s_k}^2)D_{x_1} & (c_{p_k}^2 - 2c_{s_k}^2)D_{x_2} & c_{p_k}^2 D_{x_3} \end{bmatrix}$$

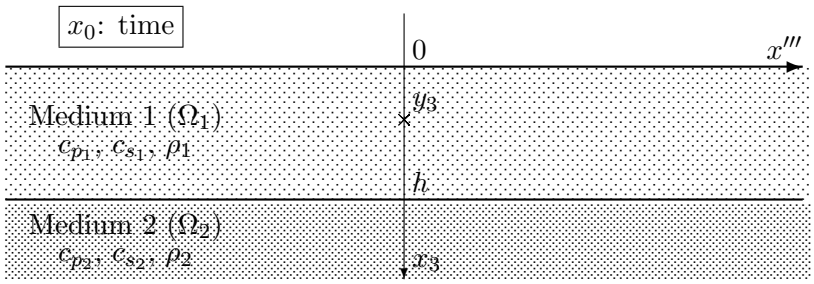

Figure 2: Notation

for $k = 1, 2$. We assume $c_{p_k} > c_{s_k}$ for $k = 1, 2$. The positive number c_{p_k} describes the speed of primary waves, c_{s_k} the speed of shear waves, ρ_k the density of the medium in Ω_k. Suppose $0 < y_3 < h$. Set $y'' := (0, 0, y_3) \in \mathbb{R}^3_{x''}$, and $y := (0, y'') \in \mathbb{R}^4$.

We discuss the following equations:

$$P_1(D_x)G(x) = \delta(x - y)I, \quad x_0 \in \mathbb{R},\ x'' \in \Omega_1, \tag{1.1}$$

$$P_2(D_x)G(x) = 0, \quad x_0 \in \mathbb{R},\ x'' \in \Omega_2, \tag{1.2}$$

$$B_1(D_x)G(x)|_{x_3=+0} = 0, \quad x' \in \mathbb{R}^3, \tag{1.3}$$

$$G(x)|_{x_3=h-0} = G(x)|_{x_3=h+0}, \quad x' \in \mathbb{R}^3, \tag{1.4}$$

$$B_1(D_x)G(x)|_{x_3=h-0} = B_2(D_x)G(x)|_{x_3=h+0}, \quad x' \in \mathbb{R}^3, \tag{1.5}$$

where I is the identity matrix of order 3. These equations describe the situation that the initial data are $(\delta, 0, 0)$, $(0, \delta, 0)$, $(0, 0, \delta)$ at a point $y'' \in \Omega_1$ at time $x_0 = 0$ with the boundary condition (1.3) and the interface or transmission conditions (1.4) and (1.5). The equation (1.4) expresses the continuity of the displacement of waves on the interface wall, and (1.5) the continuity of the stress.

The following main result says that except the special case we can reconstruct the width h of Ω_1, the speeds c_{p_2}, c_{s_2} of waves and the density ρ_2 of medium in Ω_2 from the observation data $G(x)|_{x_3=0}$ when the speeds c_{p_1}, c_{s_1} of waves and the density ρ_1 of medium in Ω_1 are known.

Main result. *Let c_{p_1}, c_{s_1}, ρ_1 and y_3 be given. Assume that the observation data $G(x)|_{x_3=+0}$ are given, where $G(x)$ denotes the solution of the equations* (1.1)–(1.5). *Then the constants c_{p_2}, c_{s_2}, ρ_2 are expressed with the given data. Moreover, the constant h is expressed with the given data unless $G(x)|_{x_3=+0} \equiv \widetilde{G}(x)|_{x_3=+0}$. Here $\widetilde{G}$ is the waves in the situation that only one medium Medium* 1 *is laying in the half-space, that is, the solution of*

$$\begin{cases} P_1(D_x)\widetilde{G}(x) = \delta(x - y)I, \quad x' \in \mathbb{R}^3,\ x_3 > 0, \\ B_1(D_x)\widetilde{G}(x)|_{x_3=+0} = 0, \quad x' \in \mathbb{R}^3. \end{cases} \tag{1.6}$$

On the other hand, if $G(x)|_{x_3=+0} \equiv \widetilde{G}(x)|_{x_3=+0}$ *then* h *is not identified.*

In Section 3, we state our main results more precisely and prove them by using the solution formula of the problem (1.1)–(1.5). The solution formula can be written by using the Fourier-Laplace transformation in the same way as Matsumura [5] and Shimizu [8]. Theoretically, this formula must describe the dependence between behavior of the solution and information of the media. However this dependence is rather intricate and is not expressed straightforwardly. In this section, we give a process of reduction to clear the dependence.

During the reduction process, we use the result in Nagayasu [6]. However, even if we use the result in [6], the speed c_{p_2} of the primary waves through Medium 2 cannot be determined. Moreover, the dependence between behavior of the solution and the constant c_{p_2} is also rather intricate and is not expressed straightforwardly. In this paper, we give a process of reduction to clear also the dependence between behavior of the solution and c_{p_2}. This is the main part of our methods.

Finally, we explain the plan of this paper. In Section 2, we rewrite the equations (1.1)–(1.5), construct the solution, and discuss some properties of the solution. In Section 3, we state the main results in this paper and give the proofs.

§2. The solution formula and some properties of the solution

In this section, we solve the mixed problem (1.1)–(1.5), and prove lemmas needed later. We mainly refer to Matsumura [5], Sakamoto [7], and Shimizu [8] in order to solve the mixed problem.

We first rewrite these equations. We define $E_1(x)$ by the fundamental solution of the forward Cauchy problem for $P_1(D_x)$ in the whole physical space $\mathbb{R}^3_{x''}$, namely, the inverse Fourier-Laplace transform of $P_1(\xi + i\eta)^{-1}$ in the sense of distribution:

$$E_1(x) = \frac{1}{(2\pi)^4} \int_{\mathbb{R}^4_\xi} e^{ix\cdot(\xi+i\eta)} P_1(\xi + i\eta)^{-1} d\xi,$$

where we determine η so as to be able to define $E_1(x)$ as the distribution (cf. Shimizu [8]).

We put $F_1(x)$ and $F_2(x)$ by

$$F_1(x) := E_1(x - y) - G(x), \quad x'' \in \Omega_1, \tag{2.1}$$

$$F_2(x) := G(x), \quad x'' \in \Omega_2, \tag{2.2}$$

respectively. Since the distribution $E_1(x-y)$ describes the first propagation of waves due to a point source, the distribution $F_1(x)$ describes the propagation in

Ω_1 of the second waves caused by the first waves, the boundary wall $\{x_3 = 0\}$, and the interface wall $\{x_3 = h\}$. By (2.1) and (2.2), the equations (1.1)–(1.5) are equivalent to the following equations:

$$P_1(D_x)F_1(x) = 0, \quad x_0 \in \mathbb{R},\ x'' \in \Omega_1, \tag{2.3}$$

$$P_2(D_x)F_2(x) = 0, \quad x_0 \in \mathbb{R},\ x'' \in \Omega_2, \tag{2.4}$$

$$B_1(D_x)F_1(x)|_{x_3=+0} = B_1(D_x)E_1(x-y)|_{x_3=+0}, \quad x' \in \mathbb{R}^3, \tag{2.5}$$

$$[E_1(x-y) - F_1(x)]|_{x_3=h-0} = F_2(x)|_{x_3=h+0}, \quad x' \in \mathbb{R}^3, \tag{2.6}$$

$$\begin{aligned}[B_1(D_x)E_1(x-y) - B_1(D_x)F_1(x)]|_{x_3=h-0} \\ = B_2(D_x)F_2(x)|_{x_3=h+0}, \quad x' \in \mathbb{R}^3.\end{aligned} \tag{2.7}$$

