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Comprehensive family of harmonic univalent functions

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Abstract. In this paper, we introduce a comprehensive family of harmonic univalent functions which contains various well-known classes of harmonic univalent functions as well as many new ones. Coefficient bounds, distortion bounds, extreme points, convolution conditions, and convex combination are determined for functions in this family. Consequently, many of our results are either extensions or new approaches to those corresponding previously known results.

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§1. Introduction and definitions

A continuous complex valued function $f = u+iv$ defined in a simply connected complex domain $\mathcal{D}$ is said to be harmonic in $\mathcal{D}$ if both u and v are real harmonic in $\mathcal{D}$. In any simply connected domain we can write $f = h+ \overline{g}$, where h and g are analytic in $\mathcal{D}$. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient for f to be locally univalent and sense preserving in $\mathcal{D}$ is that $|h'(z)| > |g'(z)|$ in $\mathcal{D}$.

Let $\mathcal{H}$ denote the family of functions $f = h+ \overline{g}$ that are harmonic univalent and sense preserving in the unit disk $\mathcal{U} = \{z : |z| < 1\}$ for which $f(0) = f_z(0)-1 = 0$. Then for $f = h+ \overline{g} \in \mathcal{H}$ we may express the analytic functions h and g as

$$(1.1) \qquad h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad |b_1| < 1.$$

The harmonic function $f = h+ \overline{g}$ for $g \equiv 0$ reduces to an analytic function $f = h$.

In 1984 Clunie and Sheil-Small [1] investigated the class $\mathcal{H}$ as well as its geometric subclasses and obtained some coefficient bounds. Since then, there has been several papers related on $\mathcal{H}$ and its subclasses. Jahangiri [2], Silverman [3], Silverman and Silvia [4] studied the harmonic starlike functions. Recently, Jahangiri [2] defined the class $\mathcal{T}_{\mathcal{H}}(\alpha)$ consisting of functions $f = h + \overline{g}$ such that h and g are of the form

$$h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \qquad g(z) = \sum_{n=1}^{\infty} |b_n| z^n \tag{1.2}$$

which satisfy the condition

$$\frac{\partial}{\partial\theta}(\arg f(re^{i\theta})) \geq \alpha, \qquad 0 \leq \alpha < 1, \quad |z| = r < 1. \tag{1.3}$$

Also Jahangiri [2] proved that if $f = h + \overline{g}$ is given by (1.1) and if

$$\sum_{n=1}^{\infty} \left(\frac{n-\alpha}{1-\alpha} |a_n| + \frac{n+\alpha}{1-\alpha} |b_n| \right) \leq 2, \quad 0 \leq \alpha < 1, \quad a_1 = 1, \tag{1.4}$$

then f is harmonic, univalent, and starlike of order α in $\mathcal{U}$. This condition is proved to be also necessary if $f \in \mathcal{T}_{\mathcal{H}}(\alpha)$. The case when $\alpha = 0$ is given in [4] and for $\alpha = b_1 = 0$, see [3].

Let $\mathcal{S}_{\mathcal{H}}(\Phi, \Psi; \alpha)$ denote the subclass of $\mathcal{H}$ consisting of functions $f = h + \overline{g} \in \mathcal{H}$ that satisfy the condition

$$\operatorname{Re}\left\{ \frac{h(z) * \Phi(z) - \overline{g(z) * \Psi(z)}}{h(z) + \overline{g(z)}} \right\} > \alpha \tag{1.5}$$

where $\alpha\,(0 \leq \alpha < 1)$, $\Phi(z) = z + \sum_{n=2}^{\infty} \lambda_n z^n$ and $\Psi(z) = z + \sum_{n=2}^{\infty} \mu_n z^n$ are analytic in $\mathcal{U}$ with the conditions $\lambda_n \geq 0$, $\mu_n \geq 0$. The operator "$*$" stands for the Hadamard product or convolution of two power series. We further let $\mathcal{TS}_{\mathcal{H}}(\Phi, \Psi; \alpha)$ denote the subclass of $\mathcal{S}_{\mathcal{H}}(\Phi, \Psi; \alpha)$ consisting of functions $f = h + \overline{g} \in \mathcal{H}$ such that h and g are of the form (1.2).

