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On sampling theory and eigenvalue problems with an eigenparameter in the boundary conditions

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Abstract. This paper is devoted to the investigation of sampling theory associated with second order eigenvalue problems with an eigenparameter appearing in the boundary conditions. We study two cases. The first is when the eigenparameter appears linearly in all boundary conditions and the second is when it appears only in one condition. We closely follow the analysis derived by C. T. Fulton (1977) to establish the needed relations for the derivations of the sampling theorems including the construction of Green's function as well as the eigenfunction expansion theorem. We derive sampling representations for transforms whose kernels are either solutions or Green's functions.

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§1. Introduction

Throughout this paper we consider the differential equation

$$\ell(y) := -y''(x) + q(x)y(x) = \lambda y(x), \quad x \in [0, 1], \tag{1.1}$$

where $q(\cdot)$ is assumed to be real valued and continuous on $[0, 1]$ and $\lambda \in \mathbb{C}$ is an eigenvalue parameter. We also consider the following two boundary conditions

$$a_1y(0) + a_2y'(0) = \lambda(a'_1y(0) + a'_2y'(0)), \tag{1.2}$$

$$b_1y(1) + b_2y'(1) = \lambda(b'_1y(1) + b'_2y'(1)), \tag{1.3}$$

where $a_i, a'_i, b_i, b'_i \in \mathbb{R}$, $i = 1, 2$. Further conditions will be imposed on the last constants to guarantee that the problem could be defined in a Hilbert space. In these boundary conditions the eigenparameter λ appears linearly in both boundary conditions. This is the only difference between problem (1.1)–(1.3) and Sturm-Liouville eigenvalue problem studied extensively in the literature, see e.g. [7, 14, 16]. There are several articles dealing with the sampling theory of signal analysis associated with Sturm-Liouville eigenvalue problems. See e.g. [9, 19, 20] where integral transforms associated with Sturm-Liouville problems are constructed from their values at the eigenvalues. In other words if we consider the Sturm-Liouville problem which consists of (1.1) together with the boundary conditions (1.2) and (1.3) when $a_i' = b_i' = 0$, $i = 1, 2$, then if $\phi(\cdot, \lambda)$ is a solution of (1.1) and $\phi(0, \lambda) = a_2$, $\phi'(0, \lambda) = -a_1$, the transform

$$f(\lambda) = \int_0^1 g(x)\phi(x, \lambda)\, dx, \quad g(\cdot) \in L^2(0, 1), \tag{1.4}$$

can be reconstructed in the sampling formula

$$f(\lambda) = \sum_{n=0}^{\infty} f(\lambda_n) \frac{\Delta(\lambda)}{(\lambda - \lambda_n)\Delta'(\lambda_n)}, \tag{1.5}$$

where $\Delta(\lambda) := b_1\phi(1, \lambda) + b_2\phi'(1, \lambda)$ is an entire function of λ, $\{\lambda_n\}_{n=0}^{\infty}$ is the sequence of eigenvalues of the Sturm-Liouville problem, which are exactly the zeros of $\Delta(\lambda)$. Series (1.5) converges absolutely on $\mathbb{C}$ and uniformly on compact sets of $\mathbb{C}$. For references concerning the sampling theory associated with second order eigenvalue problems, see also [2, 3, 8, 9].

Our purpose of this article is two-fold. The first is to derive sampling theorems associated with problem (1.1)–(1.3). For this aim, we will study briefly the spectral properties of problem (1.1)–(1.3) that we need for the derivation of the sampling theorem. We closely follow the method developed by Fulton [10], see also [17, 18] and therefore most of the proofs are omitted. This is done in the next section. In section three we derive two sampling theorems associated with problem (1.1)–(1.3). The first is of the type mentioned above and the second is by the use of Green's function. Then we indicate without proofs, the way we derive the sampling theories associated with the problem (1.1)–(1.3) when $a_i' = 0$, $i = 1, 2$, i.e. where the eigenvalue parameter appears in one boundary condition only. In this setting we use the results obtained by Fulton in [10]. It is worthy to mention here that the two cases studied here are independent. In fact while the operator associated with problem (1.1)–(1.3) is constructed in $L^2(0, 1) \oplus \mathbb{C}^2$, that of (1.1)–(1.3) when $a_i' = 0$, $i = 1, 2$, is defined in $L^2(0, 1) \oplus \mathbb{C}$. We will illustrate our results via the examples of the last section. For sampling theorems associated with eigenvalue problems with eigenvalue parameter in the boundary conditions see [4] and for a discrete analog of the theorems derived here, see [1].

§2. The eigenvalue problem

To formulate a theoretic approach to problem (1.1)–(1.3) we define the Hilbert space $\mathfrak{H} := L^2(0,1) \oplus \mathbb{C}^2$ with an inner product

$$\langle \mathfrak{f}(\cdot), \mathfrak{g}(\cdot) \rangle_{\mathfrak{H}} := \int_0^1 f(x)\overline{g(x)}\,dx + \frac{1}{\eta}\alpha\overline{\delta} + \frac{1}{\rho}\beta\overline{\gamma}, \tag{2.1}$$

where

$$\mathfrak{f}(x) = \begin{pmatrix} f(x) \\ \alpha \\ \beta \end{pmatrix}, \quad \mathfrak{g}(x) = \begin{pmatrix} g(x) \\ \delta \\ \gamma \end{pmatrix} \in \mathfrak{H},$$

$f(\cdot)$, $g(\cdot) \in L^2(0,1)$, $\alpha, \beta, \delta, \gamma \in \mathbb{C}$ and the constants η, ρ are defined by

$$\eta := \det \begin{pmatrix} a_1 & a_1' \\ a_2 & a_2' \end{pmatrix}, \qquad \rho := \det \begin{pmatrix} b_1' & b_1 \\ b_2' & b_2 \end{pmatrix}. \tag{2.2}$$

For the definiteness of the inner product of $\mathfrak{H}$, we assume that $\eta, \rho > 0$. For convenience we put

$$\begin{pmatrix} U_0(y) & U_0'(y) \\ U_1(y) & U_1'(y) \end{pmatrix} := \begin{pmatrix} a_1 y(0) + a_2 y'(0) & a_1' y(0) + a_2' y'(0) \\ b_1 y(1) + b_2 y'(1) & b_1' y(1) + b_2' y'(1) \end{pmatrix}. \tag{2.3}$$

In the following we will define the minimal closed operator in $\mathfrak{H}$ associated with the differential expression ℓ.

Let $\mathcal{D}(\mathcal{A}) \subseteq \mathfrak{H}$ be the set of all $\mathfrak{f}(x) = \begin{pmatrix} f(x) \\ U_0'(f) \\ U_1'(f) \end{pmatrix} \in \mathfrak{H}$ such that f, f' are absolutely continuous on [0,1] and $\ell(f) \in L^2(0,1)$. Define the operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \longrightarrow \mathfrak{H}$ by

$$\mathcal{A} \begin{pmatrix} f(x) \\ U_0'(f) \\ U_1'(f) \end{pmatrix} = \begin{pmatrix} \ell(f) \\ U_0(f) \\ U_1(f) \end{pmatrix}, \quad \begin{pmatrix} f(x) \\ U_0'(f) \\ U_1'(f) \end{pmatrix} \in \mathcal{D}(\mathcal{A}). \tag{2.4}$$

For $u, v \in L^2(0,1)$, where u', v' are absolutely continuous on [0,1], $\ell(u), \ell(v) \in L^2(0,1)$, we have the following Lagrange's identity

$$\int_0^1 \ell(u(x))\overline{v}(x)\,dx = \int_0^1 u(x)\ell(\overline{v}(x))\,dx + [u(x), \overline{v}(x)]_0^1. \tag{2.5}$$

