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A semi-parallel fundamental form of maximal rank for a decomposition of a vector bundle with connection

Ryuji Ishii

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Abstract. We study a subbundle with semi-parallel fundamental form. In particular, if the rank of the fundamental form is maximal, we can obtain a certain equation which plays an essential role to classify parallel affine immersions into $\mathbb{R}^{n+\frac{1}{2}n(n+1)}$.

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§0. Introduction

In Riemannian geometry, many researchers have studied submanifolds with parallel second fundamental form. In particular, Ferus [5] classified submanifolds of the Euclidean space with parallel second fundamental forms. These submanifolds are often called parallel submanifolds. Moreover, semi-parallel submanifolds which is a generalization of parallel submanifolds, have been also studied in [3] and [4], for example. In affine differential geometry, Vrancken [9] classified linearly full affine immersions from an n-dimensional manifold M to an affine space $\mathbb{R}^{n+\frac{1}{2}n(n+1)}$ with parallel affine fundamental form, where the following equation plays an essential role:

$$S_Z B_X Y = \frac{1}{n-1}(\mathrm{Ric}(X,Z)Y + \mathrm{Ric}(Y,Z)X + 2\mathrm{Ric}(X,Y)Z), \tag{0.1}$$

where S is the shape operator, B is the affine fundamental form and Ric is the Ricci tensor of the induced connection.

Our main purpose is to prove equations including (0.1) for the case of a decomposition of a vector bundle with connection, which can be regarded as a generalization of affine immersions, see [2], [6], for example. Let $V = V_1 \oplus V_2$ be a decomposition with connection ∇ on V, ∇^1 (resp. ∇^2) the induced connection on V_1 (resp. V_2) and B the fundamental form. If $\hat{R}B = 0$, where $\hat{R}$ is the curvature operator defined by ∇^1, ∇^2, and a connection D on TM, we say that B is semi-parallel. If the dimension of $\mathrm{Span}\{B_X\eta | X \in T_xM, \eta \in V_{1x}\}$ is maximal for every $x \in M$, B is said to be of maximal rank. Under the condition that the fundamental form B is semi-parallel and of maximal rank, we obtain equations including (0.1). In particular, our proof of (0.1) is relatively shorter than that in [9].

§1. Preliminaries

We assume that all objects are smooth and all vector bundles are real throughout this paper. Let M be an n-dimensional ($n \geq 2$) manifold. Let V, W be vector bundles over M, $\Gamma(V)$ the space of cross-section of V and $\mathfrak{C}(V)$ the set of covariant derivatives of connections on V. Let $\mathrm{Hom}(V, W)$ be the vector bundle of which fiber $\mathrm{Hom}(V, W)_x$ at $x \in M$ is the vector space $\mathrm{Hom}(V_x, W_x)$ of linear maps from V_x to W_x. The space of vector bundle homomorphisms from V to W is denoted by $\mathrm{HOM}(V, W)$. We note that $\mathrm{HOM}(V, W)$ can be canonically identified with the space $\Gamma(\mathrm{Hom}(V, W))$. For non-negative integer r, we denote the space of V-valued r-forms on M by $A^r(V)$ and $A^r := A^r(M \times \mathbb{R})$.

Let V_1 be a subbundle of V and $i : V_1 \to V$ the inclusion. If a subbundle V_2 of V satisfies $V_1 \oplus V_2 = V$ (direct sum), then we say that V_2 is the *transversal bundle* with respect to V_1. Take a transversal bundle V_2. We set $i_2 : V_2 \to V$ the inclusion and $p_j : V \to V_j$ the projection homomorphism for $j = 1, 2$. We note that $ip_1 + i_2p_2 = \mathrm{id}_V$. Let $\nabla \in \mathfrak{C}(V)$ be a connection on V. We set $\nabla^1 := p_1\nabla i$, where $p_1\nabla i$ is defined by $(p_1\nabla i)_X := p_1 \circ \nabla_X \circ i$ for $X \in \Gamma(TM)$. Similarly, we set $\nabla^2 := p_2\nabla i_2$, $B := p_2\nabla i$ and $S := -p_1\nabla i_2$. We call ∇^1 the *induced connection* on V_1, ∇^2 the *transversal connection* on V_2, B the *fundamental form* and S the *shape tensor.* Since $p_1 i = \mathrm{id}_{V_1}, p_2 i = 0, p_2 i_2 = \mathrm{id}_{V_2}$ and $p_1 i_2 = 0$, we have

Lemma 1.1. *For ∇^1, B, ∇^2 and S, we obtain*

$$\nabla^1 \in \mathfrak{C}(V_1),\ B \in A^1(\mathrm{Hom}(V_1, V_2)),\ \nabla^2 \in \mathfrak{C}(V_2) \text{ and } S \in A^1(\mathrm{Hom}(V_2, V_1)).$$

Let R (resp. R^1, R^2) be the curvature form of ∇ (resp. ∇^1, ∇^2).