The family $\mathcal{TS}_{\mathcal{H}}(\Phi, \Psi; \alpha)$ is of special interest because it contains various classes of well-known harmonic univalent functions as well as many new ones. For example $\mathcal{TS}_{\mathcal{H}}(\frac{z}{(1-z)^2}, \frac{z}{(1-z)^2}; \alpha) \equiv \mathcal{T}_{\mathcal{H}}(\alpha)$.

In this note, we obtain coefficient bounds, distortion bounds, extreme points, convolution conditions, and convex combination for functions in $\mathcal{TS}_{\mathcal{H}}(\Phi, \Psi; \alpha)$.

§2. Coefficients Bounds

We begin with a sufficient condition for functions in $\mathcal{S}_{\mathcal{H}}(\Phi, \Psi; \alpha)$.

Theorem 2.1. *Let the function* $f = h + \bar{g}$ *be so that* h *and* g *are given by* (1.1). *Furthermore, let*

$$\sum_{n=1}^{\infty} \left(\frac{\lambda_n - \alpha}{1-\alpha} |a_n| + \frac{\mu_n + \alpha}{1-\alpha} |b_n| \right) \leq 2, \tag{2.1}$$

where $a_1 = 1$, $0 \leq \alpha \leq 1$ *and* $n(1-\alpha) \leq \lambda_n - \alpha \leq \mu_n + \alpha$. *Then* f *is harmonic univalent in* $\mathcal{U}$, *and* $f \in \mathcal{S}_{\mathcal{H}}(\Phi, \Psi; \alpha)$.

Proof. First we note that f is locally univalent and sense-preserving in $\mathcal{U}$. This because

$$\begin{aligned} \left|h'(z)\right| &\geq 1 - \sum_{n=2}^{\infty} n |a_n| r^{n-1} > 1 - \sum_{n=2}^{\infty} n |a_n| \geq 1 - \sum_{n=2}^{\infty} \frac{\lambda_n - \alpha}{1-\alpha} |a_n| \\ &\geq \sum_{n=1}^{\infty} \frac{\mu_n + \alpha}{1-\alpha} |b_n| \geq \sum_{n=2}^{\infty} n |b_n| > \sum_{n=2}^{\infty} n |b_n| r^{n-1} \geq \left|g'(z)\right|. \end{aligned}$$

To show that f is univalent in $\mathcal{U}$, suppose $z_1, z_2 \in \mathcal{U}$ so that $z_1 \neq z_2$, then

$$\begin{aligned} \left| \frac{f(z_1) - f(z_2)}{h(z_1) - h(z_2)} \right| &\geq 1 - \left| \frac{g(z_1) - g(z_2)}{h(z_1) - h(z_2)} \right| \\ &= 1 - \left| \frac{\sum\limits_{n=1}^{\infty} a_n (z_1^n - z_2^n)}{(z_1 - z_2) + \sum\limits_{n=2}^{\infty} b_n (z_1^n - z_2^n)} \right| \\ &> 1 - \left| \frac{\sum\limits_{n=1}^{\infty} n |b_n|}{1 - \sum\limits_{n=2}^{\infty} n |a_n|} \right| \\ &\geq 1 - \frac{\sum\limits_{n=1}^{\infty} \frac{\mu_n + \alpha}{1-\alpha} |b_n|}{1 - \sum\limits_{n=2}^{\infty} \frac{\lambda_n - \alpha}{1-\alpha} |a_n|} \geq 0. \end{aligned}$$

Now, we show that $f \in \mathcal{S}_{\mathcal{H}}(\Phi, \Psi; \alpha)$. Using the fact that $\mathrm{Re} w \geq \alpha$ if and only if $|1 - \alpha + w| \geq |1 + \alpha - w|$, it suffices to show that

$$|A(z) + (1-\alpha)B(z)| - |A(z) - (1+\alpha)B(z)| \geq 0, \tag{2.2}$$

where $A(z) = h(z) * \Phi(z) - \overline{g(z) * \Psi(z)}$ and $B(z) = h(z) + \overline{g(z)}.$