Thus, we can prove in a manner similar to that of [10] that $\mathcal{A}$ is symmetric in $\mathfrak{H}$. Here

$$[u, v](x) := u(x)v'(x) - u'(x)v(x).$$

The operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \longrightarrow \mathfrak{H}$ is equivalent to the eigenvalue problem (1.1)–(1.3) in the sense that the eigenvalues of $\mathcal{A}$ are exactly those of problem (1.1)–(1.3). Let $\phi_\lambda(\cdot)$ and $\chi_\lambda(\cdot)$ be two solutions of (1.1) satisfying the following initial conditions

$$\phi_\lambda(0) = a_2 - a_2'\lambda, \quad \phi_\lambda'(0) = a_1'\lambda - a_1 \tag{2.6}$$

and

$$\chi_\lambda(1) = b_2 - b_2'\lambda, \quad \chi_\lambda'(1) = b_1'\lambda - b_1, \quad \lambda \in \mathbb{C}. \tag{2.7}$$

These functions are entire in λ for all $x \in [0, 1]$. Obviously

$$U_0'(\phi_\lambda) = -\eta, \quad U_1'(\chi_\lambda) = \rho, \quad \lambda \in \mathbb{C}. \tag{2.8}$$

Let $W_x(\phi_\lambda, \chi_\lambda)$ be the Wronskian of ϕ_λ and χ_λ which is independent of x, since the coefficient of y' in (1.1) is zero. Let

$$\begin{aligned} \omega(\lambda) :&= W_x(\phi_\lambda, \chi_\lambda) = \phi_\lambda(x)\chi_\lambda'(x) - \phi_\lambda'(x)\chi_\lambda(x) \\ &= W_1(\phi_\lambda, \chi_\lambda) = \lambda U_1'(\phi_\lambda) - U_1(\phi_\lambda). \end{aligned} \tag{2.9}$$

Then $\omega(\lambda)$ is an entire function of λ whose zeros are precisely the eigenvalues of the operator $\mathcal{A}$. Using techniques similar of those established by Titchmarsh in [16], see also [10], the zeros of $\omega(\lambda)$ are real and simple and if λ_n, $n = 0, 1, 2, \dots$ denote the zeros of $\omega(\lambda)$, then the three-component vectors

$$\Phi_n(x) := \begin{pmatrix} \phi_{\lambda_n}(x) \\ U_0'(\phi_{\lambda_n}) \\ U_1'(\phi_{\lambda_n}) \end{pmatrix} \tag{2.10}$$

are the corresponding eigenvectors of the operator $\mathcal{A}$ satisfying the orthogonality relation

$$\langle \Phi_n(\cdot), \Phi_m(\cdot) \rangle_{\mathfrak{H}} = 0 \quad \text{for}\, n \neq m. \tag{2.11}$$

Here $\{\phi_{\lambda_n}(\cdot)\}_{n=0}^{\infty}$ will be the sequence of eigenfunctions of (1.1)–(1.3) corresponding to the eigenvalues $\{\lambda_n\}_{n=0}^{\infty}$. We denote by $\Psi_n(\cdot)$ to the normalized eigenvectors

$$\Psi_n(x) := \frac{\Phi_n(x)}{\|\Phi_n(\cdot)\|_{\mathfrak{H}}} = \begin{pmatrix} \psi_n(x) \\ U_0'(\psi_n) \\ U_1'(\psi_n) \end{pmatrix}. \tag{2.12}$$

Let $k_n \neq 0$ be the real constants for which

$$\chi_{\lambda_n}(x) = k_n \phi_{\lambda_n}(x), \quad x \in [0,1],\ n = 0,1,\dots. \tag{2.13}$$

To study the completeness of the eigenvectors of $\mathcal{A}$, and hence the completeness of the eigenfunctions of (1.1)–(1.3), we construct the resolvent of $\mathcal{A}$ as well as Green's function of problem (1.1)–(1.3). We assume without any loss of generality that $\lambda = 0$ is not an eigenvalue of $\mathcal{A}$. Now let $\lambda \in \mathbb{C}$ be not an eigenvalue of $\mathcal{A}$ and consider the inhomogeneous problem

(2.14)

$$(\lambda I - \mathcal{A})\Phi(x) = \mathfrak{f}(x), \ \text{for } \mathfrak{f}(x) = \begin{pmatrix} f(x) \\ \alpha \\ \beta \end{pmatrix} \in \mathfrak{H} \text{ and } \Phi(x) = \begin{pmatrix} \phi(x) \\ U_0'(\phi) \\ U_1'(\phi) \end{pmatrix} \in \mathcal{D}(\mathcal{A}),$$

where I is the identity operator. Using the method of variation of constants, we can see after some easy calculations that

(2.15)

$$\Phi = (\lambda I - \mathcal{A})^{-1}\mathfrak{f} = \begin{pmatrix} \dfrac{\beta}{\omega(\lambda)}\phi_\lambda(x) - \dfrac{\alpha}{\omega(\lambda)}\chi_\lambda(x) + \displaystyle\int_0^1 G(x,\xi,\lambda)f(\xi)\,d\xi \\ U_0'(\phi) \\ U_1'(\phi) \end{pmatrix},$$

where

$$G(x,\xi,\lambda) = \begin{cases} \dfrac{\chi_\lambda(x)\phi_\lambda(\xi)}{\omega(\lambda)}, & 0 \le \xi \le x \le 1, \\[2ex] \dfrac{\chi_\lambda(\xi)\phi_\lambda(x)}{\omega(\lambda)}, & 0 \le x \le \xi \le 1, \end{cases} \tag{2.16}$$

is Green's function of problem (1.1)–(1.3).

Lemma 2.1. *The operator $\mathcal{A}$ is self-adjoint in $\mathfrak{H}$.*

Proof. Since $\mathcal{A}$ is a symmetric densely defined operator, then it is sufficient to show that the deficiency spaces are the null spaces and hence $\mathcal{A} = \mathcal{A}^*$, cf. [15]. Indeed, if $\mathfrak{f}(x) = \begin{pmatrix} f(x) \\ \alpha \\ \beta \end{pmatrix} \in \mathfrak{H}$ and λ is a non-real number, then letting

$$\Phi(x) = \begin{pmatrix} \phi(x) \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \dfrac{\beta}{\omega(\lambda)}\phi_\lambda(x) - \dfrac{\alpha}{\omega(\lambda)}\chi_\lambda(x) + \displaystyle\int_0^1 G(x,\xi,\lambda)f(\xi)\,d\xi \\ U_0'(\phi) \\ U_1'(\phi) \end{pmatrix},$$

implies that $\Phi \in \mathcal{D}(\mathcal{A})$. Since $G(x,\xi,\lambda)$ satisfies the conditions (1.2)–(1.3), then $(\lambda I - \mathcal{A})\Phi(x) = \mathfrak{f}(x)$. Now we prove that the inverse of $(\lambda I - \mathcal{A})$ exists. If $\mathcal{A}\Phi(x) = \lambda\Phi(x)$, then

$$\begin{aligned}(\overline{\lambda} - \lambda)\langle \Phi(\cdot), \Phi(\cdot)\rangle_{\mathfrak{H}} &= \langle \Phi(\cdot), \lambda\Phi(\cdot)\rangle_{\mathfrak{H}} - \langle \lambda\Phi(\cdot), \Phi(\cdot)\rangle_{\mathfrak{H}} \\ &= \langle \Phi(\cdot), \mathcal{A}\Phi(\cdot)\rangle_{\mathfrak{H}} - \langle \mathcal{A}\Phi(\cdot), \Phi(\cdot)\rangle_{\mathfrak{H}} \\ &= 0 \qquad\qquad \text{(since } \mathcal{A} \text{ is symmetric).}\end{aligned}$$

