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Order and radius of $(2k-1)$-connected graphs

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Abstract. We show that if k is an integer with $k \geq 3$ and G is a $(2k-1)$-connected graph with radius r, then $|V(G)| \geq 2kr - 2k - 2$.

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§1. Introduction

By a graph, we mean a finite, undirected, simple graph without loops or multiple edges. Let G be a graph. Let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. For $v, w \in V(G)$, let $d_G(v,w)$ denote the usual distance between v and w. Set

$$r(G) := \min_{v \in V(G)} \max_{w \in V(G)} d_G(v,w).$$

The number $r(G)$ is called the radius of G. A vertex $z \in V(G)$ is called a central vertex of G if $\max_{w \in V(G)} d_G(z,w) = r(G)$.

In [3], Harant and Walter proved that there is a constant $C > 0$ such that every 3-connected graph G with radius r satisfies $|V(G)| + C \log |V(G)| > 4r$. Subsequently it was proved by Harant in [2] and by Inoue in [5] that if G is a 3-connected graph with radius r, then $|V(G)| \geq 4r - 15$. In [1], Egawa and Inoue proved a more general result that if k is an integer with $k \geq 2$ and G is a $(2k-1)$-connected graph with radius r, then $|V(G)| \geq 2kr - 2k - 9$. The purpose of this paper is to prove the following refinement of the result of Egawa and Inoue.

Theorem. *Let $r \geq 1$, $k \geq 3$ be integers, and let G be a $(2k-1)$-connected graph with radius r. Then $|V(G)| \geq 2kr - 2k - 2$.*

As is remarked in [1], the following graph shows that the bound $2kr-2k-2$ in the Theorem is nearly best possible. Let $r \geq 2$ and $k \geq 3$. Let C be a cycle of order $2kr-2k+2$, and define a graph G by $V(G) = V(C)$ and $E(G) = \{vw | v, w \in V(C), v \neq w, d_C(v,w) \leq k\}$. Then G is $2k$-connected (so $(2k-1)$-connected) and $r(G) = r$.

In passing, we mention that for 3-connected graphs, it has recently been proved in [4] that every 3-connected graph G with radius r satisfies $|V(G)| \geq 4r-4$.

The organization of the paper is as follows. Section 2 contains preliminary lemmas. We prove the key proposition, Proposition 1, in Sections 3 through 5. We complete the proof of the Theorem in Section 6.

§2. Preliminary results

Throughout the rest of the paper, we let G, r be as in the Theorem. If $r \leq 2$, we clearly have $|V(G)| \geq 2k \geq 2k(r-1)-2$. Thus we may assume $r \geq 3$. For a vertex $v \in V(G)$ and a nonnegative integer i, let

$$N_i(v) := \{w | w \in V(G), d_G(v,w) = i\}.$$

We write $N(v)$ for $N_1(v)$. Fix a central vertex z, and let $X_i := N_i(z)$ for $0 \leq i \leq r$. Note that for each i with $1 \leq i \leq r-1$, we have $N(w) \subset X_{i-1} \cup X_i \cup X_{i+1}$ for every $w \in X_i$.

Lemma 1. *Let* $1 \leq i \leq r-1$. *Then* $|\{y \in X_i | N(y) \cap X_{i+1} \neq \phi\}| \geq 2k-1$.

Proof. Since $G - \{y \in X_i | N(y) \cap X_{i+1} \neq \phi\}$ is disconnected, the desired conclusion follows from the assumption that G is $(2k-1)$-connected. □

The following two lemmas immediately follow from Lemma 1.

Lemma 2. $|X_i| \geq 2k-1$ *for each* $1 \leq i \leq r-1$.

Lemma 3. *Let* $1 \leq i \leq r-1$, *and suppose that* $|X_i| = 2k-1$. *Then* $N(y) \cap X_{i+1} \neq \phi$ *for every* $y \in X_i$.

Lemma 4. $|V(G)| \geq |X_{r-1} \cup X_r| + (2k-1)r - (4k-3)$.

Proof. By Lemma 2, $|V(G)| = \sum_{i=0}^{r} |X_i| \geq 1 + (2k-1)(r-2) + |X_{r-1} \cup X_r| = |X_{r-1} \cup X_r| + (2k-1)r - (4k-3)$. □

Lemma 5. *Suppose that* $|X_{r-1} \cup X_r| \leq 2k+2$. *Then one of the following holds:*

(1) *there exists* $v_0 \in X_{r-1}$ *such that* $d_G(v_0, v) = 1$ *for every* $v \in X_r$; *or*

(2) *there exists* $v_0 \in X_{r-1}$ *such that* $d_G(v_0, v) \leq 2$ *for every* $v \in X_{r-1} \cup X_r$.