Lemma 1.2. *We have the fundamental equations as follows*:

Gauss: $\quad p_1 R_{X,Y} i = R^1_{X,Y} - S_X B_Y + S_Y B_X;$

Codazzi for B: $\quad p_2 R_{X,Y} i = B_X \nabla^1_Y - B_Y \nabla^1_X - \nabla^2_Y B_X + \nabla^2_X B_Y - B_{[X,Y]};$

Codazzi for S: $\quad p_1 R_{X,Y} i_2 = \nabla^1_Y S_X - \nabla^1_X S_Y - S_X \nabla^2_Y + S_Y \nabla^2_X + S_{[X,Y]};$

Ricci: $$p_2 R_{X,Y} i_2 = R^2_{X,Y} - B_X S_Y + B_Y S_X,$$

for $X, Y \in \Gamma(TM)$.

We apply these notions to affine immersions. Let $\tilde{M}$ be an $(n+q)$-dimensional manifold and $f : M \to \tilde{M}$ an immersion. We denote the pull-back bundle through f of $T\tilde{M}$ by $\tilde{T} := f^{\#}(T\tilde{M})$, the bundle map by $f_{\#} : \tilde{T} \to T\tilde{M}$ and its restriction to the fiber by $f_{\#x}$ for $x \in M$. We define a linear mapping $\iota_x : T_x M \to \tilde{T}_x$ by $\iota_x := (f_{\#x})^{-1} f_{*x}$ for each $x \in M$, where $f_{*x} : T_x M \to T_{f(x)}\tilde{M}$ is the differential of f at x. Thus we define a bundle homomorphism $\iota : TM \to \tilde{T}$ by $\iota|_{T_x M} := \iota_x$ and obtain the isomorphism $\tilde{\iota} : TM \to \iota(TM)$. We identify $\iota(TM)$ with TM through $\tilde{\iota}$. Let N be a subbundle of $\tilde{T}$ such that $T \oplus N = \tilde{T}$, where we set $T := TM (= \iota(TM))$. For $\tilde{D} \in \mathfrak{C}(T\tilde{M})$, there exists the pull-back connection $f^{\#}\tilde{D}$ which is denoted by $\nabla \in \mathfrak{C}(\tilde{T})$. Then we have

$$\nabla^T := p_1 \nabla i_1 \in \mathfrak{C}(T),\ \nabla^N := p_2 \nabla i_2 \in \mathfrak{C}(N),$$
$$B := p_2 \nabla i_1 \in A^1(\mathrm{Hom}(T, N)) \text{ and } S := -p_1 \nabla i_2 \in A^1(\mathrm{Hom}(N, T)).$$

We call (f, N) the *affine immersion* from (M, ∇^T) to $(\tilde{M}, \tilde{D})$, ∇^T the *induced connection,* ∇^N the *transversal connection,* B the *affine fundamental form* and S the *shape tensor.*

§2. Semi-parallel fundamental form

From now on, X, Y, Z always denote elements of $\Gamma(TM)$. Let $\nabla \in \mathfrak{C}(V)$ be a connection on V and $D \in \mathfrak{C}(TM)$ a connection on TM. We set

$$(\hat{\nabla}_X B)_Y := \nabla^2_X B_Y - B_{D_X Y} - B_Y \nabla^1_X,$$

and

$$(\hat{R}_{X,Y} B)_Z := R^2_{X,Y} B_Z - B_{R^D_{X,Y} Z} - B_Z R^1_{X,Y},$$

where R^D is the curvature form of D.

Definition 2.1. If $\hat{\nabla} B = 0$ (resp. $\hat{R} B = 0$), we say that B is *parallel* (resp. *semi-parallel*).