Since $\lambda \notin \mathbb{R}$, we have $\overline{\lambda} - \lambda \neq 0$. Thus $\langle \Phi(\cdot), \Phi(\cdot)\rangle_{\mathfrak{H}} = 0$, i.e. $\Phi = 0$. Then $R(\lambda; \mathcal{A}) := (\lambda I - \mathcal{A})^{-1}$, the resolvent operator of $\mathcal{A}$, exists. Thus

$$R(\lambda; \mathcal{A})\mathfrak{f} = (\lambda I - \mathcal{A})^{-1}\mathfrak{f} = \Phi.$$

Take $\lambda = \pm i$. The domains of $(iI - \mathcal{A})^{-1}$ and $(-iI - \mathcal{A})^{-1}$ are exactly $\mathfrak{H}$. Consequently the ranges of $(iI - \mathcal{A})$ and $(-iI - \mathcal{A})$ are also $\mathfrak{H}$. Hence the deficiency spaces of $\mathcal{A}$ are

$$N_{-i} := N(-iI - \mathcal{A}^*) = R(iI - \mathcal{A})^{\perp} = \mathfrak{H}^{\perp} = \{0\}$$

$$N_{i} := N(iI - \mathcal{A}^*) = R(-iI - \mathcal{A})^{\perp} = \mathfrak{H}^{\perp} = \{0\}.$$

Therefore $\mathcal{A}$ is self-adjoint. □

Theorem 2.2.
(i) *For* $\mathfrak{f}(\cdot) \in \mathfrak{H}$

$$\|\mathfrak{f}(\cdot)\|^2_{\mathfrak{H}} = \sum_{n=0}^{\infty} |\langle \mathfrak{f}(\cdot), \Psi_n(\cdot)\rangle_{\mathfrak{H}}|^2. \tag{2.17}$$

(ii) *For* $\mathfrak{f}(\cdot) \in \mathcal{D}(\mathcal{A})$

$$\mathfrak{f}(x) = \sum_{n=0}^{\infty} \langle \mathfrak{f}(\cdot), \Psi_n(\cdot)\rangle_{\mathfrak{H}} \Psi_n(x), \tag{2.18}$$

the series being absolutely and uniformly convergent in the first component for on $[0,1]$*, and absolutely convergent in the second and third components.*

Proof. The proof is similar to [10, p. 298-299]. □

The following corollary corresponds to [17, p. 305, Theorem 2]

Corollary 2.3. *The normalized eigenfunctions* $\psi_n(\cdot)$ *of* (2.12) *satisfy the following properties:*

(i) $\displaystyle\frac{1}{\eta}\sum_{n=0}^{\infty} U_0'(\psi_n)\psi_n(x) = 0$, *with mean-square convergence in* $[0,1]$,

(ii) $\frac{1}{\eta}\sum_{n=0}^{\infty}(U_0'(\psi_n))^2 = 1,\ \frac{1}{\eta}\sum_{n=0}^{\infty}U_0'(\psi_n)U_1'(\psi_n) = 0,$

(iii) $\frac{1}{\rho}\sum_{n=0}^{\infty}U_1'(\psi_n)\psi_n(x) = 0$ *with mean-square convergence in* $[0,1]$,

(iv) $\frac{1}{\rho}\sum_{n=0}^{\infty}(U_1'(\psi_n))^2 = 1,\ \frac{1}{\rho}\sum_{n=0}^{\infty}U_0'(\psi_n)U_1'(\psi_n) = 0,$

(v) $f(x)=\sum_{n=0}^{\infty}\left(\int_0^1 f(x)\psi_n(x)\,dx\right)\psi_n(x)$, *with mean-square convergence in* $[0,1]$ *for any* $f(\cdot)\in L^2(0,1)$,

(vi) $\sum_{n=0}^{\infty}\left(\int_0^1 f(x)\psi_n(x)\,dx\right)U_0'(\psi_n) = 0,\ \sum_{n=0}^{\infty}\left(\int_0^1 f(x)\psi_n(x)\,dx\right)U_1'(\psi_n) = 0$ *for any* $f(\cdot)\in L^2(0,1)$.

Proof. From the completeness of the eigenvectors of $\mathcal{A}$, we have for an arbitrary element $\mathfrak{f}(\cdot)\in\mathfrak{H}$

$$(2.19)\quad \mathfrak{f}(x) = \begin{pmatrix} f(x) \\ \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \sum_{n=0}^{\infty}\left(\int_0^1 f(x)\psi_n(x)\,dx + \frac{1}{\eta}\alpha U_0'(\psi_n) + \frac{1}{\rho}\beta U_1'(\psi_n)\right)\psi_n(x) \\ \sum_{n=0}^{\infty}\left(\int_0^1 f(x)\psi_n(x)\,dx + \frac{1}{\eta}\alpha U_0'(\psi_n) + \frac{1}{\rho}\beta U_1'(\psi_n)\right)U_0'(\psi_n) \\ \sum_{n=0}^{\infty}\left(\int_0^1 f(x)\psi_n(x)\,dx + \frac{1}{\eta}\alpha U_0'(\psi_n) + \frac{1}{\rho}\beta U_1'(\psi_n)\right)U_1'(\psi_n) \end{pmatrix}$$

with convergence in the $\mathfrak{H}$-norm. Properties (i) and (ii) follow from (2.19) by taking $\mathfrak{f}(x) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, properties (iii) and (iv) follow by taking $\mathfrak{f}(x) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and finally properties (v) and (vi) follow by the choice $\mathfrak{f}(x) = \begin{pmatrix} f(x) \\ 0 \\ 0 \end{pmatrix}$. □

The asymptotics of the eigenvalues and eigenfunctions can be derived similar to the classical techniques of [7, 14, 16] and [10]. We state the results briefly. Interested readers may be referred to [10].

$$\phi_\lambda(x) = (a_2 - a_2' s^2)\cos(sx) - \frac{1}{s}(a_1 - a_1' s^2)\sin(sx) + \frac{1}{s}\int_0^x \sin\{s(x-y)\}q(y)\phi_\lambda(y)\,dy, \tag{2.20}$$

$$\phi_\lambda'(x) = -s(a_2 - a_2' s^2)\sin(sx) - (a_1 - a_1' s^2)\cos(sx) + \int_0^x \cos\{s(x-y)\}q(y)\phi_\lambda(y)\,dy, \tag{2.21}$$

where $s = \sigma + it = \sqrt{\lambda}$ is the principal branch and $\phi_\lambda(\cdot)$ is the solution determined by (2.6) above. For sufficiently large λ we have, if $a_2' \neq 0$, cf. [11],

$$\phi_\lambda(x) = -a_2' s^2\cos(sx) + \mathcal{O}(|s|e^{|t|x}), \quad \phi_\lambda'(x) = a_2' s^3\sin(sx) + \mathcal{O}(|s|^2 e^{|t|x}), \tag{2.22}$$

and if $a_2' = 0$,

$$\phi_\lambda(x) = a_1' s\sin(sx) + \mathcal{O}(e^{|t|x}), \quad \phi_\lambda'(x) = a_1' s^2\cos(sx) + \mathcal{O}(|s|e^{|t|x}). \tag{2.23}$$

Then we obtain four distinct cases for the asymptotic behavior of $\omega(\lambda)$ as $|\lambda| \to \infty$, namely

$$\omega(\lambda) = \begin{cases} a_2' b_2' s^5\sin(s) + \mathcal{O}(|s|^4 e^{|t|}), & \text{if } b_2' \neq 0,\ a_2' \neq 0; \\ -a_1' b_2' s^4\cos(s) + \mathcal{O}(|s|^3 e^{|t|}), & \text{if } b_2' \neq 0,\ a_2' = 0; \\ -a_2' b_1' s^4\cos(s) + \mathcal{O}(|s|^3 e^{|t|}), & \text{if } b_2' = 0,\ a_2' \neq 0; \\ -a_1' b_1' s^3\sin(s) + \mathcal{O}(|s|^2 e^{|t|}), & \text{if } b_2' = 0,\ a_2' = 0. \end{cases} \tag{2.24}$$