Next, we take the Fourier-Laplace transformations with respect to x' for (2.3)–(2.7). In order not to vanish the Lopatinski determinant, we take it along $S_m := \{(\chi(\xi'), \xi_1, \xi_2) : \xi' \in \mathbb{R}^3_{\xi'}\}$, where $\chi(\xi') := \xi_0 - im\log(2+|\xi'|)$, and m is a positive real large enough. We remark that the Lopatinski determinant does not vanish. Indeed, there exists $\gamma > 0$ such that

$$\begin{vmatrix} |\zeta'''| & \tau^+_{s_1} & |\zeta'''| & \tau^+_{s_2} \\ -\tau^+_{p_1} & |\zeta'''| & \tau^+_{p_2} & -|\zeta'''| \\ -2\rho_1 c^2_{s_1}\tau^+_{p_1}|\zeta'''| & -\rho_1 c^2_{s_1}((\tau^+_{s_1})^2 - |\zeta'''|^2) & 2\rho_2 c^2_{s_2}\tau^+_{p_2}|\zeta'''| & \rho_2 c^2_{s_2}((\tau^+_{s_2})^2 - |\zeta'''|^2) \\ \rho_1 c^2_{s_1}((\tau^+_{s_1})^2 - |\zeta'''|^2) & -2\rho_1 c^2_{s_1}\tau^+_{s_1}|\zeta'''| & \rho_2 c^2_{s_2}((\tau^+_{s_2})^2 - |\zeta'''|^2) & -2\rho_2 c^2_{s_2}\tau^+_{s_2}|\zeta'''| \end{vmatrix} \neq 0$$

and

$$\begin{vmatrix} 2\tau^+_{p_1}|\zeta'''| & (\tau^+_{s_1})^2 - |\zeta'''|^2 \\ (\tau^+_{s_1})^2 - |\zeta'''|^2 & -2\tau^+_{s_1}|\zeta'''| \end{vmatrix} \neq 0$$

hold for all $\zeta' = (\zeta_0, \zeta_1, \zeta_2) = (\xi_0 - i\eta_0, \xi_1, \xi_2)$ ($\xi' = (\xi_0, \xi_1, \xi_2) \in \mathbb{R}^3$, $\eta_0 > \gamma$) by Achenbach [1, §5.11] and Shimizu [8, §3], respectively. Hence, we can prove that the Lopatinski determinant does not vanish by the method of Matsumura [5]. Then we obtain

$$P_1(\zeta', D_{x_3})\widehat{F}_1(\zeta', x_3) = 0, \quad 0 < x_3 < h, \tag{2.8}$$

$$P_2(\zeta', D_{x_3})\widehat{F}_2(\zeta', x_3) = 0, \quad x_3 > h, \tag{2.9}$$

$$B_1(\zeta', D_{x_3})\widehat{F}_1(\zeta', x_3)|_{x_3=0} = \frac{1}{2\pi}\int_{\mathbb{R}_{\xi_3}} e^{-iy_3\zeta_3}B_1(\zeta)P_1(\zeta)^{-1}d\xi_3, \tag{2.10}$$

$$[\widehat{F}_1(\zeta', x_3) + \widehat{F}_2(\zeta', x_3)]|_{x_3=h} = \frac{1}{2\pi}\int_{\mathbb{R}_{\xi_3}} e^{i(h-y_3)\zeta_3}P_1(\zeta)^{-1}d\xi_3, \tag{2.11}$$

$$\begin{aligned}[B_1(\zeta', D_{x_3})\widehat{F}_1(\zeta', x_3) + B_2(\zeta', D_{x_3})\widehat{F}_2(\zeta', x_3)]|_{x_3=h} \\ = \frac{1}{2\pi}\int_{\mathbb{R}_{\xi_3}} e^{i(h-y_3)\zeta_3}B_1(\zeta)P_1(\zeta)^{-1}d\xi_3,\end{aligned} \tag{2.12}$$

where $\zeta' \in S_m$ and $\zeta = (\zeta', \xi_3) \in \mathbb{C}^4$.

Moreover, we simplify these equations. Put

$$U(\zeta_1, \zeta_2) = \frac{1}{|\zeta'''|} \begin{bmatrix} \zeta_1 & -\zeta_2 & 0 \\ \zeta_2 & \zeta_1 & 0 \\ 0 & 0 & |\zeta'''| \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

where $|\zeta'''| = \sqrt{\zeta_1^2 + \zeta_2^2} (= \sqrt{\xi_1^2 + \xi_2^2})$. Then we have

$$P_k(\zeta', D_{x_3}) = -U(\zeta_1, \zeta_2) C \begin{bmatrix} P_{k1}(\zeta', D_{x_3}) & 0 \\ 0 & P_{k2}(\zeta', D_{x_3}) \end{bmatrix} (U(\zeta_1, \zeta_2) C)^{-1},$$

$$B_k(\zeta', D_{x_3}) = U(\zeta_1, \zeta_2) C \begin{bmatrix} B_{k1}(\zeta', D_{x_3}) & 0 \\ 0 & B_{k2}(\zeta', D_{x_3}) \end{bmatrix} (U(\zeta_1, \zeta_2) C)^{-1},$$

where

$$P_{k1}(\zeta', D_{x_3}) = \begin{bmatrix} \zeta_0^2 - (c_{s_k}^2 D_{x_3}^2 + c_{p_k}^2 |\zeta'''|^2) & -(c_{p_k}^2 - c_{s_k}^2)|\zeta'''| D_{x_3} \\ -(c_{p_k}^2 - c_{s_k}^2)|\zeta'''| D_{x_3} & \zeta_0^2 - (c_{p_k}^2 D_{x_3}^2 + c_{s_k}^2 |\zeta'''|^2) \end{bmatrix},$$

$$P_{k2}(\zeta', D_{x_3}) = \zeta_0^2 - c_{s_k}^2 (D_{x_3}^2 + |\zeta'''|^2),$$

$$B_{k1}(\zeta', D_{x_3}) = i\rho_k \begin{bmatrix} c_{s_k}^2 D_{x_3} & c_{s_k}^2 |\zeta'''| \\ (c_{p_k}^2 - 2c_{s_k}^2)|\zeta'''| & c_{p_k}^2 D_{x_3} \end{bmatrix},$$

$$B_{k2}(\zeta', D_{x_3}) = i\rho_k c_{s_k}^2 D_{x_3}.$$

Hence we have

$$P_{11}(\zeta', D_{x_3})\varphi_1(\zeta', x_3) = 0, \quad 0 < x_3 < h, \tag{2.13}$$

$$P_{21}(\zeta', D_{x_3})\varphi_2(\zeta', x_3) = 0, \quad x_3 > h, \tag{2.14}$$

$$B_{11}(\zeta', D_{x_3})\varphi_1(\zeta', x_3)|_{x_3=0} = -\frac{1}{2\pi} \int_{\mathbb{R}_{\xi_3}} e^{-iy_3\zeta_3} B_{11}(\zeta) P_{11}(\zeta)^{-1} d\xi_3, \tag{2.15}$$

$$[\varphi_1(\zeta', x_3) + \varphi_2(\zeta', x_3)]|_{x_3=h} = -\frac{1}{2\pi} \int_{\mathbb{R}_{\xi_3}} e^{i(h-y_3)\zeta_3} P_{11}(\zeta)^{-1} d\xi_3, \tag{2.16}$$

$$\begin{aligned} [B_{11}(\zeta', D_{x_3})\varphi_1(\zeta', x_3) + B_{21}(\zeta', D_{x_3})\varphi_2(\zeta', x_3)]|_{x_3=h} \\ = -\frac{1}{2\pi} \int_{\mathbb{R}_{\xi_3}} e^{i(h-y_3)\zeta_3} B_{11}(\zeta) P_{11}(\zeta)^{-1} d\xi_3, \end{aligned} \tag{2.17}$$

and

$$P_{12}(\zeta', D_{x_3})\psi_1(\zeta', x_3) = 0, \quad 0 < x_3 < h, \tag{2.18}$$

$$P_{22}(\zeta', D_{x_3})\psi_2(\zeta', x_3) = 0, \quad x_3 > h, \tag{2.19}$$

$$B_{12}(\zeta', D_{x_3})\psi_1(\zeta', x_3)|_{x_3=0} = -\frac{1}{2\pi} \int_{\mathbb{R}_{\xi_3}} e^{-iy_3\zeta_3} B_{12}(\zeta) P_{12}(\zeta)^{-1} d\xi_3, \tag{2.20}$$

$$[\psi_1(\zeta', x_3) + \psi_2(\zeta', x_3)]|_{x_3=h} = -\frac{1}{2\pi} \int_{\mathbb{R}_{\xi_3}} e^{i(h-y_3)\zeta_3} P_{12}(\zeta)^{-1} d\xi_3, \tag{2.21}$$

$$[B_{12}(\zeta', D_{x_3})\psi_1(\zeta', x_3) + B_{22}(\zeta', D_{x_3})\psi_2(\zeta', x_3)]|_{x_3=h} = -\frac{1}{2\pi}\int_{\mathbb{R}_{\xi_3}} e^{i(h-y_3)\zeta_3} B_{12}(\zeta) P_{12}(\zeta)^{-1} d\xi_3, \tag{2.22}$$

where $\varphi_k(\zeta', x_3)$ and $\psi_k(\zeta', x_3)$ are defined by

$$\begin{bmatrix} \varphi_k(\zeta', x_3) & 0 \\ 0 & \psi_k(\zeta', x_3) \end{bmatrix} = (U(\zeta_1, \zeta_2)C)^{-1} \widehat{F}_k(\zeta', x_3)(U(\zeta_1, \zeta_2)C) \tag{2.23}$$

for $k = 1, 2$.