If D is torsion-free, then we obtain the following equations by a straightforward calculation:

$$\begin{aligned}
&(\hat{R}_{X,Y} B)_Z \\
&\quad = \nabla^2_X (\hat{\nabla}_Y B)_Z - (\hat{\nabla}_{D_X Y} B)_Z - (\hat{\nabla}_Y B)_{D_X Z} - (\hat{\nabla}_Y B)_Z \nabla^1_X \\
&\qquad - \nabla^2_Y (\hat{\nabla}_X B)_Z + (\hat{\nabla}_{D_Y X} B)_Z + (\hat{\nabla}_X B)_{D_Y Z} + (\hat{\nabla}_X B)_Z \nabla^1_Y \\
&\quad = (\hat{\nabla}_X (\hat{\nabla}_Y B))_Z - (\hat{\nabla}_Y (\hat{\nabla}_X B))_Z - (\hat{\nabla}_{[X,Y]} B)_Z.
\end{aligned}$$

Thus we see that if D is torsion-free and B is parallel, then B is semi-parallel.

By Ricci equation, we have

Lemma 2.1. *If $p_2 Ri_2 = 0$ and B is semi-parallel, then we have the following*:

$$B_X S_Y B_Z \eta - B_Y S_X B_Z \eta = B_{R^D_{X,Y} Z}\eta + B_Z R^1_{X,Y}\eta,$$

where $\eta \in \Gamma(V_1)$.

We denote the Ricci tensor of R^D by Ric^D, i.e.,

$$\mathrm{Ric}^D(Y,Z) := \mathrm{trace}\{X \mapsto R^D_{X,Y}Z\}.$$

By using first Bianchi identity, if D is torsion-free, then we obtain

$$\mathrm{tr}(R^D_{X,Y}) = \mathrm{Ric}^D(Y,X) - \mathrm{Ric}^D(X,Y).$$

We note that there exists a local parallel volume element on V (resp. V_1) if and only if $\mathrm{tr}R = 0$ (resp. $\mathrm{tr}R^1 = 0$). If $V_1 = TM$ and ∇^1 is torsion-free, then we see that there exists a local parallel volume element on V_1 if and only if Ric is symmetric, where Ric is the Ricci tensor of R^1.

We set $m_1 := \mathrm{rank}V_1, m = \mathrm{rank}V$ and $m_2 := m - m_1 = \mathrm{rank}V_2$. Let $\mathrm{Im}B_x$ be a subspace of V_{2x} defined by $\mathrm{Im}B_x := \mathrm{Span}\{B_X\eta | X \in T_xM,\ \eta \in V_{1x}\}$ at $x \in M$. We denote $\bigcup_{x\in M} \mathrm{Im}B_x$ by $\mathrm{Im}B$.

Definition 2.2. If $\dim(\mathrm{Im}B_x)$ is maximal for every $x \in M$, the fundamental form B is said to be of *maximal rank*.

We see that B has maximal rank if and only if $\mathrm{rank}(\mathrm{Im}B) = nm_1$. In the case where B is symmetric, that is, $V_1 = TM$ and $B_XY = B_YX$ for $X, Y \in \Gamma(TM)$, then we see that B has maximal rank if and only if $\mathrm{rank}(\mathrm{Im}B) = \dfrac{1}{2}n(n+1)$. From now on, η always denote an element of $\Gamma(V_1)$. We now formulate our main result.

Theorem 2.2. *If $p_2 Ri_2 = 0$, B is semi-parallel and of maximal rank, then we have the following equations*:

$$S_X B_Y \eta = \frac{1}{n-1}(\mathrm{Ric}^D(X,Y)\eta + R^1_{Y,X}\eta),$$
$$S_X B_Y \eta - S_Y B_X \eta = -(\mathrm{tr}(R^D_{X,Y})\eta + nR^1_{X,Y}\eta).$$

For $n \geq 3$, in addition, if D is torsion-free, then we have

$$R^1_{X,Y}\eta = \frac{1}{n+1}(\mathrm{Ric}^D(X,Y) - \mathrm{Ric}^D(Y,X))\eta,$$
$$S_X B_Y \eta = \frac{1}{n^2-1}(\mathrm{Ric}^D(Y,X) + n\mathrm{Ric}^D(X,Y))\eta.$$