Substituting for $A(z)$ and $B(z)$ in (2.2) and making use of (2.1) we obtain

$$\begin{aligned}
&|A(z) + (1-\alpha)B(z)| - |A(z) - (1+\alpha)B(z)| \\
= & \left|(1-\alpha)h(z) + h(z) * \Phi(z) + \overline{(1-\alpha)g(z) - g(z) * \Psi(z)}\right| \\
& - \left|(1+\alpha)h(z) - h(z) * \Phi(z) + \overline{(1+\alpha)g(z) + g(z) * \Psi(z)}\right| \\
= & \left|(2-\alpha)z + \sum_{n=2}^{\infty}(\lambda_n + 1 - \alpha)a_n z^n - \overline{\sum_{n=1}^{\infty}(\mu_n - 1 + \alpha)b_n z^n}\right| \\
& - \left|-\alpha z + \sum_{n=2}^{\infty}(\lambda_n - 1 - \alpha)a_n z^n - \overline{\sum_{n=1}^{\infty}(\mu_n + 1 + \alpha)b_n z^n}\right| \\
\geq & (2-\alpha)|z| - \sum_{n=2}^{\infty}(\lambda_n + 1 - \alpha)|a_n||z|^n - \sum_{n=1}^{\infty}(\mu_n - 1 + \alpha)|b_n||z|^n \\
& -\alpha|z| - \sum_{n=2}^{\infty}(\lambda_n - 1 - \alpha)|a_n||z|^n - \sum_{n=1}^{\infty}(\mu_n + 1 + \alpha)|b_n||z|^n \\
= & 2(1-\alpha)|z|\left\{1 - \sum_{n=2}^{\infty}\frac{\lambda_n - \alpha}{1-\alpha}|a_n||z|^{n-1} - \sum_{n=1}^{\infty}\frac{\mu_n + \alpha}{1-\alpha}|b_n||z|^{n-1}\right\} \\
\geq & 2(1-\alpha)|z|\left\{1 - \sum_{n=2}^{\infty}\frac{\lambda_n - \alpha}{1-\alpha}|a_n| - \sum_{n=1}^{\infty}\frac{\mu_n + \alpha}{1-\alpha}|b_n|\right\} \geq 0.
\end{aligned}$$

The coefficient bound (2.1) is sharp for the function

$$f(z) = z + \sum_{n=2}^{\infty}\frac{1-\alpha}{\lambda_n - \alpha}x_n z^n + \sum_{n=1}^{\infty}\frac{1-\alpha}{\mu_n + \alpha}\overline{y_n z^n}, \tag{2.3}$$

where

$$\sum_{n=2}^{\infty}|x_n| + \sum_{n=1}^{\infty}|y_n| = 1.$$

The functions of the form (2.3) are in $\mathcal{S}_{\mathcal{H}}(\Phi, \Psi; \alpha)$ because

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left(\frac{\lambda_n - \alpha}{1-\alpha}|a_n| + \frac{\mu_n + \alpha}{1-\alpha}|b_n|\right) \\
&= 1 + \sum_{n=2}^{\infty}|x_n| + \sum_{n=1}^{\infty}|y_n| \leq 2.
\end{aligned}$$

□

We next show that the above sufficient condition is also necessary for functions in $\mathcal{TS_H}(\Phi,\Psi;\alpha)$.

Theorem 2.2. *Let the function $f = h + \overline{g}$ be so that h and g are given by* (1.2). *Then $f \in \mathcal{TS_H}(\Phi,\Psi;\alpha)$ if and only if*

$$\sum_{n=1}^{\infty}\left(\frac{\lambda_n-\alpha}{1-\alpha}\,|a_n| + \frac{\mu_n+\alpha}{1-\alpha}\,|b_n|\right) \le 2, \tag{2.4}$$

where $a_1 = 1$, $0 \le \alpha \le 1$ and $n(1-\alpha) \le \lambda_n - \alpha \le \mu_n + \alpha$ for $n \ge 2$.