In particular, we remark that the equations (2.18)–(2.22) are the Fourier-Laplace transforms of

$$(c_{s_1}^2 \Delta_{x''} - D_{x_0}^2)(-f_1)(x) = 0, \quad 0 < x_3 < h, \tag{2.24}$$

$$(c_{s_2}^2 \Delta_{x''} - D_{x_0}^2)(-f_2)(x) = 0, \quad x_3 > h, \tag{2.25}$$

$$D_{x_3}(-f_1)(x)|_{x_3=0} = D_{x_3} e_1(x-y)|_{x_3=0}, \tag{2.26}$$

$$[e_1(x-y) - (-f_1)(x)]|_{x_3=h} = (-f_2)(x)|_{x_3=h}, \tag{2.27}$$

$$\rho_1 c_{s_1}^2 D_{x_3}(e_1(x-y) - (-f_1)(x))|_{x_3=h} = \rho_2 c_{s_2}^2 D_{x_3}(-f_2)(x)|_{x_3=h} \tag{2.28}$$

which are dealt with in Nagayasu [6], where $e_1(x)$ is the fundamental solution of the forward Cauchy problem for $c_{s_1}^2 \Delta_{x''} - D_{x_0}^2$ in the whole physical space $\mathbb{R}^3$, namely, the inverse Fourier-Laplace transform of $1/P_{12}(\xi + i\eta)$ in the sense of distribution:

$$e_1(x) := \frac{1}{(2\pi)^4} \int_{\mathbb{R}^4_\xi} \frac{e^{ix\cdot(\xi+i\eta)}}{P_{12}(\xi + i\eta)} d\xi.$$

On the other hand, by the equations (2.13) and (2.14), we have

$$\begin{aligned} \begin{bmatrix} \varphi_{11l}(\zeta', x_3) \\ \varphi_{12l}(\zeta', x_3) \end{bmatrix} &= \alpha_{+pl} e^{i\tau_{p_1}^+ x_3} \begin{bmatrix} |\zeta'''| \\ \tau_{p_1}^+ \end{bmatrix} + \alpha_{-pl} e^{-i\tau_{p_1}^+ x_3} \begin{bmatrix} |\zeta'''| \\ -\tau_{p_1}^+ \end{bmatrix} \\ &\quad + \alpha_{+sl} e^{i\tau_{s_1}^+ x_3} \begin{bmatrix} \tau_{s_1}^+ \\ -|\zeta'''| \end{bmatrix} + \alpha_{-sl} e^{-i\tau_{s_1}^+ x_3} \begin{bmatrix} \tau_{s_1}^+ \\ |\zeta'''| \end{bmatrix}, \end{aligned} \tag{2.29}$$

$$\begin{bmatrix} \varphi_{21l}(\zeta', x_3) \\ \varphi_{22l}(\zeta', x_3) \end{bmatrix} = \beta_{+pl} e^{i\tau_{p_2}^+ x_3} \begin{bmatrix} |\zeta'''| \\ \tau_{p_2}^+ \end{bmatrix} + \beta_{+sl} e^{i\tau_{s_2}^+ x_3} \begin{bmatrix} \tau_{s_2}^+ \\ -|\zeta'''| \end{bmatrix} \tag{2.30}$$

for $l = 1, 2$, where

$$\varphi_k(\zeta', x_3) = \begin{bmatrix} \varphi_{k11}(\zeta', x_3) & \varphi_{k12}(\zeta', x_3) \\ \varphi_{k21}(\zeta', x_3) & \varphi_{k22}(\zeta', x_3) \end{bmatrix} \tag{2.31}$$

and $\tau_{p_k}^+(\zeta')$ [resp. $\tau_{s_k}^+(\zeta')$] is the root which has positive imaginary part of the equation in τ: $\zeta_0^2 - c_{p_k}^2(\zeta_1^2 + \zeta_2^2 + \tau^2) = 0$ [resp. $\zeta_0^2 - c_{s_k}^2(\zeta_1^2 + \zeta_2^2 + \tau^2) = 0$].

With respect to the Fourier-Laplace transforms of E_1, we remark that

$$\frac{1}{2\pi}\int_{\mathbb{R}_{\xi_3}} e^{-iy_3\zeta_3} B_{11}(\zeta)P_{11}(\zeta)^{-1}d\xi_3 \tag{2.32}$$
$$= \frac{\rho_1 c_{s_1}^2}{2\zeta_0^2}\begin{bmatrix} 2|\zeta'''|^2 e^{iy_3\tau_{p_1}^+} + ((\tau_{s_1}^+)^2 - |\zeta'''|^2)e^{iy_3\tau_{s_1}^+} \\ -\dfrac{|\zeta'''|((\tau_{s_1}^+)^2 - |\zeta'''|^2)}{\tau_{p_1}^+}e^{iy_3\tau_{p_1}^+} + 2|\zeta'''|\tau_{s_1}^+ e^{iy_3\tau_{s_1}^+} \\ -2|\zeta'''|\tau_{p_1}^+ e^{iy_3\tau_{p_1}^+} + \dfrac{|\zeta'''|((\tau_{s_1}^+)^2 - |\zeta'''|^2)}{\tau_{s_1}^+}e^{iy_3\tau_{s_1}^+} \\ ((\tau_{s_1}^+)^2 - |\zeta'''|^2)e^{iy_3\tau_{p_1}^+} + 2|\zeta'''|^2 e^{iy_3\tau_{s_1}^+} \end{bmatrix},$$

$$\frac{1}{2\pi}\int_{\mathbb{R}_{\xi_3}} e^{i(h-y_3)\zeta_3} P_{11}(\zeta)^{-1}d\xi_3 \tag{2.33}$$
$$= \frac{i}{2\zeta_0^2}\begin{bmatrix} -\dfrac{|\zeta'''|^2}{\tau_{p_1}^+}e^{i(h-y_3)\tau_{p_1}^+} - \tau_{s_1}^+ e^{i(h-y_3)\tau_{s_1}^+} \\ -|\zeta'''|e^{i(h-y_3)\tau_{p_1}^+} + |\zeta'''|e^{i(h-y_3)\tau_{s_1}^+} \\ -|\zeta'''|e^{i(h-y_3)\tau_{p_1}^+} + |\zeta'''|e^{i(h-y_3)\tau_{s_1}^+} \\ -\tau_{p_1}^+ e^{i(h-y_3)\tau_{p_1}^+} - \dfrac{|\zeta'''|^2}{\tau_{s_1}^+}e^{i(h-y_3)\tau_{s_1}^+} \end{bmatrix},$$

$$\frac{1}{2\pi}\int_{\mathbb{R}_{\xi_3}} e^{i(h-y_3)\zeta_3} B_{11}(\zeta)P_{11}(\zeta)^{-1}d\xi_3 \tag{2.34}$$
$$= \frac{\rho_1 c_{s_1}^2}{2\zeta_0^2}\begin{bmatrix} 2|\zeta'''|^2 e^{i(h-y_3)\tau_{p_1}^+} + ((\tau_{s_1}^+)^2 - |\zeta'''|^2)e^{i(h-y_3)\tau_{s_1}^+} \\ \dfrac{|\zeta'''|((\tau_{s_1}^+)^2 - |\zeta'''|^2)}{\tau_{p_1}^+}e^{i(h-y_3)\tau_{p_1}^+} - 2|\zeta'''|\tau_{s_1}^+ e^{i(h-y_3)\tau_{s_1}^+} \\ 2|\zeta'''|\tau_{p_1}^+ e^{i(h-y_3)\tau_{p_1}^+} - \dfrac{|\zeta'''|((\tau_{s_1}^+)^2 - |\zeta'''|^2)}{\tau_{s_1}^+}e^{i(h-y_3)\tau_{s_1}^+} \\ ((\tau_{s_1}^+)^2 - |\zeta'''|^2)e^{i(h-y_3)\tau_{p_1}^+} + 2|\zeta'''|e^{i(h-y_3)\tau_{s_1}^+} \end{bmatrix}$$

by the residue theorem.

Finally, we prove lemmas needed later.

