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Multipliers on modulation spaces

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Abstract. The purpose of this paper is to study the multipliers on modulation spaces $M^{p,q}(\mathbf{R}^d)$ for $0 < p, q < \infty$. In particular, it is shown in the case $0 < p < 1$ that elements of $\mathcal{B}^K$ ($K > d/(2p)$ and $K \in \mathbf{N}$), consisting of all functions $f \in C^K$ whose derivatives $\partial^\alpha f \in L^\infty$ for any multi-index α such that $|\alpha| \leqq K$, are multipliers on $M^{p,q}$.

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§1. Introduction

The modulation spaces $M^{p,q}(\mathbf{R}^d)$ for general $0 < p, q \leqq \infty$, which coincide with the usual modulation spaces when $1 \leqq p, q \leqq \infty$, have been constructed and several properties on $M^{p,q}(\mathbf{R}^d)$ have been studied in [5], [6]. The aim of this paper is the study of the boundedness of the operators

$$\sigma(D)f = \int_{\mathbf{R}^d} e^{2\pi i x\cdot\xi}\sigma(\xi)\widehat{f}(\xi)d\xi$$

on $M^{p,q}(\mathbf{R}^d)$ for $0 < p, q < \infty$.

When $1 < p, q < \infty$, it was already studied in Gröchenig and Heil [2], [4] and proved that $\sigma(D)$ has a unique bounded extension on each $M^{p,q}(\mathbf{R}^d)$ if $\sigma \in M^{\infty,1}(\mathbf{R}^d)$. However, as Gröchenig pointed it out in his paper [3], their argument doesn't cover when p or $q = 1$ or ∞, since they use the facts that $\mathcal{S}(\mathbf{R}^d)$ is dense in $M^{p,q}(\mathbf{R}^d)$ and the dual of $M^{p,q}(\mathbf{R}^d)$ is $M^{p',q'}(\mathbf{R}^d)$ for $1 \leqq p, q < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1 = \frac{1}{q} + \frac{1}{q'}$. So in this paper, we calculate the $M^{p,q}$-norm of $\sigma(D)f$ directly with our key lemma (Lemma 2.4) without using the duality, and examine what conditions on σ to guarantee the $M^{p,q}$-boundedness of $\sigma(D)$. In particular, it is shown in the case $0 < p < 1$ that elements of $\mathcal{B}^K$ ($K > d/(2p)$ and $K \in \mathbf{N}$), consisting of all functions $f \in C^K$ whose derivatives $\partial^\alpha f \in L^\infty$ for any multi-index α such that $|\alpha| \leqq K$, are multipliers on $M^{p,q}$.

§2. Preliminaries

2.1. Basic definition

The following notations will be used throughout this article. Let $\mathcal{S}(\mathbf{R}^d)$ be the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on $\mathbf{R}^d$ and $\mathcal{S}'(\mathbf{R}^d)$ be the topological dual of $\mathcal{S}(\mathbf{R}^d)$. The Fourier transform is $\hat{f}(\omega) = \int f(t)e^{-2\pi i\omega\cdot t}dt$, and the inverse Fourier transform is $\check{f}(t) = \hat{f}(-t)$. We define for $0 < p < \infty$

$$||f||_{L^p} = \Big(\int_{\mathbf{R}^d} |f(t)|^p dt\Big)^{\frac{1}{p}}$$

and $||f||_{L^\infty} = \operatorname{ess.sup}_{t\in\mathbf{R}^d} |f(t)|$. We use the pairing $\langle f, g\rangle$ between $f \in \mathcal{S}'(\mathbf{R}^d)$ and $g \in \mathcal{S}(\mathbf{R}^d)$, in a manner consistent with the inner product $\langle f, g\rangle = \int f(t)\overline{g(t)}dt$ on $L^2(\mathbf{R}^d)$. For a function f on $\mathbf{R}^d$, the translation and the modulation operators are defined by

$$T_x f(t) = f(t-x), \quad \textit{and} \quad M_\omega f(t) = e^{2\pi i\omega\cdot t}f(t) \quad (x, \omega \in \mathbf{R}^d),$$

respectively.

2.2. Modulation spaces and Basic properties

We recall the definition of the modulation spaces.