Consequently if $\lambda_0 < \lambda_1 < \cdots$ are the zeros of $\omega(\lambda)$, then we have for sufficiently large n the following asymptotic formulae

$$\begin{cases} (n-\frac{3}{2})\pi < \sqrt{\lambda_n} < (n-\frac{1}{2})\pi, & \text{if } b_2' \neq 0,\ a_2' \neq 0, \\ (n-1)\pi < \sqrt{\lambda_n} < n\pi, & \text{if } b_2' \neq 0,\ a_2' = 0, \\ (n-1)\pi < \sqrt{\lambda_n} < n\pi & \text{if } b_2' = 0,\ a_2' \neq 0, \\ (n-\frac{1}{2})\pi < \sqrt{\lambda_n} < (n+\frac{1}{2})\pi, & \text{if } b_2' = 0,\ a_2' = 0, \end{cases} \tag{2.25}$$

or equivalently

$$\sqrt{\lambda_n} = \begin{cases} (n-1)\pi + \mathcal{O}(n^{-1}), & \text{if } b_2' \neq 0,\ a_2' \neq 0, \\ (n-\frac{1}{2})\pi + \mathcal{O}(n^{-1}), & \text{if } b_2' \neq 0,\ a_2' = 0, \\ (n-\frac{1}{2})\pi + \mathcal{O}(n^{-1}), & \text{if } b_2' = 0,\ a_2' \neq 0, \\ n\pi + \mathcal{O}(n^{-1}), & \text{if } b_2' = 0,\ a_2' = 0. \end{cases} \tag{2.26}$$

The asymptotic behavior of the first component of the normalized eigenvectors (2.12) is given by

$$(2.27)\qquad \pm\psi_n(x) = \begin{cases} \sqrt{2}\cos((n-1)\pi x) + \mathcal{O}(n^{-1}), & \text{if } b_2' \neq 0,\ a_2' \neq 0, \\ \sqrt{2}\sin((n-1/2)\pi x) + \mathcal{O}(n^{-1}), & \text{if } b_2' \neq 0,\ a_2' = 0, \\ \sqrt{2}\cos((n-1/2)\pi x) + \mathcal{O}(n^{-1}), & \text{if } b_2' = 0,\ a_2' \neq 0, \\ \sqrt{2}\sin(n\pi x) + \mathcal{O}(n^{-1}), & \text{if } b_2' = 0,\ a_2' = 0. \end{cases}$$

The $\mathcal{O}$-terms are uniform for $0 \leq x \leq 1$.

§3. The Sampling Theorem

In this section we derive two sampling theorems associated with problem (1.1)–(1.3). We also give a remark concerning deriving similar results associated with problem (1.1)–(1.3) when $a_i' = 0,\ i = 1, 2$. For convenience we may assume that the eigenvectors of $\mathcal{A}$ are real-valued.

Theorem 3.1. *Consider the boundary value problem* (1.1)–(1.3), *and let* $\phi_\lambda(\cdot)$ *be the solution defined above. If*

$$(3.1)\qquad F(\lambda) = \int_0^1 g(x)\phi_\lambda(x)\,dx, \quad g(\cdot) \in L^2(0,1),$$

then $F(\lambda)$ *is an entire function of order* $1/2$ *and type* ν *with* $0 \leq \nu \leq 1$ *which admits the sampling representation*

$$(3.2)\qquad F(\lambda) = \sum_{n=0}^{\infty} F(\lambda_n)\frac{\omega(\lambda)}{(\lambda-\lambda_n)\omega'(\lambda_n)},$$

where $\omega(\lambda)$ *is the function defined in* (2.9), *which without any loss of generality may be written as*

$$(3.3)\qquad \omega(\lambda) = \begin{cases} \prod_{n=0}^{\infty}(1 - \frac{\lambda}{\lambda_n}), & \textit{if none of the eigenvalues is zero;} \\ \lambda\prod_{n=1}^{\infty}(1 - \frac{\lambda}{\lambda_n}), & \textit{if one of the eigenvalues, say } \lambda_0 = 0. \end{cases}$$

The series (3.2) *converges uniformly on any compact subset of* $\mathbb{C}$.

Proof. Recalling (2.24), $\omega(\lambda)$ is an entire function of order $1/2$ in λ whose zeros are all real, simple and located exactly at the eigenvalues $\{\lambda_n\}_{n=0}^{\infty}$. From (2.26), the product (3.3) converges and defines an entire function of order $1/2$ which will be denoted temporarily by $\tilde{\omega}(\lambda)$. By Hadamard's factorization theorem for entire functions, cf. e.g. [13], $\omega(\lambda) = h(\lambda)\tilde{\omega}(\lambda)$, where $h(\lambda)$ is an entire function of order zero with no zeros. Thus

$$\frac{\omega(\lambda)}{\omega'(\lambda_n)} = \frac{h(\lambda)\tilde{\omega}(\lambda)}{h(\lambda_n)\tilde{\omega}'(\lambda_n)}$$

and (3.1), (3.2) remain valid for the function $\dfrac{F(\lambda)}{h(\lambda)}$. Therefore without any loss of generality, we may assume that $\omega(\lambda) = \tilde{\omega}(\lambda)$. Since $g(\cdot) \in L^2(0,1)$ then relation (3.1) can be rewritten in the form

$$F(\lambda) = \langle \mathfrak{g}(\cdot), \Phi_\lambda(\cdot) \rangle_{\mathfrak{H}} = \int_0^1 g(x)\phi_\lambda(x)\, dx, \tag{3.4}$$

where

$$\mathfrak{g}(x) = \begin{pmatrix} g(x) \\ 0 \\ 0 \end{pmatrix}, \quad \Phi_\lambda(x) = \begin{pmatrix} \phi_\lambda(x) \\ U_0'(\phi_\lambda) \\ U_1'(\phi_\lambda) \end{pmatrix} \in \mathfrak{H}.$$

Since both $\mathfrak{g}(\cdot)$ and $\Phi_\lambda(\cdot)$ are in $\mathfrak{H}$, then they have the Fourier expansions

$$\mathfrak{g}(x) = \sum_{n=0}^{\infty} \widehat{\mathfrak{g}}(n) \frac{\Phi_n(x)}{\|\Phi_n(\cdot)\|_{\mathfrak{H}}^2} \quad \Phi_\lambda(x) = \sum_{n=0}^{\infty} \langle \Phi_\lambda(\cdot), \Phi_n(\cdot) \rangle_{\mathfrak{H}} \frac{\Phi_n(x)}{\|\Phi_n(\cdot)\|_{\mathfrak{H}}^2} \tag{3.5}$$

where

$$\widehat{\mathfrak{g}}(n) = \langle \mathfrak{g}(\cdot), \Phi_n(\cdot) \rangle_{\mathfrak{H}} = \int_0^1 g(x)\phi_{\lambda_n}(x)\, dx, \quad \lambda \in \mathbb{C}. \tag{3.6}$$

Applying Parseval's identity to (3.4) and using (3.6), we obtain

$$F(\lambda) = \sum_{n=0}^{\infty} F(\lambda_n) \frac{\langle \Phi_n(\cdot), \Phi_\lambda(\cdot) \rangle_{\mathfrak{H}}}{\|\Phi_n(\cdot)\|_{\mathfrak{H}}^2}. \tag{3.7}$$

Now we calculate $\langle \Phi_n(\cdot), \Phi_\lambda(\cdot) \rangle_{\mathfrak{H}}$ and $\|\Phi_n(\cdot)\|_{\mathfrak{H}}$. Let $\lambda \in \mathbb{C}$ be not an eigenvalue and $n \in \mathbb{N}$. To prove (3.2) we need to show that

$$\frac{\langle \Phi_n(\cdot), \Phi_\lambda(\cdot) \rangle_{\mathfrak{H}}}{\|\Phi_n(\cdot)\|_{\mathfrak{H}}^2} = \frac{\omega(\lambda)}{(\lambda - \lambda_n)\omega'(\lambda)} \quad n = 0, 1, 2, \cdots. \tag{3.8}$$