Lemma 1. *Let* $\{\varphi_{kml}\}$ *be the solution of the equations* (2.13)–(2.17)*, where* $\{\varphi_{kml}\}$ *are defined by* (2.31)*. If* $\alpha_{-p1} \equiv \alpha_{-s1} \equiv 0$ *then* $\alpha_{+p1} \not\equiv (i|\zeta'''|/2\zeta_0^2\tau_{p_1}^+)\times e^{-iy_3\tau_{p_1}^+}$*, where* $\{\alpha_{\pm pl}, \alpha_{\pm sl}\}$ *are defined by* (2.29)*.*

Proof. We use a reduction to absurdity. Assume $\alpha_{-p1} \equiv \alpha_{-s1} \equiv 0$ and

$$\alpha_{+p1} \equiv \frac{i|\zeta'''|}{2\zeta_0^2\tau_{p_1}^+}e^{-iy_3\tau_{p_1}^+}.$$

Then we have

$$\alpha_{+s1} = \frac{i|\zeta'''|}{\zeta_0^2((\tau_{s_1}^+)^2 - |\zeta'''|^2)}(e^{iy_3\tau_{p_1}^+} - e^{-iy_3\tau_{p_1}^+}) + \frac{i}{2\zeta_0^2}e^{iy_3\tau_{s_1}^+} \tag{2.35}$$

by the $(1,1)$-components of (2.15) and (2.32). In a similar way, we have

$$\alpha_{+s1} = \frac{i((\tau_{s_1}^+)^2 - |\zeta'''|^2)}{4\tau_{p_1}^+\tau_{s_1}^+\zeta_0^2}(e^{iy_3\tau_{p_1}^+} + e^{-iy_3\tau_{p_1}^+}) - \frac{i}{2\zeta_0^2}e^{iy_3\tau_{s_1}^+} \tag{2.36}$$

by the $(2,1)$-components of (2.15) and (2.32). By (2.35) and (2.36), we have

$$\begin{aligned} &- \frac{|\zeta'''|}{(\tau_{s_1}^+)^2 - |\zeta'''|^2}(e^{iy_3\tau_{p_1}^+} - e^{-iy_3\tau_{p_1}^+}) \\ &\quad + \frac{(\tau_{s_1}^+)^2 - |\zeta'''|^2}{4\tau_{p_1}^+\tau_{s_1}^+}(e^{iy_3\tau_{p_1}^+} + e^{-iy_3\tau_{p_1}^+}) = e^{iy_3\tau_{s_1}^+}. \end{aligned} \tag{2.37}$$

Hence we obtain

$$\begin{aligned} &\frac{c_{p_1}}{4c_{s_1}}\left\{e^{-iy_3\xi_0/c_{p_1}}(2+|\xi_0|)^{-y_3m/c_{p_1}} + e^{iy_3\xi_0/c_{p_1}}(2+|\xi_0|)^{y_3m/c_{p_1}}\right\} \\ &\qquad = e^{-iy_3\xi_0/c_{s_1}}(2+|\xi_0|)^{-y_3m/c_{s_1}} \end{aligned} \tag{2.38}$$

from substituting $\xi_1 = \xi_2 = 0$ into (2.37). Since

$$|(\text{the left-hand side of (2.38)})| \to +\infty$$

and

$$|(\text{the right-hand side of (2.38)})| \to 0$$

as $\xi_0 \to +\infty$, we have a contradiction. □

Lemma 2. *Let $\{\varphi_{kml}\}$ be the solution of the equations* (2.13)–(2.17)*, where $\{\varphi_{kml}\}$ are defined by* (2.31)*. Then $\beta_{+p1} \not\equiv 0$ holds, where $\{\beta_{+pl}, \beta_{+sl}\}$ are defined by* (2.30)*.*

Proof. We use a reduction to absurdity. Assume that $\beta_{+p1} \equiv 0$. We define $\{\alpha_{\pm pl}, \alpha_{\pm sl}\}$ by (2.29). By (2.29) and (2.32), and the $(1,1)$ and $(2,1)$-components of equation (2.15), we have

$$\begin{aligned} &\alpha_{+p1}2\tau_{p_1}^+|\zeta'''| - \alpha_{-p1}2\tau_{p_1}^+|\zeta'''| \\ &\quad + \alpha_{+s1}((\tau_{s_1}^+)^2 - |\zeta'''|^2) - \alpha_{-s1}((\tau_{s_1}^+)^2 - |\zeta'''|^2) \\ &\quad = \frac{i}{2\zeta_0^2}\left\{2|\zeta'''|^2e^{iy_3\tau_{p_1}^+} + ((\tau_{s_1}^+)^2 - |\zeta'''|^2)e^{iy_3\tau_{s_1}^+}\right\}, \end{aligned} \tag{2.39}$$

$$\begin{aligned} &\alpha_{+p1}((\tau_{s_1}^+)^2 - |\zeta'''|^2) + \alpha_{-p1}((\tau_{s_1}^+)^2 - |\zeta'''|^2) \\ &\quad - \alpha_{+s1}2\tau_{s_1}^+|\zeta'''| - \alpha_{-s1}2\tau_{s_1}^+|\zeta'''| \\ &\quad = \frac{i}{2\zeta_0^2}\left\{-\frac{|\zeta'''|((\tau_{s_1}^+)^2 - |\zeta'''|^2)}{\tau_{p_1}^+}e^{iy_3\tau_{p_1}^+} + 2|\zeta'''|\tau_{s_1}^+e^{iy_3\tau_{s_1}^+}\right\}. \end{aligned} \tag{2.40}$$

On the other hand, by (2.29), (2.30), (2.33), (2.34) and the assumption $\beta_{+p1} \equiv 0$, we can rewrite the $(1,1)$ and $(2,1)$-components of the equations (2.16) and (2.17) to the four equations which include $\alpha_{\pm p1}$, $\alpha_{\pm s1}$, β_{+s1}. These four equations are simultaneous linear equations for $\alpha_{\pm p1}$ and $\alpha_{\pm s1}$, and

$$\begin{vmatrix} |\zeta'''| & |\zeta'''| & \tau_{s_1}^+ & \tau_{s_1}^+ \\ \tau_{p_1}^+ & -\tau_{p_1}^+ & -|\zeta'''| & |\zeta'''| \\ 2\tau_{p_1}^+|\zeta'''| & -2\tau_{p_1}^+|\zeta'''| & (\tau_{s_1}^+)^2 - |\zeta'''|^2 & -(\tau_{s_1}^+)^2 + |\zeta'''|^2 \\ (\tau_{s_1}^+)^2 - |\zeta'''|^2 & (\tau_{s_1}^+)^2 - |\zeta'''|^2 & -2\tau_{s_1}^+|\zeta'''| & -2\tau_{s_1}^+|\zeta'''| \end{vmatrix} = \frac{4\tau_{p_1}^+\tau_{s_1}^+\zeta_0^2}{s_1^2} \neq 0.$$

Hence we can solve these four equations for $\alpha_{\pm p1}$ and $\alpha_{\pm s1}$. Then we substitute these $\alpha_{\pm p1}$ and $\alpha_{\pm s1}$ into equations (2.39) and (2.40). Moreover we substitute $\xi_1 = \xi_2 = 0$ into these equations and simplify these. Then we have

$$\beta_{+s1}|_{\xi_1=\xi_2=0} = 0$$

and

$$e^{2iy_3\xi_0/c_{s_1}}(2 + |\xi_0|)^{2y_3m/c_{s_1}} = 1. \tag{2.41}$$

Hence we have $(2 + |\xi_0|)^{2y_3m/c_{s_1}} = 1$ by taking the absolute value of the equation (2.41). Therefore we have a contradiction since m is a positive large enough. □

§3. The main theorems and their proofs

In Section 2, we rewrite equations (1.1)–(1.5) to equations (2.13)–(2.17) and (2.18)–(2.22). By (2.1), (2.2) and (2.23), we remark that the following are equivalent:

- $G(x)|_{x_3=+0}$ is given.
- $\varphi_1(\zeta', x_3)|_{x_3=0}$ and $\psi_1(\zeta', x_3)|_{x_3=0}$ are given.