First for $\alpha > 0$ we define $\Phi^\alpha(\mathbf{R}^d)$ to be the space of all $g \in \mathcal{S}(\mathbf{R}^d)$ satisfying

$$\operatorname{supp} \widehat{g} \subset \{\xi \mid |\xi| \leqq 1\}, \ \textit{and} \ \sum_{k\in\mathbf{Z}^d} \widehat{g}(\xi - \alpha k) \equiv 1, \ \forall\xi \in \mathbf{R}^d.$$

In the following, we choose a sufficiently small $\alpha > 0$ so that the function space $\Phi^\alpha(\mathbf{R}^d)$ is not empty. With this, we have defined the modulation spaces as follows:

Definition 2.1. Given a $g \in \Phi^\alpha(\mathbf{R}^d)$, and $0 < p, q \leqq \infty$, we define the modulation space $M^{p,q}(\mathbf{R}^d)$ to be the space of all tempered distributions $f \in \mathcal{S}'(\mathbf{R}^d)$ such that the quasi-norm

$$||f||_{M^{p,q}} := \Big(\int_{\mathbf{R}^d}\Big(\int_{\mathbf{R}^d}\big|f * (M_\omega g)(x)\big|^p dx\Big)^{\frac{q}{p}} d\omega\Big)^{\frac{1}{q}}$$

is finite, with obvious modifications if p or $q = \infty$.

We state basic properties of modulation spaces, which will play an important role in this article (see [5]).

Proposition 2.2. *Let* $0 < p, q \leqq \infty$ *and* $g \in \Phi^\alpha(\mathbf{R}^d)$. *Then*

(a) $$\Big(\sum_{k\in\mathbf{Z}^d}\Big(\int_{\mathbf{R}^d}\big|f*\big(M_{\alpha k}g\big)(x)\big|^p dx\Big)^{\frac{q}{p}}\Big)^{\frac{1}{q}}$$

is an equivalent quasi-norm on $M^{p,q}(\mathbf{R}^d)$ *with modifications if* p *or* $q = \infty$.
(*b*) *Different test functions* $g_1, g_2 \in \Phi^\alpha(\mathbf{R}^d)$ *define the same space and equivalent quasi-norms on* $M^{p,q}(\mathbf{R}^d)$.
(*c*) *Let* $0 < p_0 \leqq p_1 \leqq \infty$ *and* $0 < q_0 \leqq q_1 \leqq \infty$. *Then*

$$M^{p_0,q_0}(\mathbf{R}^d) \subset M^{p_1,q_1}(\mathbf{R}^d).$$

(*d*) *We have the continuous embeddings*

$$\mathcal{S}(\mathbf{R}^d) \subset M^{p,q}(\mathbf{R}^d) \subset \mathcal{S}'(\mathbf{R}^d)$$

for $0 < p, q \leqq \infty$.
(*e*) $M^{p,q}(\mathbf{R}^d)$ *is a quasi-Banach space if* $0 < p, q \leqq \infty$ (*Banach space if* $1 \leqq p, q \leqq \infty$).
(*f*) *If* $0 < p, q < \infty$, *then* $\mathcal{S}(\mathbf{R}^d)$ *is dense in* $M^{p,q}(\mathbf{R}^d)$.

These facts have been derived from the following.

Let $0 < p \leqq \infty$, and Γ be a compact subset of $\mathbf{R}^d$. Then L^p_Γ is defined by

$$L^p_\Gamma = \{f \in \mathcal{S}'(\mathbf{R}^d) \mid \exists \xi_0 \in \mathbf{R}^d,\ \mathrm{supp}\widehat{f} \subset \xi_0 + \Gamma,\ ||f||_{L^p} < \infty\}.$$

Lemma 2.3 ([5] Theorem 2.5). *Let* Γ *be a compact subset of* $\mathbf{R}^d$ *and let* $0 < p \leqq q \leqq \infty$. *Then there exists a positive constant* C (*which depends only on the diameter of* Γ *and* p) *such that*

$$||f||_{L^q} \leqq C||f||_{L^p}$$

holds for all $f \in L^p_\Gamma$.

Lemma 2.4 ([5] Lemma 2.6). *Let* $0 < p \leqq 1$ *and* Γ, Γ' *be compact subsets of* $\mathbf{R}^d$. *Then there exists a positive constant* C (*which depends only on the diameters of* Γ, Γ' *and* p) *such that*

$$\Big|\Big||f|*|g|\Big|\Big|_{L^p} \leqq C||f||_{L^p}||g||_{L^p}$$

holds for all $f \in L^p_\Gamma$ *and all* $g \in L^p_{\Gamma'}$.

In the sequel, we shall not distinguish between equivalent quasi-norms of a given quasi-normed space.