Proof. The if part, follows from Theorem 2.1. To prove the only if part, let $f \in \mathcal{TS_H}(\Phi,\Psi;\alpha)$ then from (1.5) we have

$$\begin{aligned}
&\operatorname{Re}\left\{\frac{h(z)*\Phi(z) - \overline{g(z)*\Psi(z)}}{h(z)+\overline{g(z)}} - \alpha\right\}\\
= &\operatorname{Re}\left\{\frac{(1-\alpha)z - \sum\limits_{n=2}^{\infty}(\lambda_n-\alpha)\,|a_n|\,z^n - \sum\limits_{n=1}^{\infty}(\mu_n+\alpha)\,|b_n|\,\overline{z}^{\,n}}{z - \sum\limits_{n=2}^{\infty}|a_n|\,z^n + \sum\limits_{n=1}^{\infty}|b_n|\,\overline{z}^{\,n}}\right\} > 0.
\end{aligned}$$

If we choose z to be real and $z \to 1^-$, we get

$$\frac{(1-\alpha) - \sum\limits_{n=2}^{\infty}(\lambda_n-\alpha)\,|a_n| - \sum\limits_{n=1}^{\infty}(\mu_n+\alpha)\,|b_n|}{1 - \sum\limits_{n=2}^{\infty}|a_n| + \sum\limits_{n=1}^{\infty}|b_n|} \ge 0,$$

or, equivalently,

$$\sum_{n=2}^{\infty}(\lambda_n-\alpha)\,|a_n| + \sum_{n=1}^{\infty}(\mu_n+\alpha)\,|b_n| \le 1-\alpha$$

which is the required condition (2.4). □

Taking different choices of $\Phi(z)$ and $\Psi(z)$ in Theorem 2.2, we obtain the following corollaries:

Corollary 2.3 ([2]). *Let the function $f = h + \overline{g}$ be so that h and g are given by* (1.2). *Then $f \in \mathcal{TS_H}(\frac{z}{(1-z)^2}, \frac{z}{(1-z)^2};\alpha)$ if and only if*

$$\sum_{n=1}^{\infty}\left(\frac{n-\alpha}{1-\alpha}\,|a_n| + \frac{n+\alpha}{1-\alpha}\,|b_n|\right) \le 2, \tag{2.5}$$

where $a_1 = 1$, $0 \le \alpha \le 1$.

Corollary 2.4. *Let the function* $f = h + \overline{g}$ *be so that* h *and* g *are given by* (1.2). *Then* $f \in \mathcal{TS_H}(\frac{z+z^2}{(1-z)^3}, \frac{z}{(1-z)^2}; \alpha)$ *if and only if*

$$\sum_{n=1}^{\infty} \left(\frac{n^2-\alpha}{1-\alpha} |a_n| + \frac{n+\alpha}{1-\alpha} |b_n| \right) \le 2, \tag{2.6}$$

where $a_1 = 1$, $0 \le \alpha \le 1$.

Corollary 2.5. *Let the function* $f = h + \overline{g}$ *be so that* h *and* g *are given by* (1.2). *Then* $f \in \mathcal{TS_H}(\frac{z}{(1-z)^2}, z; \alpha)$ *if and only if*

$$\sum_{n=1}^{\infty} \left(\frac{n-\alpha}{1-\alpha} |a_n| + \frac{\alpha}{1-\alpha} |b_n| \right) \le 2, \tag{2.7}$$

where $a_1 = 1$, $0 \le \alpha \le 1$.

Corollary 2.6. *Let the function* $f = h + \overline{g}$ *be so that* h *and* g *are given by* (1.2). *Then* $f \in \mathcal{TS_H}(\frac{z+(1-2\alpha)z^2}{(1-z)^{3-2\alpha}}, \frac{z}{(1-z)^{2-2\alpha}}; \alpha)$ *if and only if*

$$\sum_{n=1}^{\infty} \left(\frac{nC(\alpha,n)-\alpha}{1-\alpha} |a_n| + \frac{C(\alpha,n)+\alpha}{1-\alpha} |b_n| \right) \le 2, \tag{2.8}$$

where $a_1 = 1$, $0 \le \alpha \le 1$ *and* $C(\alpha, n) = \prod_{i=2}^{n} (i-2\alpha)/(n-1)$.

§3. Distortion Theorems

Our next theorem is on the distortion bounds for functions in $\mathcal{TS_H}(\Phi, \Psi; \alpha)$, which yields a covering result for this family.