We first prove that we obtain the solution $F_1(x)$ in Ω_1 when observation data $\varphi_1(\zeta', x_3)|_{x_3=0}$ and $\psi_1(\zeta', x_3)|_{x_3=0}$ are given, that is, the following lemma:

Lemma 3. *Let c_{p_1}, c_{s_1}, ρ_1, and y_3 be given. Assume that observation data $N(\zeta') := \varphi_1(\zeta', x_3)|_{x_3=0}$ are given. Then $\varphi_1(\zeta', x_3)$ is expressed in the form of*

$$\varphi_1(\zeta', x_3) = \begin{bmatrix} \varphi_{111}^N(\zeta', x_3) & \varphi_{112}^N(\zeta', x_3) \\ \varphi_{121}^N(\zeta', x_3) & \varphi_{122}^N(\zeta', x_3) \end{bmatrix}$$

where $\{\varphi^N_{1ml}(\zeta', x_3)\}$ *are defined by*

$$\begin{aligned}(3.1)\quad \begin{bmatrix} \varphi^N_{11l}(\zeta', x_3) \\ \varphi^N_{12l}(\zeta', x_3) \end{bmatrix} &= \alpha^N_{+pl} e^{i\tau^+_{p_1} x_3} \begin{bmatrix} |\zeta'''| \\ \tau^+_{p_1} \end{bmatrix} + \alpha^N_{-pl} e^{-i\tau^+_{p_1} x_3} \begin{bmatrix} |\zeta'''| \\ -\tau^+_{p_1} \end{bmatrix} \\ &\quad + \alpha^N_{+sl} e^{i\tau^+_{s_1} x_3} \begin{bmatrix} \tau^+_{s_1} \\ -|\zeta'''| \end{bmatrix} + \alpha^N_{-sl} e^{-i\tau^+_{s_1} x_3} \begin{bmatrix} \tau^+_{s_1} \\ |\zeta'''| \end{bmatrix}\end{aligned}$$

for $l = 1, 2$ *and*

$$\begin{aligned}(3.2)\quad &\begin{bmatrix} \alpha^N_{+p1} & \alpha^N_{+p2} \\ \alpha^N_{-p1} & \alpha^N_{-p2} \\ \alpha^N_{+s1} & \alpha^N_{+s2} \\ \alpha^N_{-s1} & \alpha^N_{-s2} \end{bmatrix} \\ &= \begin{bmatrix} |\zeta'''| & |\zeta'''| & \tau^+_{s_1} & \tau^+_{s_1} \\ \tau^+_{p_1} & -\tau^+_{p_1} & -|\zeta'''| & |\zeta'''| \\ 2\tau^+_{p_1}|\zeta'''| & -2\tau^+_{p_1}|\zeta'''| & (\tau^+_{s_1})^2 - |\zeta'''|^2 & |\zeta'''|^2 - (\tau^+_{s_1})^2 \\ (\tau^+_{s_1})^2 - |\zeta'''|^2 & (\tau^+_{s_1})^2 - |\zeta'''|^2 & -2\tau^+_{s_1}|\zeta'''| & -2\tau^+_{s_1}|\zeta'''| \end{bmatrix}^{-1} \\ &\quad \cdot \begin{bmatrix} N(\zeta') \\ -\dfrac{1}{2\pi} \dfrac{1}{i\rho_1 c^2_{s_1}} \displaystyle\int_{\mathbb{R}_{\xi_3}} e^{-iy_3\zeta_3} B_{11}(\zeta) P_{11}(\zeta)^{-1} d\xi_3 \end{bmatrix}.\end{aligned}$$

Proof. We remark that $\varphi_{1ml}(\zeta', x_3)$ can be expressed in the form of (2.29). By $\varphi_1(\zeta', x_3)|_{x_3=0} = N(\zeta')$ and the equation (2.15), we have

$$\begin{aligned}(3.3)\quad &\begin{bmatrix} |\zeta'''| & |\zeta'''| & \tau^+_{s_1} & \tau^+_{s_1} \\ \tau^+_{p_1} & -\tau^+_{p_1} & -|\zeta'''| & |\zeta'''| \\ 2\tau^+_{p_1}|\zeta'''| & -2\tau^+_{p_1}|\zeta'''| & (\tau^+_{s_1})^2 - |\zeta'''|^2 & -(\tau^+_{s_1})^2 + |\zeta'''|^2 \\ (\tau^+_{s_1})^2 - |\zeta'''|^2 & (\tau^+_{s_1})^2 - |\zeta'''|^2 & -2\tau^+_{s_1}|\zeta'''| & -2\tau^+_{s_1}|\zeta'''| \end{bmatrix} \\ &\quad \cdot \begin{bmatrix} \alpha_{+p1} & \alpha_{+p2} \\ \alpha_{-p1} & \alpha_{-p2} \\ \alpha_{+s1} & \alpha_{+s2} \\ \alpha_{-s1} & \alpha_{-s2} \end{bmatrix} = \begin{bmatrix} N(\zeta') \\ -\dfrac{1}{2\pi} \dfrac{1}{i\rho_1 c^2_{s_1}} \displaystyle\int_{\mathbb{R}_{\xi_3}} e^{-iy_3\zeta_3} B_{11}(\zeta) P_{11}(\zeta)^{-1} d\xi_3 \end{bmatrix}.\end{aligned}$$

Since

$$\begin{aligned}&\begin{vmatrix} |\zeta'''| & |\zeta'''| & \tau^+_{s_1} & \tau^+_{s_1} \\ \tau^+_{p_1} & -\tau^+_{p_1} & -|\zeta'''| & |\zeta'''| \\ 2\tau^+_{p_1}|\zeta'''| & -2\tau^+_{p_1}|\zeta'''| & (\tau^+_{s_1})^2 - |\zeta'''|^2 & -(\tau^+_{s_1})^2 + |\zeta'''|^2 \\ (\tau^+_{s_1})^2 - |\zeta'''|^2 & (\tau^+_{s_1})^2 - |\zeta'''|^2 & -2\tau^+_{s_1}|\zeta'''| & -2\tau^+_{s_1}|\zeta'''| \end{vmatrix} \\ &\qquad = \frac{4\tau^+_{p_1}\tau^+_{s_1}\zeta_0^2}{s_1^2} \neq 0,\end{aligned}$$

we can solve the equation (3.3), and we obtain (3.2). □

We mention that the case when the observation data $G(x)|_{x_3=+0}$ is identically equal to $\widetilde{G}(x)|_{x_3=+0}$, where $\widetilde{G}(x)$ is the solution of equations (1.6).

Corollary 4. *If the observation data $G(x)|_{x_3=+0} \equiv \widetilde{G}(x)|_{x_3=+0}$, then $\alpha^N_{-p1} \equiv \alpha^N_{-s1} \equiv 0$ holds.*

Proof. We solve the equations (1.6) in a similar way to the equations (1.1)–(1.5). Then we have

$$\begin{bmatrix} \widetilde{\varphi}_{11l}(\zeta', x_3) \\ \widetilde{\varphi}_{12l}(\zeta', x_3) \end{bmatrix} = \widetilde{\alpha}_{+pl} e^{i\tau^+_{p_1} x_3} \begin{bmatrix} |\zeta'''| \\ \tau_{p_1} \end{bmatrix} + \widetilde{\alpha}_{+sl} e^{i\tau^+_{s_1} x_3} \begin{bmatrix} \tau^+_{s_1} \\ -|\zeta'''| \end{bmatrix},$$

where

$$\widetilde{\varphi}_1(\zeta', x_3) = \begin{bmatrix} \widetilde{\varphi}_{111}(\zeta', x_3) & \widetilde{\varphi}_{112}(\zeta', x_3) \\ \widetilde{\varphi}_{121}(\zeta', x_3) & \widetilde{\varphi}_{122}(\zeta', x_3) \end{bmatrix}$$

and

$$\begin{bmatrix} \widetilde{\varphi}_1(\zeta', x_3) & 0 \\ 0 & \widetilde{\psi}_1(\zeta', x_3) \end{bmatrix} = (U(\zeta_1, \zeta_2)C)^{-1}[E_1(x-y) - \widetilde{G}(x)]^\wedge(\zeta')(U(\zeta_1, \zeta_2)C).$$

In particular,

$$\begin{bmatrix} 2\tau^+_{p_1}|\zeta'''| & (\tau^+_{s_1})^2 - |\zeta'''|^2 \\ (\tau^+_{s_1})^2 - |\zeta'''|^2 & -2\tau^+_{s_1}|\zeta'''| \end{bmatrix} \begin{bmatrix} \widetilde{\alpha}_{+p1} \\ \widetilde{\alpha}_{+s1} \end{bmatrix} = \frac{i}{2\zeta_0^2} \begin{bmatrix} 2|\zeta'''|^2 e^{iy_3\tau^+_{p_1}} + ((\tau^+_{s_1})^2 - |\zeta'''|^2) e^{iy_3\tau^+_{s_1}} \\ -\dfrac{|\zeta'''|((\tau^+_{s_1})^2 - |\zeta'''|^2)}{\tau^+_{p_1}} e^{iy_3\tau^+_{p_1}} + 2|\zeta'''|\tau^+_{s_1} e^{iy_3\tau^+_{s_1}} \end{bmatrix} \tag{3.4}$$

and we can solve the equation (3.4) for $\widetilde{\alpha}_{+p1}$ and $\widetilde{\alpha}_{+s1}$. If the observation data $G(x)|_{x_3=0} \equiv \widetilde{G}(x)|_{x_3=0}$, then

$$N(\zeta') = \begin{bmatrix} \widetilde{\alpha}_{+p1}|\zeta'''| + \widetilde{\alpha}_{+s1}\tau^+_{s_1} \\ \widetilde{\alpha}_{+p1}\tau^+_{p_1} - \widetilde{\alpha}_{+s1}|\zeta'''| \end{bmatrix},$$

where $\widetilde{\alpha}_{+p1}$ and $\widetilde{\alpha}_{+s1}$ are defined by the equation (3.4). In this case, we have $\alpha^N_{-p1} \equiv \alpha^N_{-s1} \equiv 0$ by calculating the equation (3.2). □

We remark that we also obtain $G(x)|_{x_3=+0} \not\equiv \widetilde{G}(x)|_{x_3=+0}$ when $\alpha_{-p1} \not\equiv 0$ by the method of proof of Corollary 4.