By the definition of the inner product of $\mathfrak{H}$, we have

$$\begin{aligned} &\langle \Phi_\lambda(\cdot), \Phi_n(\cdot) \rangle_{\mathfrak{H}} \\ &\quad = \int_0^1 \phi_\lambda(x)\phi_{\lambda_n}(x)\, dx + \frac{1}{\eta} U_0'(\phi_\lambda) U_0'(\phi_{\lambda_n}) + \frac{1}{\rho} U_1'(\phi_\lambda) U_1'(\phi_{\lambda_n}). \end{aligned} \tag{3.9}$$

Lagrange's identity (2.5) and initial conditions (2.6) imply

$$\begin{aligned}(\lambda-\lambda_n)\int_0^1 \phi_\lambda(x)\phi_{\lambda_n}(x)\,dx &= [\phi_\lambda,\phi_{\lambda_n}](1)-[\phi_\lambda,\phi_{\lambda_n}](0)\\ &= -W_1(\phi_{\lambda_n},\phi_\lambda)-(\phi_\lambda(0)\phi'_{\lambda_n}(0)-\phi'_\lambda(0)\phi_{\lambda_n}(0))\\ &= -W_1(\phi_{\lambda_n},\phi_\lambda)+(\lambda_n-\lambda)\eta.\end{aligned}$$

Thus

$$\int_0^1 \phi_\lambda(x)\phi_{\lambda_n}(x)\,dx = \frac{W_1(\phi_{\lambda_n},\phi_\lambda)}{\lambda_n-\lambda}-\eta. \tag{3.10}$$

From (2.13), (2.7) and (2.3), the Wronskian of ϕ_{λ_n} and ϕ_λ at $x=1$ will be

$$\begin{aligned}W_1(\phi_{\lambda_n},\phi_\lambda) &= \phi_\lambda(1)\phi'_{\lambda_n}(1)-\phi'_\lambda(1)\phi_{\lambda_n}(1)\\ &= k_n^{-1}[\chi_{\lambda_n}(1)\phi'_\lambda(1)-\chi'_{\lambda_n}(1)\phi_\lambda(1)]\\ &= k_n^{-1}[(b_2-b'_2\lambda_n)\phi'_\lambda(1)-(b'_1\lambda_n-b_1)\phi_\lambda(1)]\\ &= -k_n^{-1}[\omega(\lambda)+(\lambda_n-\lambda)U'_1(\phi_\lambda)].\end{aligned} \tag{3.11}$$

Relation (2.13) and the linearity of the boundary conditions yield

$$\frac{1}{\rho}U'_1(\phi_\lambda)U'_1(\phi_{\lambda_n}) = \frac{k_n^{-1}}{\rho}U'_1(\phi_\lambda)U'_1(\chi_{\lambda_n}). \tag{3.12}$$

From (2.8) and (3.12), we obtain

$$\frac{1}{\rho}U'_1(\phi_\lambda)U'_1(\phi_{\lambda_n}) = k_n^{-1}U'_1(\phi_\lambda),\quad \frac{1}{\eta}U'_0(\phi_\lambda)U'_0(\phi_{\lambda_n}) = \eta. \tag{3.13}$$

Substituting from (3.10), (3.11) and (3.13) into (3.9), we get

$$\langle\Phi_\lambda(\cdot),\Phi_n(\cdot)\rangle_{\mathfrak{H}} = k_n^{-1}\frac{\omega(\lambda)}{\lambda-\lambda_n}. \tag{3.14}$$

Letting $\lambda\to\lambda_n$ in (3.14) and since the zeros of $\omega(\lambda)$ are simple, we have

$$\langle\Phi_n(\cdot),\Phi_n(\cdot)\rangle_{\mathfrak{H}} = \|\Phi_n(\cdot)\|^2_{\mathfrak{H}} = k_n^{-1}\omega'(\lambda_n). \tag{3.15}$$

Therefore from (3.14) and (3.15) we establish (3.8). Since λ and n are arbitrary, then (3.2) is proved with a pointwise convergence on $\mathbb{C}$, since the case $\lambda=\lambda_n$ is trivial.

Now we investigate the convergence of (3.2). First we prove that it is absolutely convergent on $\mathbb{C}$. Using Cauchy-Schwarz' inequality for $\lambda\in\mathbb{C}$,

$$\begin{aligned}&\sum_{n=0}^{\infty}\left|F(\lambda_n)\frac{\omega(\lambda)}{(\lambda-\lambda_n)\omega'(\lambda_n)}\right|\\ &\quad\le\left(\sum_{n=0}^{\infty}\frac{\left|\langle\mathfrak{g}(\cdot),\Phi_n(\cdot)\rangle_{\mathfrak{H}}\right|^2}{\|\Phi_n(\cdot)\|^2_{\mathfrak{H}}}\right)^{1/2}\left(\sum_{n=0}^{\infty}\frac{\left|\langle\Phi_n(\cdot),\Phi_\lambda(\cdot)\rangle_{\mathfrak{H}}\right|^2}{\|\Phi_n(\cdot)\|^2_{\mathfrak{H}}}\right)^{1/2}.\end{aligned} \tag{3.16}$$

Since $\mathfrak{g}(\cdot), \Phi_\lambda(\cdot) \in \mathfrak{H}$, then both series in the right-hand side of (3.16) converge. Thus series (3.2) converges absolutely on $\mathbb{C}$. For uniform convergence let $M \subset \mathbb{C}$ be compact. Let $\lambda \in M$ and $N > 0$. Define $\sigma_N(\lambda)$ to be

$$\sigma_N(\lambda) := \left| F(\lambda) - \sum_{n=0}^{N} F(\lambda_n) \frac{\omega(\lambda)}{(\lambda - \lambda_n)\omega'(\lambda_n)} \right|. \tag{3.17}$$

Using the same method developed above

$$\sigma_N(\lambda) \le \left(\sum_{n=N+1}^{\infty} \frac{\left|\langle \mathfrak{g}(\cdot), \Phi_n(\cdot)\rangle_{\mathfrak{H}}\right|^2}{\|\Phi_n(\cdot)\|_{\mathfrak{H}}^2} \right)^{1/2} \left(\sum_{n=N+1}^{\infty} \frac{\left|\langle \Phi_n(\cdot), \Phi_\lambda(\cdot)\rangle_{\mathfrak{H}}\right|^2}{\|\Phi_n(\cdot)\|_{\mathfrak{H}}^2} \right)^{1/2}. \tag{3.18}$$

Therefore

$$\sigma_N(\lambda) \le \|\Phi_\lambda(\cdot)\|_{\mathfrak{H}} \left(\sum_{n=N+1}^{\infty} \frac{\left|\langle \mathfrak{g}(\cdot), \Phi_n(\cdot)\rangle_{\mathfrak{H}}\right|^2}{\|\Phi_n(\cdot)\|_{\mathfrak{H}}^2} \right)^{1/2}. \tag{3.19}$$

Since $[0,1] \times M$ is compact, then, cf. e.g. [6, p. 225], we can find a positive constant C_M such that

$$\|\Phi_\lambda(\cdot)\|_{\mathfrak{H}} \le C_M, \quad \text{for all}\, \lambda \in M. \tag{3.20}$$

Then

$$\sigma_N(\lambda) \le C_M \left(\sum_{n=N+1}^{\infty} \frac{\left|\langle \mathfrak{g}(\cdot), \Phi_n(\cdot)\rangle_{\mathfrak{H}}\right|^2}{\|\Phi_n(\cdot)\|_{\mathfrak{H}}^2} \right)^{1/2}. \tag{3.21}$$

uniformly on M. In view of Parseval's equality,

$$\left(\sum_{n=N+1}^{\infty} \frac{\left|\langle \mathfrak{g}(\cdot), \Phi_n(\cdot)\rangle_{\mathfrak{H}}\right|^2}{\|\Phi_n(\cdot)\|_{\mathfrak{H}}^2} \right)^{1/2} \longrightarrow 0 \quad \text{as } N \longrightarrow \infty.$$