Theorem 3.1. *Let* $f \in \mathcal{TS_H}(\Phi, \Psi; \alpha)$ *and* $\lambda_2 - \alpha \le \lambda_n - \alpha \le \mu_n + \alpha$ *for* $n \ge 2$. *Then we have,*

$$|f(z)| \le (1+|b_1|)r + \left(\frac{1-\alpha}{\lambda_2-\alpha} - \frac{\mu_1+\alpha}{\lambda_2-\alpha} |b_1| \right) r^2, \qquad |z| = r < 1, \tag{3.1}$$

and

$$|f(z)| \ge (1-|b_1|)r - \left(\frac{1-\alpha}{\lambda_2-\alpha} - \frac{\mu_1+\alpha}{\lambda_2-\alpha} |b_1| \right) r^2, \qquad |z| = r < 1, \tag{3.2}$$

Proof. Let $f \in \mathcal{TS_H}(\Phi, \Psi; \alpha)$, then we have

$$\begin{aligned}
|f(z)| &\leq (1+|b_1|)r + \sum_{n=2}^{\infty}(|a_n|+|b_n|)r^n \\
&\leq (1+|b_1|)r + \sum_{n=2}^{\infty}(|a_n|+|b_n|)r^2 \\
&= (1+|b_1|)r + \frac{1-\alpha}{\lambda_2-\alpha}\sum_{n=2}^{\infty}\left(\frac{\lambda_2-\alpha}{1-\alpha}|a_n| + \frac{\lambda_2-\alpha}{1-\alpha}|b_n|\right)r^2 \\
&\leq (1+|b_1|)r + \frac{1-\alpha}{\lambda_2-\alpha}\sum_{n=2}^{\infty}\left(\frac{\lambda_n-\alpha}{1-\alpha}|a_n| + \frac{\mu_n+\alpha}{1-\alpha}|b_n|\right)r^2 \\
&\leq (1+|b_1|)r + \frac{1-\alpha}{\lambda_2-\alpha}\left(1 - \frac{\mu_1+\alpha}{1-\alpha}|b_1|\right)r^2, \quad \text{by (2.4)}, \\
&= (1+|b_1|)r + \left(\frac{1-\alpha}{\lambda_2-\alpha} - \frac{\mu_1+\alpha}{\lambda_2-\alpha}|b_1|\right)r^2
\end{aligned}$$

and

$$\begin{aligned}
|f(z)| &\geq (1-|b_1|)r + \sum_{n=2}^{\infty}(|a_n|+|b_n|)r^n \\
&\geq (1-|b_1|)r - \sum_{n=2}^{\infty}(|a_n|+|b_n|)r^2 \\
&= (1-|b_1|)r - \frac{1-\alpha}{\lambda_2-\alpha}\sum_{n=2}^{\infty}\left(\frac{\lambda_2-\alpha}{1-\alpha}|a_n| + \frac{\lambda_2-\alpha}{1-\alpha}|b_n|\right)r^2 \\
&\geq (1-|b_1|)r - \frac{1-\alpha}{\lambda_2-\alpha}\sum_{n=2}^{\infty}\left(\frac{\lambda_n-\alpha}{1-\alpha}|a_n| + \frac{\mu_n+\alpha}{1-\alpha}|b_n|\right)r^2 \\
&\geq (1-|b_1|)r - \frac{1-\alpha}{\lambda_2-\alpha}\left(1 - \frac{\mu_1+\alpha}{1-\alpha}|b_1|\right)r^2, \quad \text{by (2.4)}, \\
&= (1-|b_1|)r - \left(\frac{1-\alpha}{\lambda_2-\alpha} - \frac{\mu_1+\alpha}{\lambda_2-\alpha}|b_1|\right)r^2.
\end{aligned}$$

The bounds (3.1) and (3.2) are sharp for the functions given by

$$f(z) = z + |b_1|\,\overline{z} + \left(\frac{1-\alpha}{\lambda_2-\alpha} - \frac{\mu_1+\alpha}{\lambda_2-\alpha}|b_1|\right)\overline{z}^2 \tag{3.3}$$

and

$$f(z) = (1-|b_1|)z - \left(\frac{1-\alpha}{\lambda_2-\alpha} - \frac{\mu_1+\alpha}{\lambda_2-\alpha}|b_1|\right)z^2 \tag{3.4}$$

for $|b_1| \leq (1-\alpha)/(\mu_1+\alpha)$. □

The following covering result follows from the left hand inequality in Theorem 3.1.