On the other hand, we remark that the equations (2.18)–(2.22) are the Fourier-Laplace transforms of (2.24)–(2.28) (see Section 2). By Nagayasu [6], we obtain the following lemma and proposition:

Lemma 5 ([6, Lemma 5]). *Let c_{s_1}, ρ_1 and y_3 be given. Assume that the observation data $M(\zeta') := \psi_1(\zeta', x_3)|_{x_3=0}$ are given. Then $\psi_1(\zeta', x_3)$ is explicitly expressed by the known data c_{s_1}, ρ_1, y_3 and $M(\zeta')$.*

Proposition 6 ([6, Main result]). *Let c_{s_1}, ρ_1 and y_3 be given. Assume that the observation data $M(\zeta') := \psi_1(\zeta', x_3)|_{x_3=0}$ are given. Then the constants c_{s_2} and ρ_2 are explicitly expressed with the known data c_{s_1}, ρ_1, y_3 and $M(\zeta')$. Moreover, if $M(\zeta') \not\equiv \widetilde{\psi}_1(\zeta', x_3)|_{x_3=0}$, then the constant h is explicitly expressed with c_{s_1}, ρ_1, y_3 and $M(\zeta')$, where $\widetilde{\psi}_1(\zeta', x_3)$ is the solution of*

$$
(3.5)\quad \begin{cases} P_{12}(\zeta', D_{x_3})\widetilde{\psi}_1(\zeta', x_3) = 0, \quad x_3 > 0, \\ B_{12}(\zeta', D_{x_3})\widetilde{\psi}_1(\zeta', x_3)|_{x_3=0} = -\dfrac{1}{2\pi}\displaystyle\int_{\mathbb{R}_{\xi_3}} e^{i(h-y_3)\zeta_3} B_{12}(\zeta) P_{12}(\zeta)^{-1} d\xi_3. \end{cases}
$$

On the other hand, if $M(\zeta') \equiv \widetilde{\psi}_1(\zeta', x_3)|_{x_3=0}$, then $c_{s_2} = c_{s_1}$ and $\rho_2 = \rho_1$.

We remark that we obtain the solution $F_1(x)$ in Ω_1 when the observation data $G(x)|_{x_3=+0}$ are given by Lemmas 3 and 5. Hereafter, we define $\{\varphi^N_{1ml}\}$ and $\{\alpha^N_{\pm pl}, \alpha^N_{\pm sl}\}$ by (3.1) and (3.2) for the observation data $N(\zeta') = \varphi_1(\zeta', x_3)|_{x_3=0}$.

Now, we determine the unknown constants. We first consider the case when $\psi_1(\zeta', x_3)|_{x_3=0} \not\equiv \widetilde{\psi}_1(\zeta', x_3)|_{x_3=0}$, where $\widetilde{\psi}_1(\zeta', x_3)$ is the solution of the equations (3.5). In this case, we need only to prove the following theorem because we can express c_{s_2}, ρ_2, and h with the known data by Proposition 6:

Theorem 7. *Let c_{p_1}, c_{s_1}, ρ_1 and y_3 be given. Let c_{s_2}, ρ_2, and h be known. Assume that the observation data $N(\zeta') := \varphi_1(\zeta', x_3)|_{x_3=0}$ are given. Then the constant c_{p_2} is expressed as*

$$
(3.6)\qquad c^2_{p_2} = \frac{\zeta_0^2}{(K^N_6(\zeta')/K^N_5(\zeta'))^2 + |\zeta'''|^2} \quad \text{on } V,
$$

where

$$
(3.7)\qquad K^N_1(\zeta') := -\varphi^N_{111}(\zeta', x_3)|_{x_3=h} + \frac{i}{2\zeta_0^2}\left\{ \frac{|\zeta'''|^2}{\tau^+_{p_1}} e^{i(h-y_3)\tau^+_{p_1}} + \tau_{s_1^+} e^{i(h-y_3)\tau^+_{s_1}} \right\},
$$

$$
(3.8)\qquad K^N_2(\zeta') := -\varphi^N_{121}(\zeta', x_3)|_{x_3=0} + \frac{i}{2\zeta_0^2}\left\{ |\zeta'''| e^{i(h-y_3)\tau^+_{p_1}} - |\zeta'''| e^{i(h-y_3)\tau^+_{s_1}} \right\},
$$

$$
\begin{aligned}
(3.9)\qquad & K_3^N(\zeta') \\
&:= -\frac{\rho_1}{\rho_2 c_{s_2}^2}\left\{c_{s_1}^2 (D_{x_3}\varphi_{111}^N)(\zeta', x_3)|_{x_3=h} + c_{s_1}^2 |\zeta'''| \varphi_{121}^N(\zeta', x_3)|_{x_3=h}\right\} \\
&\quad + \frac{i\rho_1 c_{s_1}^2}{2\zeta_0^2 \rho_2 c_{s_2}^2}\left\{2|\zeta'''|^2 e^{i(h-y_3)\tau_{p_1}^+} + ((\tau_{s_1}^+)^2 - |\zeta'''|^2) e^{i(h-y_3)\tau_{s_1}^+}\right\},
\end{aligned}
$$

$$
\begin{aligned}
(3.10)\qquad & K_4^N(\zeta') \\
&:= -\frac{\rho_1}{\rho_2 c_{s_2}^2}\big\{(c_{p_1}^2 - 2c_{s_1}^2)|\zeta'''| \varphi_{111}^N(\zeta', x_3)|_{x_3=h} \\
&\qquad\qquad + c_{p_1}^2 (D_{x_3}\varphi_{111}^N)(\zeta', x_3)|_{x_3=h}\big\} \\
&\quad + \frac{i\rho_1 c_{s_1}^2 |\zeta'''|}{2\zeta_0^2 \rho_2 c_{s_2}^2}\left\{\frac{(\tau_{s_1}^+)^2 - |\zeta'''|^2}{\tau_{p_1}^+} e^{i(h-y_3)\tau_{p_1}^+} - 2\tau_{s_1}^+ e^{i(h-y_3)\tau_{s_1}^+}\right\},
\end{aligned}
$$

$$
(3.11)\qquad K_5^N(\zeta') := 2|\zeta'''| K_1^N(\zeta') + K_4^N(\zeta'),
$$

$$
(3.12)\qquad K_6^N(\zeta') := ((\tau_{s_1}^+)^2 - |\zeta'''|^2) K_2^N(\zeta') + |\zeta'''| K_3^N(\zeta')
$$

and $V := \{\zeta' = (\chi(\xi'), \xi_1, \xi_2) : K_5^N(\zeta') \neq 0,\ \xi' \in \mathbb{R}^3\}$.