Proof. In this proof, we do not use Einstein's convention. Let $X_1, X_2, \cdots, X_n$ (resp. $\eta_1, \eta_2, \cdots, \eta_{m_1}$) be a basis of T_xM (resp. V_{1x}) and $X^1, X^2, \cdots, X^n$ (resp. $\eta^1, \eta^2, \cdots, \eta^{m_1}$) its dual basis. From Lemma 2.1, we have

$$B_{X_i}S_{X_j}B_{X_k}\eta_a = B_{X_j}S_{X_i}B_{X_k}\eta_a + B_{R^D_{X_i,X_j}X_k}\eta_a + B_{X_k}R^1_{X_i,X_j}\eta_a \tag{2.1}$$

for $1 \le i,j,k \le n$, $1 \le a \le m_1$. Let b be an index, where $1 \le b \le m_1$. Comparing the coefficient of $B_{X_i}\eta_b$ in the right hand side with that in the left hand side in (2.1), we have

$$\begin{aligned}&\eta^b(S_{X_j}B_{X_k}\eta_a)\\ &\quad = X^i(X_j)\eta^b(S_{X_i}B_{X_k}\eta_a) + X^i(R^D_{X_i,X_j}X_k)\eta^b(\eta_a) + X^i(X_k)\eta^b(R^1_{X_i,X_j}\eta_a).\end{aligned}$$

Hence we obtain

$$S_{X_j}B_{X_k}\eta_a = \delta^i_j S_{X_i}B_{X_k}\eta_a + X^i(R^D_{X_i,X_j}X_k)\eta_a + \delta^i_k R^1_{X_i,X_j}\eta_a.$$

Summing up the index i, we have

$$nS_{X_j}B_{X_k}\eta_a = S_{X_j}B_{X_k}\eta_a + \mathrm{Ric}^D(X_j,X_k)\eta_a + R^1_{X_k,X_j}\eta_a.$$

Thus we see that

$$S_XB_Y\eta = \frac{1}{n-1}(\mathrm{Ric}^D(X,Y)\eta + R^1_{Y,X}\eta), \tag{2.2}$$

$$S_XB_Y\eta - S_YB_X\eta = \frac{1}{n-1}(\mathrm{Ric}^D(X,Y)\eta - \mathrm{Ric}^D(Y,X)\eta + 2R^1_{Y,X}\eta).$$

Comparing the coefficient of $B_{X_k}\eta_b$ in the right hand side with that in the left hand side in (2.1), we have

$$S_XB_Y\eta - S_YB_X\eta = -\mathrm{tr}(R^D_{X,Y})\eta - nR^1_{X,Y}\eta. \tag{2.3}$$

Thus we have the first assertion.

If $n = 2$, then we have $\mathrm{Ric}^D(Y,X) - \mathrm{Ric}^D(X,Y) = \mathrm{tr}R^D_{X,Y}$. Combining (2.2) with (2.3) for $n \ge 3$, we have

$$-\frac{1}{n-1}(\mathrm{Ric}^D(X,Y)\eta - \mathrm{Ric}^D(Y,X)\eta + 2R^1_{Y,X}\eta) = \mathrm{tr}(R^D_{X,Y})\eta + nR^1_{X,Y}\eta.$$

If D is torsion-free, then we have

$$(n+1)R^1_{X,Y}\eta = \mathrm{Ric}^D(X,Y)\eta - \mathrm{Ric}^D(Y,X)\eta.$$

Hence we see that

$$S_XB_Y\eta = \frac{1}{n^2-1}(\mathrm{Ric}^D(Y,X) + n\mathrm{Ric}^D(X,Y))\eta.$$

□

Corollary 2.3. *If $p_2Ri_2 = 0$, B is semi-parallel, of maximal rank and in addition, $p_1Ri = 0$, then we have*

$$R^1_{X,Y}\eta = \frac{1}{n+1}(\mathrm{Ric}^D(X,Y) - \mathrm{Ric}^D(Y,X))\eta,$$

$$S_XB_Y\eta = \frac{1}{n^2-1}(\mathrm{Ric}^D(Y,X) + n\mathrm{Ric}^D(X,Y))\eta.$$

Proof. From Gauss equation and (2.2), we see that

$$R^1_{X,Y}\eta = S_X B_Y \eta - S_Y B_X \eta$$
$$= \frac{1}{n-1}(\mathrm{Ric}^D(X,Y)\eta - \mathrm{Ric}^D(Y,X)\eta + 2R^1_{Y,X}\eta).$$
□

In the case where $V_1 = TM$, we set

$$(\hat{R}_{X,Y}B)_Z := R^2_{X,Y}B_Z - B_{R^1_{X,Y}Z} - B_Z R^1_{X,Y}.$$

If $\hat{R}B = 0$, we say that B is *semi-parallel.* The following theorem specializes to Theorem 2.2 if B is symmetric.