2.3. Multiplier operators and Symbol classes

DEFINITION 2.5. Let $0 < p, q < \infty$ and $\sigma \in \mathcal{S}'(\mathbf{R}^d)$. If the operator $\sigma(D)$, initially defined in $\mathcal{S}(\mathbf{R}^d)$ by the relation

$$\sigma(D)f = (\sigma \cdot \widehat{f})^\vee, \tag{2.1}$$

satisfies the inequality

$$\|\sigma(D)f\|_{M^{p,q}} \leqq C\|f\|_{M^{p,q}}, \quad f \in M^{p,q}(\mathbf{R}^d), \tag{2.2}$$

where C is independent of f, we say that σ is a multiplier on $M^{p,q}$ and $\sigma(D)$ is a multiplier operator on $M^{p,q}$.

DEFINITION 2.6. For $g \in \Phi^\alpha(\mathbf{R}^d)$ and $0 < p < \infty$, we define $S(p)$ to be the space of all tempered distributions $\sigma \in \mathcal{S}'(\mathbf{R}^d)$ such that

$$\|\sigma\|_{S(p)} := \|\check{\sigma}\|_{M^{p,\infty}} = \sup_{k \in \mathbf{Z}^d} \Big(\int_{\mathbf{R}^d} |(\sigma \cdot T_{\alpha k}\widehat{g})^\vee(x)|^p dx \Big)^{\frac{1}{p}} < \infty. \tag{2.3}$$

2.4. Main results

We now formulate our results.
(i) Let $1 \leqq p < \infty$, $0 < q < \infty$ and $\sigma \in S(1)$. Then $\sigma(D)$ is a multiplier operator on $M^{p,q}(\mathbf{R}^d)$.
(ii) Let $0 < p < 1$, $0 < q < \infty$ and $\sigma \in S(p)$. Then $\sigma(D)$ is a multiplier operator on $M^{p,q}(\mathbf{R}^d)$.

Precise statements of these results and their proof are stated in §3.

2.5. Examples

THEOREM 2.7. *Let $0 < p \leqq 1$ and δ_{x_n} be the Dirac measure at a point $x_n \in \mathbf{R}^d$. Then, for a sequence of complex numbers $\{c_n\}_{n=-\infty}^{\infty} \in l^p(\mathbf{Z})$,*

$$\sigma = \Big(\sum_{n=-\infty}^{\infty} c_n \delta_{x_n} \Big)^\wedge$$

belongs to $S(p)$.

PROOF. A direct calculation shows that for each $k \in \mathbf{Z}^d$,

$$\check{\sigma} * M_{\alpha k} g(x) = \sum_{n=-\infty}^{\infty} c_n \langle \delta_{x_n}, \overline{M_{\alpha k} g(x - \cdot)} \rangle = \sum_{n=-\infty}^{\infty} c_n M_{\alpha k} g(x - x_n).$$

Hence it follows that

$$\begin{aligned}||\check{\sigma} * M_{\alpha k}g(x)||_{L^p}^p &= \int_{\mathbf{R}^d} \Big| \sum_{n=-\infty}^{\infty} c_n M_{\alpha k} g(x - x_n)\Big|^p dx \\ &\leqq \sum_{n=-\infty}^{\infty} |c_n|^p \int |e^{2\pi i \alpha k\cdot(x-x_n)} g(x-x_n)|^p dx \\ &= \sum_{n=-\infty}^{\infty} |c_n|^p ||g||_{L^p}^p < \infty.\end{aligned}$$

By taking $\frac{1}{p}$-th power and l^∞-norm, we see that $\sigma \in S(p)$. □

Remark. Oberlin in [7] has proved that every bounded linear operator T on $L^p(\mathbf{R}^d)$ $(0 < p < 1)$ which commutes with translations is represented by $Tf = \sigma(D)f$ with $\sigma = (\sum c_n \delta_{x_n})^\wedge$, where $\{c_n\} \in l^p(\mathbf{Z})$.

THEOREM 2.8. *Let* $1 \leqq p < \infty$. *Then we have* $M^{\infty,p}(\mathbf{R}^d) \subset S(p)$.