Proof. By the $(1,1)$-component of (2.16) and the $(2,1)$-component of (2.17), we have

$$
\begin{bmatrix} |\zeta'''| & 1 \\ (\tau_{s_2}^+)^2 - |\zeta'''|^2 & -2|\zeta'''| \end{bmatrix}
\begin{bmatrix} \beta_{+p1} e^{i\tau_{p_2}^+ h} \\ \beta_{+s1} e^{i\tau_{s_2}^+ h} \tau_{s_2}^+ \end{bmatrix}
= \begin{bmatrix} K_1^N(\zeta') \\ K_4^N(\zeta') \end{bmatrix},
$$

where $K_1^N(\zeta')$ and $K_4^N(\zeta')$ are defined by (3.7) and (3.10). Since

$$
\begin{vmatrix} |\zeta'''| & 1 \\ (\tau_{s_2}^+)^2 - |\zeta'''|^2 & -2|\zeta'''| \end{vmatrix} = -\frac{\zeta_0^2}{c_{s_2}^2} \neq 0,
$$

we can solve this equation, and obtain

$$
(3.13)\qquad \beta_{+p1} e^{i\tau_{p_2}^+ h} = \frac{c_{s_2}^2}{\zeta_0^2} K_5^N(\zeta')
$$

in particular, where $K_5^N(\zeta')$ is defined by (3.11). In a similar way, by the $(2,1)$-component of (2.16) and the $(1,1)$-component of (2.17), we have

$$
\begin{bmatrix} 1 & -|\zeta'''| \\ 2|\zeta'''| & (\tau_{s_2}^+)^2 - |\zeta'''|^2 \end{bmatrix}
\begin{bmatrix} \beta_{+p1} e^{i\tau_{p_2}^+ h} \tau_{p_2}^+ \\ \beta_{+s1} e^{i\tau_{s_2}^+ h} \end{bmatrix}
= \begin{bmatrix} K_2^N(\zeta') \\ K_3^N(\zeta') \end{bmatrix},
$$

where $K_2^N(\zeta')$ and $K_3^N(\zeta')$ are defined by (3.8) and (3.9). Since

$$
\begin{vmatrix} 1 & -|\zeta'''| \\ 2|\zeta'''| & (\tau_{s_2}^+)^2 - |\zeta'''|^2 \end{vmatrix} = \frac{\zeta_0^2}{c_{s_2}^2} \neq 0,
$$

we can solve this equation, and obtain

$$\beta_{+p1}e^{i\tau_{p_2}^+ h}\tau_{p_2}^+ = \frac{c_{s_2}^2}{\zeta_0^2}K_6^N(\zeta') \tag{3.14}$$

in particular, where $K_6^N(\zeta')$ is defined by (3.12). We remark that $K_5^N(\zeta') \not\equiv 0$ by Lemma 2. From the equalities (3.13) and (3.14), we obtain

$$\tau_{p_2}^+ = \frac{K_6^N(\zeta')}{K_5^N(\zeta')} \quad \text{on } V, \tag{3.15}$$

where $V := \{\zeta' = (\chi(\xi'), \xi_1, \xi_2) : K_5^N(\zeta') \neq 0,\ \xi' \in \mathbb{R}^3\}$. Squaring the equality (3.15) and simplifying it, we have the equality (3.6). □

Next, we consider the case when $\psi_1(\zeta', x_3)|_{x_3=0} \equiv \widetilde{\psi}_1(\zeta', x_3)|_{x_3=0}$, where $\widetilde{\psi}_1(\zeta', x_3)$ is the solution of the equations (3.5). In this case, we need only to prove the following theorem because in this situation we obtain $c_{s_2} = c_{s_1}$ and $\rho_2 = \rho_1$ from Proposition 6. We remark that if $G(x)|_{x_3=+0} \equiv \widetilde{G}(x)|_{x_3=+0}$ then $\psi_1(\zeta', x_3)|_{x_3=0} \equiv \widetilde{\psi}_1(\zeta', x_3)|_{x_3=0}$ and $\alpha_{-p1}^N \equiv 0$ by Corollary 4.

Theorem 8. *Let c_{p_1}, c_{s_1}, ρ_1 and y_3 be given. Let $c_{s_2} = c_{s_1}$ and $\rho_2 = \rho_1$. Assume that the observation data $N(\zeta') := \varphi_1(\zeta', x_3)|_{x_3=0}$ are given. If $\alpha_{-p1}^N \not\equiv 0$, then the constants c_{p_2} and h are expressed as*

$$c_{p_2}^2 = \frac{12\chi(\zeta')^3(D_{\xi_0}\chi)(\zeta') - (D_{\xi_0}K_{10}^N)(\zeta')}{10\chi(\zeta')(D_{\xi_0}\chi)(\zeta')|\zeta'''|^2} \quad \text{on } |\zeta'''| \neq 0,\ \alpha_{-p1}^N \neq 0, \tag{3.16}$$

$$h = \frac{ic_{p_1}^2}{2}\left\{\frac{|\zeta'''|}{c_{p_2}^2(\tau_{p_2}^+)^3} - \frac{(D_{\xi_0}K_8^N)(\zeta')}{2\chi(\zeta')(D_{\xi_0}\chi)(\zeta')}\right\}, \tag{3.17}$$

where

$$K_7^N(\zeta') := \frac{\alpha_{-p1}^N}{\alpha_{+p1} - (i|\zeta'''|/2\chi(\zeta')^2\tau_{p_1}^+)e^{-iy_3\tau_{p_1}^+}}, \tag{3.18}$$

$$K_8^N(\zeta') := \frac{\chi(\zeta')\tau_{p_1}^+(D_{\xi_0}K_7^N)(\zeta')}{(D_{\xi_0}\chi)(\zeta')K_7^N(\zeta')}, \tag{3.19}$$

$$K_9^N(\zeta') := \frac{\chi(\zeta')}{|\zeta'''|}\left\{\frac{(D_{\xi_0}K_8^N)(\zeta')}{2(D_{\xi_0}\chi)(\zeta')} - \frac{\tau_{p_1}^+(D_{\xi_0}K_7^N)(\zeta')}{(D_{\xi_0}\chi)(\zeta')K_7^N(\zeta')}\right\}, \tag{3.20}$$

$$K_{10}^N(\zeta') = -\frac{3|\zeta'''|\chi(\zeta')(D_{\xi_0}\chi)(\zeta')K_9^N(\zeta')}{D_{\xi_0}((D_{\xi_0}K_8^N)(\zeta')/2\chi(\zeta')(D_{\xi_0}\chi)(\zeta'))}. \tag{3.21}$$

If $\alpha_{-p1}^N \equiv 0$, then $c_{p_2} = c_{p_1}$ holds and h is not identified.

Proof. By the $(1,1)$-component of (2.16) and the $(2,1)$-component of (2.17), we have

$$\begin{aligned}&\begin{bmatrix} |\zeta'''| & \tau^+_{s_1} \\ (\tau^+_{s_1})^2 - |\zeta'''|^2 & -2\tau^+_{s_1}|\zeta'''| \end{bmatrix} \begin{bmatrix} \alpha^N_{+p1}e^{i\tau^+_{p_1}h} + \alpha^N_{-p1}e^{-i\tau^+_{p_1}h} + \beta_{+p1}e^{i\tau^+_{p_2}h} \\ \alpha^N_{+s1}e^{i\tau^+_{s_1}h} + \alpha^N_{-s1}e^{-i\tau^+_{s_1}h} + \beta_{+s1}e^{i\tau^+_{s_1}h} \end{bmatrix} \\ &= \frac{i}{2\zeta_0^2} \begin{bmatrix} \dfrac{|\zeta'''|^2}{\tau^+_{p_1}}e^{i(h-y_3)\tau^+_{p_1}} + \tau^+_{s_1}e^{i(h-y_3)\tau^+_{s_1}} \\ \dfrac{|\zeta'''|((\tau^+_{s_1})^2 - |\zeta'''|^2)}{\tau^+_{p_1}}e^{i(h-y_3)\tau^+_{p_1}} - 2|\zeta'''|\tau^+_{s_1}e^{i(h-y_3)\tau^+_{s_1}} \end{bmatrix}.\end{aligned}$$

Hence, we obtain

$$\begin{bmatrix} \alpha^N_{+p1}e^{i\tau^+_{p_1}h} + \alpha^N_{-p1}e^{-i\tau^+_{p_1}h} + \beta_{+p1}e^{i\tau^+_{p_2}h} \\ \alpha^N_{+s1}e^{i\tau^+_{s_1}h} + \alpha^N_{-s1}e^{-i\tau^+_{s_1}h} + \beta_{+s1}e^{i\tau^+_{s_1}h} \end{bmatrix} = \frac{i}{2\zeta_0^2} \begin{bmatrix} \dfrac{|\zeta'''|}{\tau^+_{p_1}}e^{i(h-y_3)\tau^+_{p_1}} \\ e^{i(h-y_3)\tau^+_{s_1}} \end{bmatrix} \tag{3.22}$$

because

$$\begin{vmatrix} |\zeta'''| & \tau^+_{s_1} \\ (\tau^+_{s_1})^2 - |\zeta'''|^2 & -2\tau^+_{s_1}|\zeta'''| \end{vmatrix} = -\tau^+_{s_1}\frac{\zeta_0^2}{c^2_{s_1}} \neq 0.$$

