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Regularity of a Noetherian local ring with a *p*-basis

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Abstract. In this note, we shall show the following: Let R be a Noetherian local ring of prime characteristic p. If R has a p-basis over R^p and R is generically reduced, then R is a regular local ring.

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§1. Introduction

Let R be a local ring that is essentially of finite type over a field of prime characteristic p. In this situation, in [2, 7.5], E.Kunz gave, among others, the following theorem:

Theorem 1. Let R be a local ring that is essentially of finite type over a field of prime characteristic p. If R has a p-basis over R^p and R is generically reduced, then R is a regular local ring.

In this note we generalize this theorem to an arbitrary Noetherian local ring R of prime characteristic p without the assumption that R is essentially of finite type over a subfield as follows:

Theorem 2. Let R be a Noetherian local ring of prime characteristic p. If R has a p-basis over R^p and R is generically reduced, then R is a regular local ring.

In this result the assumption that R is generically reduced can not be omitted (cf. [2, p.121]). Furthermore the converse is not true in general. That is, there is a regular local ring R such that R/R^p has no p-basis (cf. [3, Example 3.8]).

§2. Preliminaries

All rings in this note are commutative rings with identity element. Let P be a ring and R a P-algebra of prime characteristic p. Let R^p denote the subring $\{x^p \mid x \in R\}$ of R and PR^p the subring of R generated by R^p and the image of P in R. We denote by $\left(\Omega_{R/P},\ d_{R/P}\right)$ the module of differentials of R/P (P-algebra R is denoted simply by R/P) (cf. [4, p.182]) $\left(\text{in the notation of [2] it is denoted by } \left(\Omega^1_{R/P},\ d_{R/P}\right)\right)$. In case $P = R^p$ we write simply $(\Omega_R,\ d_R)$ for $\left(\Omega_{R/P},\ d_{R/P}\right)$. A ring R is called generically reduced, if $R_{\mathfrak{q}}$ is a field (or equivalently $\mathfrak{q}R_{\mathfrak{q}} = (0)$) for every minimal prime ideal $\mathfrak{q}$ of R (cf.[2, p.118]). A subset B of R is said to be p-independent (in R) over PR^p if the monomials ${b_1}^{e_1} \cdots {b_m}^{e_m}$, where $b_1, \dots, b_m$ are distinct elements of B and $0 \le e_i \le p-1$, are linearly independent over PR^p. A subset B of R is called a p-basis of R/P (or B is a p-basis of R over P) if it is p-independent over PR^p and $R = PR^p[\mathrm{B}]$.

§3. Main result

The main result of this note is the following:

Theorem. Let $(R, \mathfrak{m}, L)$ be a Noetherian local ring of prime characteristic p. If R has a p-basis over R^p and R is generically reduced, then R is a regular local ring.

Proof. Let R have a p-basis over R^p. By [1, 3.2], R/R^p has a p-basis of the form $\{b_1, \dots, b_r\} \cup \{x_j \mid j \in J\}$ such that $\mathfrak{m} = (b_1, \dots, b_r)$ $\left(r := \dim_L(\mathfrak{m}/\mathfrak{m}^2)\right)$ and $\{\overline{x_j} \mid j \in J\}$ is a p-basis of L/L^p, where $\overline{x_j} := x_j + \mathfrak{m}$. Put $X := \{x_j \mid j \in J\}$. Then we see that $k[X]$ is a polynomial ring with variables X over k and moreover that $k[X] \cap \mathfrak{m} = (0)$, where k is the prime field contained in R. Thus R contains the quotient filed K of $k[X]$. It is easy to see that K is a quasi-coefficient field of R and $\{b_1, \dots, b_r\}$ is a p-basis of R/K, and thus $\Omega_{R/K}$ is a finitely generated free R-module with a basis $\left\{d_{R/K}(b_1), \dots, d_{R/K}(b_r)\right\}$.

Let $\{y_1, \dots, y_r\}$ be any subset of $\mathfrak{m}$ with $\mathfrak{m} = (y_1, \dots, y_r)$. Then we have the following canonical exact sequence of L-modules:

$$0 \longrightarrow \mathfrak{m}/\mathfrak{m}^2 \longrightarrow \Omega_{R/K}/\mathfrak{m}\Omega_{R/K} \longrightarrow \Omega_{L/K} \longrightarrow 0.$$

Since K is a quasi-coefficient field of R, we have that $\Omega_{L/K} = (0)$. Thus we see that $\mathfrak{m}/\mathfrak{m}^2 \cong \Omega_{R/K}/\mathfrak{m}\Omega_{R/K}$. Therefore from Nakayama's lemma, $\left\{d_{R/K}(y_1), \dots, d_{R/K}(y_r)\right\}$ is a basis of the free R-module $\Omega_{R/K}$.