PROOF. Since $\big(\sigma \cdot T_\omega \widehat{g}\big)^\vee(x) = e^{2\pi i\omega\cdot x}\sigma * \big(M_{-x}\mathcal{I}\widehat{g}\big)(\omega)$, where $\mathcal{I}\widehat{g}(\xi) = \widehat{g}(-\xi)$ and $M^{\infty,p}(\mathbf{R}^d)$ $(1 \leqq p < \infty)$ is independent of the choice of a window $g \in \mathcal{S}(\mathbf{R}^d) \setminus \{0\}$ (see [2] Proposition 11.3.2), it follows that

$$\begin{aligned}||\sigma||_{S(p)} &\leqq c \sup_{\omega\in\mathbf{R}^d} \Big(\int_{\mathbf{R}^d} \big|\big(\sigma\cdot T_\omega\widehat{g}\big)^\vee(x)\big|^p dx\Big)^{\frac{1}{p}} \\ &\leqq c\Big(\int_{\mathbf{R}^d}\Big(\sup_{\omega\in\mathbf{R}^d}\big|\sigma * \big(M_{-x}\mathcal{I}\widehat{g}\big)(\omega)\big|\Big)^p dx\Big)^{\frac{1}{p}} \leqq c'||\sigma||_{M^{\infty,p}}.\end{aligned}$$

□

THEOREM 2.9. *Let* $0 < p < \infty$ *and* K *be a positive integer. If* $K > \frac{d}{2p}$ *then*

$$\mathcal{B}^K := \big\{f \in C^K(\mathbf{R}^d) \;\big|\; \sum_{|\alpha|\leqq K} ||\partial^\alpha f||_{L^\infty} < \infty\big\}$$

belongs to $S(p)$.

Proof. Let $f \in \mathcal{B}^K$ and denote $\Delta_\xi = \sum_{j=1}^d (\partial^2/\partial\xi_j^2)$. Then we have

$$\begin{aligned}
&(1+4\pi^2|x|^2)^K \big|\big(f \cdot T_{\alpha k}\widehat{g}\big)^\vee(x)\big| \\
&= (1+4\pi^2|x|^2)^K \Big| \int_{\mathbf{R}^d} f(\xi)\widehat{g}(\xi-\alpha k)e^{2\pi i x\cdot\xi}d\xi \Big| \\
&= \Big| \int_{\mathbf{R}^d} f(\xi)\widehat{g}(\xi-\alpha k)(1-\Delta_\xi)^K e^{2\pi i x\cdot\xi}d\xi \Big| \\
&= \Big| \int_{\mathbf{R}^d} \sum_{|\alpha+\beta|\leqq 2K} C_{\alpha,\beta}\partial^\alpha f(\xi)\partial^\beta \widehat{g}(\xi-\alpha k)e^{2\pi i x\cdot\xi}d\xi \Big| \\
&\leqq \sum_{|\alpha+\beta|\leqq 2K} C_{\alpha,\beta}||\partial^\alpha f||_{L^\infty} \int_{\mathbf{R}^d} |\partial^\beta \widehat{g}(\xi)|d\xi.
\end{aligned}$$

Since $K > \frac{d}{2p}$, we have

$$||f||_{S(p)} \leqq C \sum_{|\alpha|\leqq 2K} ||\partial^\alpha f||_{L^\infty}.$$

□

§3. Proof of the main results

We now consider the behavior of $\sigma(D)$ on $M^{p,q}(\mathbf{R}^d)$. Throughout this section, g denotes a function in $\Phi^\alpha(\mathbf{R}^d)$.

3.1. The case $1 \leqq p < \infty$

Theorem 3.1. *Let $1 \leqq p < \infty$, $0 < q < \infty$ and $\sigma \in S(1)$. Then the linear operator $\sigma(D)$, initially defined in the dense subspace $\mathcal{S}(\mathbf{R}^d)$ of $M^{p,q}(\mathbf{R}^d)$, has a unique bounded extension on $M^{p,q}(\mathbf{R}^d)$ and satisfies*

$$||\sigma(D)f||_{M^{p,q}} \leqq c||\sigma||_{S(1)}||f||_{M^{p,q}}. \tag{3.1}$$

Proof. First note that there exists a constant N (depending only on the size of supp $\widehat{g}$, $\alpha > 0$ and dimension d) such that $T_{\alpha k}\widehat{g} = \sum_{|r|\leqq N} T_{\alpha(k+r)}\widehat{g} \cdot T_{\alpha k}\widehat{g}$ for all $k \in \mathbf{Z}^d$. Then for $f \in \mathcal{S}(\mathbf{R}^d)$, we have

$$\begin{aligned}
(\sigma \cdot \widehat{f})^\vee * \big(M_{\alpha k}g\big)(x) &= (\sigma \cdot \widehat{f} \cdot T_{\alpha k}\widehat{g})^\vee(x) = \sum_{|r|\leqq N} (\sigma \cdot T_{\alpha k}\widehat{g} \cdot \widehat{f} \cdot T_{\alpha(k+r)}\widehat{g})^\vee(x) \\
&= \sum_{|r|\leqq N} (\sigma \cdot T_{\alpha k}\widehat{g})^\vee * (\widehat{f} \cdot T_{\alpha(k+r)}\widehat{g})^\vee(x)
\end{aligned}$$