Corollary 3.2. *Let* $f \in \mathcal{TS_H}(\Phi, \Psi; \alpha)$ *and* $\lambda_2 - \alpha \leq \lambda_n - \alpha \leq \mu_n + \alpha$ *for* $n \geq 2$. *Then we have,*

$$\left\{ w : |w| < \frac{1}{\lambda_2 - \alpha} \left(\lambda_2 - 1 + (\mu_1 - \lambda_2 + 2\alpha) |b_1|\right) \right\} \subset f(\mathcal{U}).$$

§4. Extreme Points

In this section we determine the extreme points of $\mathcal{TS_H}(\Phi, \Psi; \alpha)$.

Theorem 4.1. *Let* $h_1(z) = z$, $h_n(z) = z - \frac{1-\alpha}{\lambda_n - \alpha} z^n$ $(n \geq 2)$ *and* $g_n(z) = z + \frac{1-\alpha}{\mu_n + \alpha} \overline{z}^n$ $(n \geq 1)$. *Then* $f \in \mathcal{TS_H}(\Phi, \Psi; \alpha)$ *if and only if it can be expressed as*

$$f(z) = \sum_{n=1}^{\infty} (x_n h_n + y_n g_n), \tag{4.1}$$

where $x_n \geq 0$, $y_n \geq 0$, $\sum\limits_{n=1}^{\infty} (x_n + y_n) = 1$. *In particular, the extreme points of* $\mathcal{TS_H}(\Phi, \Psi; \alpha)$ *are* $\{h_n\}$ *and* $\{g_n\}$.

Proof. Suppose

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} (x_n h_n + y_n g_n) \\ &= \sum_{n=1}^{\infty} (x_n + y_n) z - \sum_{n=2}^{\infty} \frac{1-\alpha}{\lambda_n - \alpha} x_n z^n + \sum_{n=1}^{\infty} \frac{1-\alpha}{\mu_n + \alpha} y_n \overline{z}^n. \end{aligned}$$

Then

$$\begin{aligned} &\sum_{n=2}^{\infty} \frac{\lambda_n - \alpha}{1-\alpha} \left(\frac{1-\alpha}{\lambda_n - \alpha} x_n \right) + \sum_{n=2}^{\infty} \frac{\mu_n + \alpha}{1-\alpha} \left(\frac{1-\alpha}{\mu_n + \alpha} y_n \right) \\ = &\sum_{n=2}^{\infty} x_n + \sum_{n=1}^{\infty} y_n = 1 - x_1 \leq 1 \end{aligned}$$

and so $f \in \mathcal{TS_H}(\Phi, \Psi; \alpha)$.

Conversely, if $f \in \mathcal{TS_H}(\Phi, \Psi; \alpha)$, then $|a_n| \leq \frac{1-\alpha}{\lambda_n - \alpha}$ and $|b_n| \leq \frac{1-\alpha}{\mu_n + \alpha}$. Setting $x_n = \frac{\lambda_n - \alpha}{1-\alpha}$ $(n \geq 2)$ and $y_n = \frac{\mu_n + \alpha}{1-\alpha}$ $(n \geq 1)$. Then note that by Theorem 2.2, $0 \leq x_n \leq 1$ $(n \geq 2)$ and $0 \leq y_n \leq 1$ $(n \geq 1)$. We define $x_1 = 1 - \sum\limits_{n=2}^{\infty} x_n -$

$\sum_{n=1}^{\infty} y_n \geq 0$, by Theorem 2.2. Consequently, we can see that $f(z)$ can be expressed in the form (4.1). This completes the proof of the Theorem 4.1. □

§5. Convolution and Convex Combinations

In this section, we show that the class $\mathcal{TS}_{\mathcal{H}}(\Phi, \Psi; \alpha)$ is invariant under convolution and convex combinations of its members. For harmonic functions $f(z) = z - \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n \overline{z}^n$ and $F(z) = z - \sum_{n=2}^{\infty} A_n z^n + \sum_{n=1}^{\infty} B_n \overline{z}^n$ we define the convolution of two harmonic functions f and F as

$$(f * F)(z) = f(z) * F(z) = z - \sum_{n=2}^{\infty} a_n A_n z^n + \sum_{n=1}^{\infty} b_n B_n \overline{z}^n.$$