Thus $\sigma_N(\lambda) \to 0$ uniformly on M. Hence (3.2) converges uniformly on M. Thus $F(\lambda)$ is analytic on compact subsets of $\mathbb{C}$ and hence it is entire. Moreover $F(\lambda)$ is of order $1/2$ and type ν with $0 \le \nu \le 1$ since

$$|F(\lambda)| \le \|g(\cdot)\|_{L^2(0,1)} \max_{0 \le x \le 1} |\phi_\lambda(x)|$$

and $\phi_\lambda(x)$ has these properties, cf. (2.22). This completes the proof. □

The next theorem is devoted to give interpolation sampling expansions associated with problem (1.1)–(1.3) for integral transforms whose kernels defined in terms of Green's function. As we see in (2.16), Green's function $G(x, \xi, \lambda)$ of problem (1.1)–(1.3) has simple poles at $\{\lambda_n\}_{n=0}^{\infty}$. Define the function $G(x, \lambda)$ to be $G(x, \lambda) := \omega(\lambda) G(x, \xi_0, \lambda)$, where $\xi_0 \in [0, 1]$ is a fixed point and $\omega(\lambda)$ is the function defined in (2.9) or it is the canonical product (3.3).

Theorem 3.2. *Let $g(\cdot) \in L^2(0,1)$ and $\mathcal{F}(\lambda)$ be the integral transform*

$$\mathcal{F}(\lambda) = \int_0^1 G(x,\lambda)\overline{g}(x)\,dx. \tag{3.22}$$

Then $\mathcal{F}(\lambda)$ is an entire function of order $1/2$ and type ν with $0 \le \nu \le 1$ which admits the sampling representation

$$\mathcal{F}(\lambda) = \sum_{n=0}^{\infty} \mathcal{F}(\lambda_n)\frac{\omega(\lambda)}{(\lambda-\lambda_n)\omega'(\lambda_n)}. \tag{3.23}$$

Series (3.23) *converges absolutely on $\mathbb{C}$ and uniformly on compact subsets of $\mathbb{C}$.*

Proof. The integral transform (3.22) can be written as

$$\mathcal{F}(\lambda) = \langle \mathcal{G}(\cdot,\lambda), \mathfrak{g}(\cdot)\rangle_{\mathfrak{H}}\,, \tag{3.24}$$

$$\mathfrak{g}(x) = \begin{pmatrix} g(x) \\ 0 \\ 0 \end{pmatrix}, \quad \mathcal{G}(x,\lambda) = \begin{pmatrix} G(x,\lambda) \\ U_0'(G(x,\lambda)) \\ U_1'(G(x,\lambda)) \end{pmatrix} \in \mathfrak{H}.$$

Applying Parseval's identity to (3.24) with respect to $\{\Phi_n(\cdot)\}_{n=1}^{\infty}$, we obtain

$$\mathcal{F}(\lambda) = \sum_{n=0}^{\infty} \langle \mathcal{G}(\cdot,\lambda), \Phi_n(\cdot)\rangle_{\mathfrak{H}} \frac{\overline{\langle \mathfrak{g}(\cdot), \Phi_n(\cdot)\rangle_{\mathfrak{H}}}}{\|\Phi_n(\cdot)\|_{\mathfrak{H}}^2}. \tag{3.25}$$

Let $\lambda \ne \lambda_n$. Since each $\Phi_n(\cdot)$ is an eigenvectors of $\mathcal{A}$, then

$$(\lambda I - \mathcal{A})\Phi_n(x) = (\lambda - \lambda_n)\Phi_n(x).$$

Thus

$$(\lambda I - \mathcal{A})^{-1}\Phi_n(x) = \frac{1}{\lambda-\lambda_n}\Phi_n(x). \tag{3.26}$$

From (2.15) and (3.26) we obtain

$$\frac{U_1'(\phi_{\lambda_n})}{\omega(\lambda)}\phi_\lambda(\xi_0) - \frac{U_0'(\phi_{\lambda_n})}{\omega(\lambda)}\chi_\lambda(\xi_0) + \int_0^1 G(x,\xi_0,\lambda)\phi_{\lambda_n}(x)\,dx = \frac{1}{\lambda-\lambda_n}\phi_{\lambda_n}(\xi_0). \tag{3.27}$$

Using (2.8) and (2.13), (3.27) becomes

$$\frac{\rho k_n^{-1}}{\omega(\lambda)}\phi_\lambda(\xi_0) + \frac{\eta}{\omega(\lambda)}\chi_\lambda(\xi_0) + \int_0^1 G(x,\xi_0,\lambda)\phi_{\lambda_n}(x)\,dx = \frac{1}{\lambda-\lambda_n}\phi_{\lambda_n}(\xi_0). \tag{3.28}$$

Since $\lambda \neq \lambda_n$, then,

$$\rho k_n^{-1}\phi_\lambda(\xi_0) + \eta\chi_\lambda(\xi_0) + \int_0^1 G(x,\lambda)\phi_{\lambda_n}(x)\,dx = \frac{\omega(\lambda)}{\lambda-\lambda_n}\phi_{\lambda_n}(\xi_0). \tag{3.29}$$

From the definition of $\mathcal{G}(\cdot,\lambda)$, we have

$$\begin{aligned}\langle \mathcal{G}(\cdot,\lambda), \Phi_n(\cdot)\rangle_{\mathfrak{H}} &= \int_0^1 G(x,\lambda)\phi_{\lambda_n}(x)\,dx \\ &\quad + \frac{1}{\eta}U_0'(G(x,\lambda))U_0'(\phi_{\lambda_n}) + \frac{1}{\rho}U_1'(G(x,\lambda))U_1'(\phi_{\lambda_n}).\end{aligned} \tag{3.30}$$

From formula (2.16), we get

$$U_0'(G(x,\lambda)) = \chi_\lambda(\xi_0)U_0'(\phi_\lambda), \qquad U_1'(G(x,\lambda)) = \phi_\lambda(\xi_0)U_1'(\chi_\lambda). \tag{3.31}$$

Combining (3.31), (2.8) and (2.13) together with (3.30), yields

$$\langle \mathcal{G}(\cdot,\lambda), \Phi_n(\cdot)\rangle_{\mathfrak{H}} = \eta\chi_\lambda(\xi_0) + \rho k_n^{-1}\phi_\lambda(\xi_0) + \int_0^1 G(x,\lambda)\phi_{\lambda_n}(x)\,dx. \tag{3.32}$$

Substituting from (3.29) and (3.32) gives

$$\langle \mathcal{G}(\cdot,\lambda), \Phi_n(\cdot)\rangle_{\mathfrak{H}} = \frac{\omega(\lambda)}{\lambda-\lambda_n}\phi_{\lambda_n}(\xi_0). \tag{3.33}$$

As an element of $\mathfrak{H}$, $\mathcal{G}(\cdot,\lambda)$ has the eigenvectors expansion

$$\begin{aligned}\mathcal{G}(x,\lambda) &= \sum_{i=0}^{\infty} \langle \mathcal{G}(\cdot,\lambda), \Phi_i(\cdot)\rangle_{\mathfrak{H}} \frac{\Phi_i(x)}{\|\Phi_i(\cdot)\|_{\mathfrak{H}}^2} \\ &= \sum_{i=0}^{\infty} \frac{\omega(\lambda)}{(\lambda-\lambda_i)}\phi_{\lambda_i}(\xi_0)\frac{\Phi_i(x)}{\|\Phi_i(\cdot)\|_{\mathfrak{H}}^2}.\end{aligned} \tag{3.34}$$