From this and Young's inequality, we have

$$||(\sigma \cdot \widehat{f})^{\vee} * M_{\alpha k} g(x)||_{L^p} \leqq \sum_{|r| \leqq N} ||(\sigma \cdot T_{\alpha k} \widehat{g})^{\vee}(x)||_{L^1} ||(\widehat{f} \cdot T_{\alpha(k+r)} \widehat{g})^{\vee}(x)||_{L^p}.$$

Taking the l^q-norm on both sides, we obtain

$$||\sigma(D)f||_{M^{p,q}} \leqq c \sup_{k \in \mathbf{Z}^d} ||(\sigma \cdot T_{\alpha k} \widehat{g})^{\vee}||_{L^1} ||f||_{M^{p,q}}$$

Then, since $\mathcal{S}(\mathbf{R}^d)$ is dense and $M^{p,q}(\mathbf{R}^d)$ is a quasi-Banach space, we have the desired result. □

3.2. The case $0 < p < 1$

THEOREM 3.2. *Let $0 < p < 1$, $0 < q < \infty$ and $\sigma \in S(p)$. Then the linear operator $\sigma(D)$, initially defined in the dense subspace $\mathcal{S}(\mathbf{R}^d)$ of $M^{p,q}(\mathbf{R}^d)$, has a unique bounded extension on $M^{p,q}(\mathbf{R}^d)$ and satisfies*

$$||\sigma(D)f||_{M^{p,q}} \leqq C||\sigma||_{S(p)} ||f||_{M^{p,q}}. \tag{3.2}$$

PROOF. Let $f \in \mathcal{S}(\mathbf{R}^d)$. Then we have

$$(\sigma \cdot \widehat{f})^{\vee} * \big(M_{\alpha k} g\big)(x) = \sum_{|r| \leqq N} (\sigma \cdot T_{\alpha k} \widehat{g})^{\vee} * (\widehat{f} \cdot T_{\alpha(k+r)} \widehat{g})^{\vee}(x).$$

From this and Lemma 2.4, we have

$$||(\sigma \cdot \widehat{f})^{\vee} * M_{\alpha k} g(x)||_{L^p} \leqq C \sum_{|r| \leqq N} ||(\sigma \cdot T_{\alpha k} \widehat{g})^{\vee}(x)||_{L^p} ||(\widehat{f} \cdot T_{\alpha(k+r)} \widehat{g})^{\vee}(x)||_{L^p}.$$

Taking the l^q-norm on both sides, we obtain

$$||\sigma(D)f||_{M^{p,q}} \leqq C' \sup_{k \in \mathbf{Z}^d} ||(\sigma \cdot T_{\alpha k} \widehat{g})^{\vee}||_{L^p} ||f||_{M^{p,q}}.$$

Then, since $\mathcal{S}(\mathbf{R}^d)$ is dense and $M^{p,q}(\mathbf{R}^d)$ is a quasi-Banach space, we have the desired result. □

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References

[1] H. G. Feichtinger, *A new class of function spaces,* in Proc. Conf. “Constructive Function Theory”, Kiew 1983, (1984).

[2] K. Gröchenig, *Foundations of Time-Frequency Analysis,* Birkhäuser, 2001.

[3] K. Gröchenig, *A Pedestrian Approach to Pseudodifferential Operators,* preprint.

[4] K. Gröchenig and C. Heil, *Modulation spaces and pseudodifferential operators,* Integral Equations Operator Theory, 34(4):439-457,1999.

[5] M. Kobayashi, *Modulation spaces* $M^{p,q}$ *for* $0 < p, q \leqq \infty$*,* J. Function Spaces Appl. 4(3) (2006) 329-341.

[6] M. Kobayashi, *Dual of Modulation spaces,* to appear in J. Function Spaces Appl.

[7] D. M. Oberlin, *Translation-invariant operators on* $L^p(G)$*,* Canad. J. Math. 29(3) (1977), 119–122.

[8] H. Triebel, *Theory of Function Spaces,* Birkhäuser, Boston, 1983.

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