In a similar way, by the $(2,1)$-component of (2.16) and the $(1,1)$-component of (2.17), we have

$$\begin{aligned}&\begin{bmatrix} 1 & -|\zeta'''| \\ 2|\zeta'''| & (\tau^+_{s_1})^2 - |\zeta'''|^2 \end{bmatrix} \begin{bmatrix} (\alpha^N_{+p1}e^{i\tau^+_{p_1}h} - \alpha^N_{-p1}e^{-i\tau^+_{p_1}h})\tau^+_{p_1} + \beta_{+p1}e^{i\tau^+_{p_2}h}\tau^+_{p_2} \\ \alpha^N_{+s1}e^{i\tau^+_{s_1}h} - \alpha^N_{-s1}e^{-i\tau^+_{s_1}h} + \beta_{+s1}e^{i\tau^+_{s_1}h} \end{bmatrix} \\ &= \frac{i}{2\zeta_0^2} \begin{bmatrix} |\zeta'''|e^{i(h-y_3)\tau^+_{p_1}} - |\zeta'''|e^{i(h-y_3)\tau^+_{s_1}} \\ 2|\zeta'''|^2e^{i(h-y_3)\tau^+_{p_1}} + ((\tau^+_{s_1})^2 - |\zeta'''|^2)e^{i(h-y_3)\tau^+_{s_1}} \end{bmatrix}.\end{aligned}$$

Hence, we obtain

$$\begin{aligned}&\begin{bmatrix} (\alpha^N_{+p1}e^{i\tau^+_{p_1}h} - \alpha^N_{-p1}e^{-i\tau^+_{p_1}h})\tau^+_{p_1} + \beta_{+p1}e^{i\tau^+_{p_2}h}\tau^+_{p_2} \\ \alpha^N_{+s1}e^{i\tau^+_{s_1}h} - \alpha^N_{-s1}e^{-i\tau^+_{s_1}h} + \beta_{+s1}e^{i\tau^+_{s_1}h} \end{bmatrix} \\ &\qquad = \frac{i}{2\zeta_0^2} \begin{bmatrix} |\zeta'''|e^{i(h-y_3)\tau^+_{p_1}} \\ e^{i(h-y_3)\tau^+_{s_1}} \end{bmatrix}\end{aligned} \tag{3.23}$$

because

$$\begin{vmatrix} 1 & -|\zeta'''| \\ 2|\zeta'''| & (\tau^+_{s_1})^2 - |\zeta'''|^2 \end{vmatrix} = \frac{\zeta_0^2}{c^2_{s_1}} \neq 0.$$

We remark that we have $\alpha^N_{-s1} \equiv 0$ by the second components of the equalities (3.22) and (3.23). On the other hand, we have

$$\alpha^N_{-p1}e^{-i\tau^+_{p_1}h}(\tau^+_{p_2} + \tau^+_{p_1}) = \left(\alpha^N_{+p1} - \frac{i|\zeta'''|}{2\zeta_0^2\tau^+_{p_1}}e^{-iy_3\tau^+_{p_1}}\right)e^{ih\tau^+_{p_1}}(\tau^+_{p_1} - \tau^+_{p_2}) \tag{3.24}$$

by subtracting the first component of the equality (3.23) from the first component of the equality (3.22) multiplied by $\tau_{p_2}^+$. If $\alpha_{-p1}^N \equiv 0$ then $c_{p_2} = c_{p_1}$ by the equality (3.24) and Lemma 1, and it is easy to check that we cannot identify h in this case. Hereafter we assume $\alpha_{-p1}^N \not\equiv 0$. Then we have $c_{p_2} \neq c_{p_1}$ by the equality (3.24). On the other hand, we can rewrite the equality (3.24) to

$$\frac{\tau_{p_1}^+ - \tau_{p_2}^+}{\tau_{p_1}^+ + \tau_{p_2}^+} = K_7^N(\zeta')e^{-2i\tau_{p_1}^+ h}(\neq 0) \quad \text{on } \alpha_{-p1}^N \neq 0, \tag{3.25}$$

where $K_7^N(\zeta')$ is defined by (3.18). We remark that

$$D_{\xi_0}\tau_{p_k}^+ = \frac{\chi(\zeta')(D_{\xi_0}\chi)(\zeta')}{c_{p_k}^2\tau_{p_k}^+}.$$

Hence we have

$$\begin{aligned}\frac{2(c_{p_1}^2 - c_{p_2}^2)|\zeta'''|\chi(\zeta')(D_{\xi_0}\chi)(\zeta')}{c_{p_1}^2c_{p_2}^2\tau_{p_1}^+\tau_{p_2}^+(\tau_{p_1}^+ + \tau_{p_2}^+)^2} \\ = \left(D_{\xi_0}K_7^N(\zeta') - 2ih\frac{\chi(\zeta')(D_{\xi_0}\chi)(\zeta')}{c_{p_1}^2\tau_{p_1}^+}K_7^N(\zeta')\right)e^{-2i\tau_{p_1}^+ h}\end{aligned} \tag{3.26}$$

by applying D_{ξ_0} to the equality (3.25). By the equalities (3.25) and (3.26), we have

$$-\frac{2|\zeta'''|}{\tau_{p_2}^+\chi(\zeta')^2} = \frac{\tau_{p_1}^+(D_{\xi_0}K_7^N)(\zeta')}{\chi(\zeta')(D_{\xi_0}\chi)(\zeta')K_7^N(\zeta')} - 2ih\frac{1}{c_{p_1}^2}. \tag{3.27}$$

Hence we have

$$\frac{|\zeta'''|}{c_{p_2}^2(\tau_{p_2}^+)^3} = \frac{(D_{\xi_0}K_8^N)(\zeta')}{2\chi(\zeta')(D_{\xi_0}\chi)(\zeta')} - 2ih\frac{1}{c_{p_1}^2} \tag{3.28}$$

by multiplying the equality (3.27) by $\chi(\zeta')^2$ and applying D_{ξ_0} to it, where $K_8^N(\zeta')$ is defined by (3.19). Subtracting the equality (3.27) from the equality (3.28), we obtain

$$\frac{3\chi(\zeta')^2 - 2c_{p_2}^2|\zeta'''|^2}{c_{p_2}^2(\tau_{p_2}^+)^3} = K_9^N(\zeta'), \tag{3.29}$$

where $K_9^N(\zeta')$ is defined by (3.20). Applying D_{ξ_0} to the equality (3.28), we have

$$-3\frac{|\zeta'''|\chi(\zeta')(D_{\xi_0}\chi)(\zeta')}{c_{p_2}^4(\tau_{p_2}^+)^5} = D_{\xi_0}\left(\frac{D_{\xi_0}K_8^N(\zeta')}{2\chi(\zeta')D_{\xi_0}\chi(\zeta')}\right). \tag{3.30}$$

Hence we have

$$3\chi(\zeta')^4 - 5c_{p_2}^2\chi(\zeta')^2|\zeta'''|^2 + 2c_{p_2}^4|\zeta'''|^4 = K_{10}^N(\zeta') \tag{3.31}$$

by substituting the equality (3.29) into the equality (3.30), where $K_{10}^N(\zeta')$ is defined by (3.21). Therefore we have (3.16) by applying D_{ξ_0} to the equality (3.31). We obtain (3.17) from the equality (3.28). □

Finally, we make a remark concerning the value of m.

Remark 9. We deal with m as a fixed number. Indeed, this m depends on the unknown constants. However, we can check whether this m is so large that the Lopatinski determinant does not vanish for the determined constants or not after we determine the unknown constants in the above way. If this m is large enough then there is no problem. If this m is not large enough, then we take larger number as new m instead of this m, and determine the unknown constants once again. This procedure is sure to conclude since there exists a large enough number m certainly.

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References

[1] J. D. Achenbach, *Wave propagation in elastic solids*, North-Holland, Amsterdam, 1973.

[2] A. Bartoloni, C. Lodovici and F. Zirilli, *Inverse problem for a class of one-dimensional wave equations with piecewise constant coefficients*, J. Optim. Theory Appl. **76** (1993), 13–32.

[3] L. Fatone, P. Maponi, C. Pignotti and F. Zirilli, *An inverse problem for the two-dimensional wave equation in a stratified medium, Inverse problems of wave propagation and diffraction*, 263–274, Lecture Notes in Phys. 486, Springer, Berlin, 1997.

[4] S. Hansen, *Solution of a hyperbolic inverse problem by linearization*, Comm. Partial Differential Equations **16** (1991), 291–309.

[5] M. Matsumura, *On the Singularities of the Riemann Functions of Mixed Problems for the Wave Equation in Plane-Stratified Media I*, Proc. Japan Acad. **52** (1976), 289–292.

[6] S. Nagayasu, *An inverse problem for the wave equation in plane-stratified media*, Osaka J. Math. **42** (2005), no.3, 613–632.

[7] R. Sakamoto, *Hyperbolic boundary value problems*, Cambridge University Press, Cambridge-New York, 1982.

[8] S. Shimizu, *Eigenfunction expansions for elastic wave propagation problems in stratified media* $\mathbb{R}^3$, Tsukuba J. Math. **18** (1994), 283–350.

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