Theorem 5.1. *If $f \in \mathcal{TS}_{\mathcal{H}}(\Phi, \Psi; \alpha)$ and $F \in \mathcal{TS}_{\mathcal{H}}(\Phi, \Psi; \alpha)$ then $f * F \in \mathcal{TS}_{\mathcal{H}}(\Phi, \Psi; \alpha)$.*

Proof. Let $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n + \sum_{n=1}^{\infty} |b_n| \overline{z}^n$ and $F(z) = z - \sum_{n=2}^{\infty} |A_n| z^n + \sum_{n=1}^{\infty} |B_n| \overline{z}^n$ be in $\mathcal{TS}_{\mathcal{H}}(\Phi, \Psi; \alpha)$, Then by Theorem 2.2, we have

$$\sum_{n=1}^{\infty} \left(\frac{\lambda_n - \alpha}{1-\alpha} |a_n| + \frac{\mu_n + \alpha}{1-\alpha} |b_n| \right) \leq 2, \tag{5.1}$$

and

$$\sum_{n=1}^{\infty} \left(\frac{\lambda_n - \alpha}{1-\alpha} |A_n| + \frac{\mu_n + \alpha}{1-\alpha} |B_n| \right) \leq 2. \tag{5.2}$$

So for the coefficients of $f * F$ we can write

$$\begin{aligned} & \sum_{n=1}^{\infty} \left(\frac{\lambda_n - \alpha}{1-\alpha} |a_n A_n| + \frac{\mu_n + \alpha}{1-\alpha} |b_n B_n| \right) \\ \leq & \sum_{n=1}^{\infty} \left(\frac{\lambda_n - \alpha}{1-\alpha} |a_n| + \frac{\mu_n + \alpha}{1-\alpha} |b_n| \right) \leq 2. \end{aligned}$$

Thus $f * F \in \mathcal{TS}_{\mathcal{H}}(\Phi, \Psi; \alpha)$. □

Finally, we prove

Corollary 5.2. *The class $\mathcal{TS_H}(\Phi,\Psi;\alpha)$ is closed under convex combination.*

Proof. For $i=1,2,3,\ldots$ suppose that $f_i(z)\in\mathcal{TS_H}(\Phi,\Psi;\alpha)$ where f_i is given by

$$f_i(z)=z-\sum_{n=2}^{\infty}|a_{i_n}|\,z^n+\sum_{n=1}^{\infty}|b_{i_n}|\,\overline{z}^n.$$

Then by (2.4),

$$\sum_{n=1}^{\infty}\left(\frac{\lambda_n-\alpha}{1-\alpha}\,|a_{i_n}|+\frac{\mu_n+\alpha}{1-\alpha}\,|b_{i_n}|\right)\le 2. \tag{5.3}$$

For $\sum\limits_{i=1}^{\infty}t_i=1$, $0\le t_i\le 1$, the convex combination of f_i may be written as

$$\sum_{i=1}^{\infty}t_if_i(z)=z-\sum_{n=2}^{\infty}\left(\sum_{i=1}^{\infty}t_i\,|a_{i_n}|\right)z^n+\sum_{n=1}^{\infty}\left(\sum_{i=1}^{\infty}t_i\,|b_{i_n}|\right)\overline{z}^n.$$

Then by (5.3),

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left[\frac{\lambda_n-\alpha}{1-\alpha}\left|\sum_{i=1}^{\infty}t_i\,|a_{i_n}|\right|+\frac{\mu_n+\alpha}{1-\alpha}\left|\sum_{i=1}^{\infty}t_i\,|b_{i_n}|\right|\right]\\
=\;&\sum_{n=1}^{\infty}t_i\left\{\sum_{n=1}^{\infty}\left[\frac{\lambda_n-\alpha}{1-\alpha}\,|a_{i_n}|+\frac{\mu_n+\alpha}{1-\alpha}\,|b_{i_n}|\right]\right\}\\
\le\;&2\sum_{n=1}^{\infty}t_i=2
\end{aligned}$$

and so by Theorem 2.2, we have $\sum\limits_{i=1}^{\infty}t_if_i(z)\in\mathcal{TS_H}(\Phi,\Psi;\alpha)$. □

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