Theorem 2.4. *We assume that B is symmetric. If $p_2 R i_2 = 0$, B is semi-parallel and of maximal rank, then we have the following equations*:

$$\begin{aligned}&nS_X B_Y Z\\ &\quad = \mathrm{tr}(S.B_Y Z)X + \mathrm{Ric}(X,Y)Z + R^1_{Z,X}Y + \mathrm{Ric}(X,Z)Y + R^1_{Y,X}Z,\\ &S_X B_Y Z + \mathrm{tr}(S_X B.Z)Y\\ &\quad = S_Y B_X Z + \mathrm{tr}(S_Y B.Z)X + \mathrm{tr}(R^1_{Y,X})Z + (n+2)R^1_{Y,X}Z,\end{aligned}$$

where $\mathrm{tr}(S.B_Y Z) = \mathrm{trace}\{X \mapsto S_X B_Y Z\}$, $\mathrm{tr}(S_Y B.Z) = \mathrm{trace}\{X \mapsto S_Y B_X Z\}$. *For $n \geq 3$, in addition, if ∇^1 is torsion-free, then we have*

$$\mathrm{Ric}\ \textit{is symmetric,}$$
$$R^1_{X,Y}Z = \frac{1}{n-1}(\mathrm{Ric}(Y,Z)X - \mathrm{Ric}(X,Z)Y).$$

Proof. We can now proceed analogously to the proof of Theorem 2.2. Let $X_1, X_2, \cdots, X_n$ be a basis of T_xM and $X^1, X^2, \cdots, X^n$ its dual basis. From Lemma 2.1, we have

$$B_{X_i} S_{X_j} B_{X_k} X_l = B_{X_j} S_{X_i} B_{X_k} X_l + (B_{R^1_{X_i,X_j}X_k} X_l + B_{X_k} R^1_{X_i,X_j} X_l) \tag{2.4}$$

for $1 \leq i,j,k,l \leq n$. Let s be an index, where $1 \leq s \leq n$. Comparing the coefficient of $B_{X_i}X_s$ in the right hand side with that in the left hand side in (2.4), we have

$$\begin{aligned}&X^s(S_{X_j}B_{X_k}X_l) + X^s(X_i)X^i(S_{X_j}B_{X_k}X_l)\\ &\quad = X^i(X_j)X^s(S_{X_i}B_{X_k}X_l) + X^s(X_j)X^i(S_{X_i}B_{X_k}X_l) + X^i(R^1_{X_i,X_j}X_k)X^s(X_l)\\ &\qquad + X^s(R^1_{X_i,X_j}X_k)X^i(X_l) + X^i(R^1_{X_i,X_j}X_l)X^s(X_k) + X^s(R^1_{X_i,X_j}X_l)X^i(X_k).\end{aligned}$$

Hence we obtain

$$\begin{aligned}&S_{X_j}B_{X_k}X_l\\ &\quad = -X^i(S_{X_j}B_{X_k}X_l)X_i + \delta^i_j S_{X_i}B_{X_k}X_l + X^i(S_{X_i}B_{X_k}X_l)X_j\\ &\qquad + X^i(R^1_{X_i,X_j}X_k)X_l + \delta^i_l R^1_{X_i,X_j}X_k + X^i(R^1_{X_i,X_j}X_l)X_k + \delta^i_k R^1_{X_i,X_j}X_l.\end{aligned}$$

Summing up the index i, we have

$$nS_{X_j}B_{X_k}X_l$$

$$= \mathrm{tr}(S.B_{X_k}X_l)X_j + \mathrm{Ric}(X_j, X_k)X_l + R^1_{X_l,X_j}X_k + \mathrm{Ric}(X_j, X_l)X_k + R^1_{X_k,X_j}X_l.$$

Thus we have

$$\begin{aligned}(2.5)\quad & nS_XB_YZ \\ &= \mathrm{tr}(S.B_YZ)X + \mathrm{Ric}(X, Y)Z + R^1_{Z,X}Y + \mathrm{Ric}(X, Z)Y + R^1_{Y,X}Z.\end{aligned}$$

Comparing the coefficient of $B_{X_k}X_s$ in the right hand side with that in the left hand side in (2.4), we have

$$\begin{aligned}(2.6)\quad & S_XB_YZ + \mathrm{tr}(S_XB.Z)Y \\ &= S_YB_XZ + \mathrm{tr}(S_YB.Z)X + \mathrm{tr}(R^1_{Y,X})Z + (n + 2)R^1_{Y,X}Z.\end{aligned}$$

Thus we have the first assertion.