Taking the limit when $\lambda \longrightarrow \lambda_n$ in (3.24), we get

$$\mathcal{F}(\lambda_n) = \lim_{\lambda\to\lambda_n} \langle \mathcal{G}(\cdot,\lambda), \mathfrak{g}(\cdot)\rangle_{\mathfrak{H}}. \tag{3.35}$$

The interchange of the limit and summation processes is justified by the uniform convergence of the eigenvector expansion of $\mathcal{G}(x,\lambda)$ on $[0,1]$ for any $\lambda \in \mathbb{C}$. Making use of (3.34), we may rewrite (3.35) as

$$\begin{aligned}\mathcal{F}(\lambda_n) &= \lim_{\lambda\to\lambda_n} \sum_{i=0}^{\infty} \frac{\omega(\lambda)}{(\lambda-\lambda_i)}\phi_{\lambda_i}(\xi_0)\frac{\langle \Phi_i(\cdot), \mathfrak{g}(\cdot)\rangle_{\mathfrak{H}}}{\|\Phi_i(\cdot)\|_{\mathfrak{H}}^2} \\ &= \omega'(\lambda_n)\phi_{\lambda_n}(\xi_0)\frac{\langle \Phi_n(\cdot), \mathfrak{g}(\cdot)\rangle_{\mathfrak{H}}}{\|\Phi_n(\cdot)\|_{\mathfrak{H}}^2}.\end{aligned} \tag{3.36}$$

The interchange of the limit and summation is justified by the asymptotic behavior of $\Phi_i(x)$ and $\omega(\lambda)$. If $\phi_{\lambda_n}(\xi_0) \neq 0$, then (3.36) gives

$$\frac{\overline{\langle \mathfrak{g}(\cdot), \Phi_n(\cdot) \rangle_{\mathfrak{H}}}}{\|\Phi_n(\cdot)\|^2_{\mathfrak{H}}} = \frac{\mathcal{F}(\lambda_n)}{\omega'(\lambda_n)\phi_{\lambda_n}(\xi_0)}. \tag{3.37}$$

Combining (3.33), (3.37) and (3.25) we get (3.23) under the assumption that $\phi_{\lambda_n}(\xi_0) \neq 0$ for all n. If $\phi_{\lambda_n}(\xi_0) = 0$, for some n, the same expansion holds with $\mathcal{F}(\lambda_n) = 0$. The convergence properties as well as the analytic and growth properties can be established as in Theorem 3.1 above. □

Remark 3.3. Now we indicate how to derive sampling theorems associated with problem (1.1)–(1.3) when $a'_i = 0$, $i = 1, 2$. This problem contains only one boundary condition with an eigenvalue parameter in the boundary condition. In this case, cf. [10], the eigenvalue problem is equivalent to the operator $\mathcal{B} : \mathcal{D}_{\mathcal{B}} \longrightarrow \mathcal{H}$, $\mathcal{H} = L^2(0,1) \oplus \mathbb{C}$ with

$$\langle \mathfrak{f}, \mathfrak{g} \rangle_{\mathcal{H}} := \int_0^1 f(x)\overline{g}(x)\,dx + \frac{1}{\rho}\alpha\overline{\beta}, \quad \mathfrak{f}(x) = \begin{pmatrix} f(x) \\ \alpha \end{pmatrix}, \quad \mathfrak{g}(x) = \begin{pmatrix} g(x) \\ \beta \end{pmatrix} \in \mathcal{H},$$

$\mathcal{D}_{\mathcal{B}} \subseteq \mathcal{H}$ is the set of all $\mathfrak{f}(x) = \begin{pmatrix} f(x) \\ U'_1(f) \end{pmatrix} \in \mathcal{H}$ such that f, f' are absolutely continuous on [0,1] and $\ell(f) \in L^2(0,1)$, $U_0(f) = 0$, and

$$\mathcal{B}\begin{pmatrix} f \\ U'_1(f) \end{pmatrix} = \begin{pmatrix} \ell(f) \\ U_1(f) \end{pmatrix}, \quad \begin{pmatrix} f \\ U'_1(f) \end{pmatrix} \in \mathcal{D}_{\mathcal{B}}.$$

In this case $\eta = 0$. This indicates that the present problem cannot be considered as a special case of problem (1.1)–(1.3) above. In this problem we define the solutions $\theta_\lambda(x)$ and $\chi_\lambda(x)$ of (1.1) via the following initial conditions

$$\theta_\lambda(0) = a_2, \quad \theta_\lambda(0) = -a_1$$

and

$$\chi_\lambda(1) = b_2 - b'_2\lambda, \quad \chi_\lambda(1) = b'_1\lambda - b_1, \quad \lambda \in \mathbb{C}.$$

As we mentioned this problem is studied by Fulton in [10], see also [12, 17, 18]. Among the results obtained in [10] is, the asymptotics of eigenvalues $\{\mu_n\}_{n=0}^{\infty} \subseteq \mathbb{R}$ [10, p. 300], the completeness of the eigenfunctions, $\{\theta_{\mu_n}(\cdot)\}_{n=0}^{\infty}$ or $\{\chi_{\mu_n}(\cdot)\}_{n=0}^{\infty}$. Moreover all eigenvalues are real and simple. Green's function of this problem has the form [10, p. 297]

$$K(x,\xi,\lambda) = \begin{cases} \dfrac{\chi_\lambda(x)\theta_\lambda(\xi)}{\gamma(\lambda)}, & 0 \le \xi \le x \le 1, \\[2ex] \dfrac{\chi_\lambda(\xi)\theta_\lambda(x)}{\gamma(\lambda)}, & 0 \le x \le \xi \le 1, \end{cases} \tag{3.38}$$

where $\gamma(\lambda)$ is the characteristic determinant of the problem, i.e,

$$\gamma(\lambda) := U_0(\chi_\lambda), \text{ or } \quad \gamma(\lambda) := \lambda U_1'(\theta_\lambda) - U_1(\theta_\lambda). \tag{3.39}$$

Similar to Theorem 3.1 and Theorem 3.2 above we state without proofs the sampling theorems associated with the considered problem.

Theorem 3.4. *Consider the boundary value problem* (1.1)–(1.3) *with* $a_i' = 0$, $i = 1, 2$, *and let* $\theta_\lambda(x)$ *be the solution defined above. If*

$$F(\lambda) = \int_0^1 g(x)\theta_\lambda(x)\,dx, \quad g(\cdot) \in L^2(0,1), \tag{3.40}$$

then $F(\lambda)$ *is an entire function of order* $1/2$ *and type* ν *with* $0 \le \nu \le 1$ *which admits the sampling representation*

$$F(\lambda) = \sum_{n=0}^{\infty} F(\mu_n) \frac{\gamma(\lambda)}{(\lambda - \mu_n)\gamma'(\mu_n)}, \tag{3.41}$$

where $\gamma(\lambda)$ *is the function defined in* (3.39), *which without any loss of generality may be written as*

$$\gamma(\lambda) = \begin{cases} \prod_{n=0}^{\infty}(1 - \frac{\lambda}{\mu_n}), & \text{if none of the eigenvalues is zero;} \\ \lambda \prod_{n=1}^{\infty}(1 - \frac{\lambda}{\mu_n}), & \text{if one of the eigenvalues, say } \mu_0 = 0. \end{cases} \tag{3.42}$$

The series (3.41) *converges uniformly on any compact subset of* $\mathbb{C}$.