Next we shall show that $\mathfrak{m}^2 \supset \mathfrak{q}$ for every minimal prime ideal $\mathfrak{q}$ of R. If there exists an element c of $\mathfrak{q}$ with $c \notin \mathfrak{m}^2$, then $c + \mathfrak{m}^2 \neq 0$ in $\mathfrak{m}/\mathfrak{m}^2$. Thus there are elements $c_2, \ldots, c_r$ of $\mathfrak{m}$ such that $\mathfrak{m} = (c, c_2, \ldots, c_r)$. Therefore $\Omega_{R/K}$ has a basis $\{d_{R/K}(c), d_{R/K}(c_2), \ldots, d_{R/K}(c_r)\}$, and thus $\{d(c/1), d(c_2/1), \ldots, d(c_r/1)\}$ is a basis of the free $R_\mathfrak{q}$-module $\Omega_{R_\mathfrak{q}/K} = R_\mathfrak{q} \otimes_R \Omega_{R/K}$, where $d := d_{R_\mathfrak{q}/K}$. On the other hand, since R is generically reduced, $c/1 = 0$ in $R_\mathfrak{q}$ and thus $d(c/1) = 0$. This is a contradiction. Thus we have that $\mathfrak{m}^2 \supset \mathfrak{q}$.

Choose a minimal prime ideal $\mathfrak{q}$ of R with $\dim R = \dim R/\mathfrak{q}$. Put $R_1 := R/\mathfrak{q}$ and $\mathfrak{m}_1 := \mathfrak{m}/q$. Then $(R_1, \mathfrak{m}_1, L)$ is a Notherian local domain and $\mathfrak{m}_1/{\mathfrak{m}_1}^2 = \mathfrak{m}/\left(\mathfrak{m}^2 + \mathfrak{q}\right) = \mathfrak{m}/\mathfrak{m}^2$.

Let $A := \{a_i \mid i \in I\}$ be any p-basis of R/R^p. Then it is known that Ω_R is a free R-module with a basis $\{d_R(a_i) \mid i \in I\}$. Now we shall show that $d(x) \in \mathfrak{q}\Omega_R$ for every $x \in \mathfrak{q}$. For any $x \in \mathfrak{q}$, $x/1 = 0$ in $R_\mathfrak{q}$. Thus there exists $y \in R$-$\mathfrak{q}$ such that $xy = 0$ in R. Hence $d_R(x)y + xd_R(y) = 0$ and thus $yd_R(x) \in \mathfrak{q}\Omega_R$. Since Ω_R is the free R-module with a basis $\{d_R(a_i) \mid i \in I\}$, we can write $d_R(x) = \sum_i w_i d_R(a_i)$ $(w_i \in R)$. Thus $yd_R(x) = \sum_i yw_i d_R(a_i) \in \mathfrak{q}\Omega_R$. Therefore $yw_i \in \mathfrak{q}$ and $w_i \in \mathfrak{q}$. Hence $d_R(x) \in \mathfrak{q}\Omega_R$.

Since $\Omega_{R_1} = \Omega_R/\left(\mathfrak{q}\Omega_R + Rd_R(\mathfrak{q})\right) = \Omega_R/\mathfrak{q}\Omega_R = R_1 \otimes_R \Omega_R$, Ω_{R_1} is the free R_1-module with a basis $\{d_{R_1}(\overline{a_i}) \mid i \in I\}$, where $\overline{a_i} := a_i + \mathfrak{q}$. Furthermore $R_1 = {R_1}^p[\overline{A}]$, where $\overline{A} := \{\overline{a_i} \mid i \in I\}$. Thus $\overline{A}$ is a p-basis of $R_1/{R_1}^p$ by [2, 5.6]. Therefore R_1 is a domain that is flat over ${R_1}^p$. Hence R_1 is regular by Kunz's theorem (cf.[4, Theorem107]). Thus $\dim R = \dim R_1 = \dim_L \mathfrak{m}_1/{\mathfrak{m}_1}^2 = \dim_L \mathfrak{m}/\mathfrak{m}^2$, and thus R is regular. ■

We remark that we can further generalize Theorem 2 as follows:

Corollary. Let R be a Noetherian ring of prime characteristic p. If R has locally p-bases over R^p and R is generically reduced, then R is a regular ring.

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