The trace of (2.6) by X (resp. Z) are the following:

$$\begin{aligned}(2.7)\qquad \mathrm{tr}(S.B_YZ) &= n\mathrm{tr}(S_YB.Z) - (n + 3)\mathrm{Ric}(Y, Z) + \mathrm{Ric}(Z, Y), \\ \mathrm{tr}(S_XB_Y\cdot) &= \mathrm{tr}(S_YB_X\cdot) + (n + 1)\mathrm{tr}(R^1_{Y,X}).\end{aligned}$$

Now we assume that ∇^1 is torsion-free. Since $S.B_YZ = S.B_ZY$, we have

$$\begin{aligned} n(\mathrm{tr}(S_YB.Z) - \mathrm{tr}(S_ZB.Y)) &= -(n + 4)(\mathrm{Ric}(Z, Y) - \mathrm{Ric}(Y, Z)) \\ &= -n(n + 1)\mathrm{tr}(R^1_{Y,Z}) = -n(n + 1)(\mathrm{Ric}(Z, Y) - \mathrm{Ric}(Y, Z)).\end{aligned}$$

If $n = 2$, then the previous equation is trivial. If $n \geq 3$, then we see that Ric is symmetric. We consider the case $n \geq 3$. By using (2.5)–(2.7) and first Bianchi identity, we have the following equations:

$$\begin{aligned} & n(S_XB_YZ - S_YB_XZ) \\ &= (\mathrm{tr}(S.B_YZ) - \mathrm{Ric}(Y, Z))X - (\mathrm{tr}(S.B_XZ) - \mathrm{Ric}(X, Z))Y + 3R^1_{Y,X}Z \\ &= (n\mathrm{tr}(S_YB.Z) - (n+3)\mathrm{Ric}(Y, Z))X - (n\mathrm{tr}(S_XB.Z) - (n+3)\mathrm{Ric}(X, Z))Y \\ &\quad + 3R^1_{Y,X}Z \\ &= n(\mathrm{tr}(S_YB.Z)X - \mathrm{tr}(S_XB.Z)Y + (n + 2)R^1_{Y,X}Z).\end{aligned}$$

Thus we have

$$\mathrm{Ric}(Y, Z)X - \mathrm{Ric}(X, Z)Y = (n - 1)R^1_{X,Y}Z.$$

□

We can now state the analogue of (0.1).

Corollary 2.5. *We assume that B is symmetric. For $n \geq 3$, if $p_2Ri_2 = 0$, B is semi-parallel, of maximal rank, ∇^1 is torsion-free and in addition, $p_1Ri = 0$, then we have*

$$S_XB_YZ = \frac{1}{n - 1}(\mathrm{Ric}(X, Y)Z + \mathrm{Ric}(X, Z)Y + 2\mathrm{Ric}(Y, Z)X).$$

Proof. From Gauss equation and (2.5), we have

$$nR^1_{X,Y}Z = n(S_XB_YZ - S_YB_XZ)$$

$$= (\mathrm{tr}(S.B_Y Z) - \mathrm{Ric}(Y,Z))X - (\mathrm{tr}(S.B_X Z) - \mathrm{Ric}(X,Z))Y + 3R^1_{Y,X}Z.$$

By these equations and Theorem 2.4, we have

$$(n+3)R^1_{Y,X}Z = \frac{n+3}{n-1}(\mathrm{Ric}(X,Z)Y - \mathrm{Ric}(Y,Z)X)$$
$$= (-\mathrm{tr}(S.B_Y Z) + \mathrm{Ric}(Y,Z))X - (-\mathrm{tr}(S.B_X Z) + \mathrm{Ric}(X,Z))Y.$$

Hence we obtain

$$\mathrm{tr}(S.B_Y Z) = \frac{2(n+1)}{n-1}\mathrm{Ric}(Y,Z).$$

Substituting this equation to (2.5), we have the assertion. □

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Ryuji Ishii
Department of Mathematics, Faculty of Science, Tokyo University of Science
26 Wakamiya-cho, Shinjuku-ku Tokyo 162-0827, Japan