Now let $\xi_0 \in [0, 1]$. Let $K(x, \lambda)$ be the function

$$K(x, \lambda) := \gamma(\lambda) K(x, \xi_0, \lambda). \tag{3.43}$$

Theorem 3.5. *Let* $g(\cdot) \in L^2(0,1)$ *and* $\mathcal{F}(\lambda)$ *be the integral transform*

$$\mathcal{F}(\lambda) = \int_0^1 K(x, \lambda)\overline{g}(x)\,dx. \tag{3.44}$$

Then $\mathcal{F}(\lambda)$ *is an entire function of order* $1/2$ *and type* ν *with* $0 \le \nu \le 1$ *which admits the sampling representation*

$$\mathcal{F}(\lambda) = \sum_{n=0}^{\infty} \mathcal{F}(\mu_n) \frac{\gamma(\lambda)}{(\lambda - \mu_n)\gamma'(\mu_n)}. \tag{3.45}$$

Series (3.45) *converges absolutely on* $\mathbb{C}$ *and uniformly on compact subsets of* $\mathbb{C}$.

§4. Examples

In this section we give some examples exhibiting the obtained results.

Example 4.1. Consider the boundary value problem

$$y''(x) = -\lambda y(x), \quad 0 \le x \le 1, \tag{4.1}$$

$$y'(0) = -\lambda y(0), \quad y'(1) = \lambda y(1). \tag{4.2}$$

This problem is a special case of problem (1.1)–(1.3) when $q \equiv 0$, $a_1 = a_2' = b_1 = b_2' = 0$, $b_2 = b_1' = a_2 = 1$, $a_1' = -1$. Then $\rho = \eta = 1 > 0$. In the previous notations

$$\phi_\lambda(x) = \cos\sqrt{\lambda}x - \sqrt{\lambda}\sin\sqrt{\lambda}x, \qquad \chi_\lambda(x) = \sqrt{\lambda}\sin\sqrt{\lambda}(x-1) + \cos\sqrt{\lambda}(x-1).$$

Green's function of problem (4.1)–(4.2) is given by

$$G(x,\xi,\lambda) = \begin{cases} \dfrac{\left[\sqrt{\lambda}\sin\sqrt{\lambda}(x-1) + \cos\sqrt{\lambda}(x-1)\right]\left[\cos\sqrt{\lambda}\xi - \sqrt{\lambda}\sin\sqrt{\lambda}\xi\right]}{\omega(\lambda)} & 0 \le \xi \le x \le 1, \\[2ex] \dfrac{\left[\sqrt{\lambda}\sin\sqrt{\lambda}(\xi-1) + \cos\sqrt{\lambda}(\xi-1)\right]\left[\cos\sqrt{\lambda}x - \sqrt{\lambda}\sin\sqrt{\lambda}x\right]}{\omega(\lambda)} & 0 \le x \le \xi \le 1, \end{cases} \tag{4.3}$$

where

$$\omega(\lambda) = 2\lambda\cos\sqrt{\lambda} + (\sqrt{\lambda} - \lambda^{3/2})\sin\sqrt{\lambda} \tag{4.4}$$

By Theorem 3.1, the transform

$$F(\lambda) = \int_0^1 g(x)\left[\cos\sqrt{\lambda}x - \sqrt{\lambda}\sin\sqrt{\lambda}x\right] dx, \qquad g(\cdot) \in L^2(0,1) \tag{4.5}$$

has the following expansion

$$F(\lambda) = \sum_{n=0}^{\infty} F(\lambda_n)\frac{2\sqrt{\lambda_n}\left[2\lambda\cos\sqrt{\lambda} + (\sqrt{\lambda} - \lambda^{3/2})\sin\sqrt{\lambda}\right]}{(\lambda-\lambda_n)\left[(5\sqrt{\lambda_n} - \lambda_n^{3/2})\cos\sqrt{\lambda_n} + (1-5\lambda_n)\sin\sqrt{\lambda_n}\right]}, \tag{4.6}$$

where $\{\lambda_n\}_{n=0}^{\infty}$ are the zeros of $\omega(\lambda)$. In the view of Theorem 3.2, let $\xi_0 = 0$ and $g(\cdot) \in L^2(0,1)$. Then the function $G(x,\lambda)$ will be

$$G(x,\lambda) = \sqrt{\lambda}\sin\sqrt{\lambda}(x-1) + \cos\sqrt{\lambda}(x-1)$$

and the transform

$$\mathcal{F}(\lambda) = \int_0^1 g(x)\left(\sqrt{\lambda}\sin\sqrt{\lambda}(x-1) + \cos\sqrt{\lambda}(x-1)\right)\,dx,$$

will have a sampling formula of the type (4.6) above. The choice $\xi_0 = 1$ will lead to the case (4.5)–(4.6).

Example 4.2. Consider the boundary value problem which consists of (4.1) and the boundary conditions

$$y(0) = \lambda y'(0), \quad y(1) = -\lambda y'(1). \tag{4.7}$$

In this problem we have $q \equiv 0$, $a_1' = a_2 = b_1' = b_2 = 0$, $b_1 = a_1 = a_2' = 1$, $b_2' = -1$. Then $\rho = \eta = 1 > 0$. Hence

$$\phi_\lambda(x) = \frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} + \lambda\cos\sqrt{\lambda}x, \qquad \chi_\lambda(x) = \lambda\cos\sqrt{\lambda}(x-1) - \frac{\sin\sqrt{\lambda}(x-1)}{\sqrt{\lambda}}.$$

Green's function of problem (4.1) and (4.7) will be

$$G(x,\xi,\lambda) = \begin{cases} \dfrac{\left[\lambda\cos\sqrt{\lambda}(x-1) - \dfrac{\sin\sqrt{\lambda}(x-1)}{\sqrt{\lambda}}\right]\left[\dfrac{\sin\sqrt{\lambda}\xi}{\sqrt{\lambda}} + \lambda\cos\sqrt{\lambda}\xi\right]}{\omega(\lambda)} & 0 \le \xi \le x \le 1, \\[2ex] \dfrac{\left[\lambda\cos\sqrt{\lambda}(\xi-1) - \dfrac{\sin\sqrt{\lambda}(\xi-1)}{\sqrt{\lambda}}\right]\left[\dfrac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} + \lambda\cos\sqrt{\lambda}x\right]}{\omega(\lambda)} & 0 \le x \le \xi \le 1, \end{cases} \tag{4.8}$$

where

$$\omega(\lambda) = -2\lambda\cos\sqrt{\lambda} + (\lambda^{5/2} - \lambda^{-1/2})\sin\sqrt{\lambda} \tag{4.9}$$

By Theorem 3.1, the transform,

$$F(\lambda) = \int_0^1 g(x)\left[\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}} + \lambda\cos\sqrt{\lambda}x\right]dx, \quad g(\cdot) \in L^2(0,1) \tag{4.10}$$

can be recovered via the sampling representation

$$F(\lambda) = \sum_{n=0}^{\infty} F(\lambda_n)\frac{2\lambda_n^{3/2}\left[-2\lambda\cos\sqrt{\lambda} + (\lambda^{5/2} - \lambda^{-1/2})\sin\sqrt{\lambda}\right]}{(\lambda-\lambda_n)\left[(\lambda_n^{7/2} - 4\lambda_n^{3/2} - \sqrt{\lambda_n})\cos\sqrt{\lambda_n} + (5\lambda_n^3 + 2\lambda_n^2 + 1)\sin\sqrt{\lambda_n}\right]}, \tag{4.11}$$

where $\{\lambda_n\}_{n=0}^{\infty}$ are the zeros of $\omega(\lambda)$. In the view of Theorem 3.2, let $\xi_0 = 0$ and $g(\cdot) \in L^2(0,1)$. Then the function $G(x,\lambda)$ will be

$$G(x,\lambda) = \lambda^2 \cos\sqrt{\lambda}(x-1) - \sqrt{\lambda}\sin\sqrt{\lambda}(x-1)$$

and the transform

$$\mathcal{F}(\lambda) = \int_0^1 g(x)\left(\lambda^2 \cos\sqrt{\lambda}(x-1) - \sqrt{\lambda}\sin\sqrt{\lambda}(x-1)\right) dx$$

will have a sampling formula similar to (4.11) above.

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