Proof. Note that $|X_{r-1}| \geq 2k-1$ by Lemma 2. Hence $|X_r| \leq 3$ by the assumption of the lemma. Write $X_r = \{v_1, v_2, \dots, v_{|X_r|}\}$. If $|X_r| = 1$, then (1) holds with $v_0 \in N(v_1) \cap X_{r-1}$. Assume for the moment that $|X_r| = 2$. Then $2k-1 \leq |X_{r-1}| \leq 2k$. By the assumption that G is $(2k-1)$-connected, we have $|N(v_i) \cap X_{r-1}| \geq |N(v_i)| - (|X_r| - 1) \geq (2k-1) - 1 = 2k-2$ for each $i = 1, 2$. Hence $|N(v_1) \cap N(v_2) \cap X_{r-1}| \geq |N(v_1) \cap X_{r-1}| + |N(v_2) \cap X_{r-1}| - |X_{r-1}| \geq 2(2k-2) - 2k > 0$. Consequently (1) holds with $v_0 \in N(v_1) \cap N(v_2) \cap X_{r-1}$. Thus we may assume $|X_r| = 3$. Then $|X_{r-1}| = 2k-1$, and $|N(v_i) \cap X_{r-1}| \geq 2k-3$ for every $1 \leq i \leq 3$. If $|N(v_i) \cap N(v_j) \cap X_{r-1}| \geq |N(v_i) \cap X_{r-1}| - 1$ for every i, j, then $|N(v_1) \cap N(v_2) \cap N(v_3) \cap X_{r-1}| \geq |N(v_1) \cap N(v_2) \cap X_{r-1}| + |N(v_1) \cap N(v_3) \cap X_{r-1}| - |N(v_1) \cap X_{r-1}| \geq 2(|N(v_1) \cap X_{r-1}| - 1) - |N(v_1) \cap X_{r-1}| = |N(v_1) \cap X_{r-1}| - 2 \geq 2k-5 > 0$, and hence (1) holds with $v_0 \in N(v_1) \cap N(v_2) \cap N(v_3) \cap X_{r-1}$. Thus we may assume that some two of the v_i, say v_1 and v_2, satisfy $|N(v_1) \cap N(v_2) \cap X_{r-1}| \leq |N(v_1) \cap X_{r-1}| - 2$. Since $|X_{r-1}| = 2k-1$ and $|N(v_2) \cap X_{r-1}| \geq 2k-3$, this forces

$$|N(v_2) \cap X_{r-1}| = 2k-3, \tag{2.1}$$

$$(N(v_1) \cap X_{r-1}) \cup (N(v_2) \cap X_{r-1}) = X_{r-1}, \tag{2.2}$$

and $|N(v_1) \cap N(v_2) \cap X_{r-1}| = |N(v_1) \cap X_{r-1}| - 2 > 0$. Take $v_0 \in N(v_1) \cap N(v_2) \cap X_{r-1}$. Then by (2.2), $d_G(v_0, v) \leq 2$ for every $v \in X_{r-1}$. Further since $|N(v_2)| \geq 2k-1$, it follows from (2.1) that $v_1, v_3 \in N(v_2)$, and hence $d_G(v_0, v) \leq 2$ for every $v \in X_r$. Therefore (2) holds. This completes the proof of the lemma. □

§3. Statement of key proposition

We continue with the notation of the preceding section. The bulk of the proof of the Theorem is devoted to the verification of the following proposition, which roughly says that the average of the $|X_i|$ is only slightly less than $2k$, if it is less than $2k$.

Proposition 1. *Let* a, b *be integers with* $a + 2 \leq b$*, and suppose that* $|X_a| = |X_b| = 2k-1$ *and* $|X_i| > 2k-1$ *for each* $a+2 \leq i \leq b-1$.

(1) *Suppose that* $r \geq 9$, $a \geq 4$ *and* $b \leq r-3$. *Then* $\sum_{i=a}^{b-1} |X_i| \geq 2k(b-a)$.

(2) *Suppose that* $r \geq 7$, $a \geq 3$, $b \leq r-2$ *and* $|X_{r-1} \cup X_r| \leq 2k+2$. *Then* $\sum_{i=a}^{b-1} |X_i| \geq 2k(b-a)$.

(3) *Suppose that* $r \geq 6$, $a \geq 3$, $b \leq r-1$, $|X_{a-1}| \leq 2k+2$ *and* $|X_{r-1} \cup X_r| = 2k$. *Then one of the following holds:*

(i) $\sum_{i=a}^{b-1} |X_i| \geq 2k(b-a)$; *or*

(ii) $b = r-1$, $|X_{a-1}| = 2k+2$ *and* $\sum_{i=a}^{b-1} |X_i| \geq 2k(b-a)-1$.

Proof. We prove (1), (2) and (3) simultaneously. Suppose that $\sum_{i=a}^{b-1} |X_i| < 2k(b-a)$. Then one of the following two situations must occur:

(A) $|X_i| = 2k$ for each $a+1 \leq i \leq b-1$; or

(B) $|X_{a+1}| = 2k-1$, and $|X_i| = 2k$ or $2k+1$ for each $a+2 \leq i \leq b-1$, and the number of X_i with $|X_i| = 2k+1$ is at most one.

We aim at showing that either we get a contradiction, or (3)(ii) holds. We introduce a graph structure $\mathscr{G}$ on X_{a+1} by joining v and w if and only if $d_G(v,w) \leq 2$ and $v \neq w$. Let α denote the independence number of $\mathscr{G}$.

Claim 3.1. $\alpha \leq 2$.

Proof. Suppose that $\alpha \geq 3$. Then there exist $v_1, v_2, v_3 \in X_{a+1}$ such that $d_G(v_i, v_j) \geq 3$ for all $1 \leq i < j \leq 3$. This implies that $(\{v_i\} \cup N(v_i)) \cap (\{v_j\} \cup N(v_j)) = \phi$ for all $1 \leq i < j \leq 3$. On the other hand, whether (A) holds or (B) holds, $|X_{a+1}|+|X_{a+2}| \leq 4k$. Since $|N(v_i)| \geq 2k-1$ for each i by the assumption that G is $(2k-1)$-connected, we now obtain $6k \leq \sum_{1\leq i\leq 3} |\{v_i\} \cup N(v_i)| = |\cup_{1\leq i\leq 3} (\{v_i\} \cup N(v_i))| \leq |X_a \cup X_{a+1} \cup X_{a+2}| \leq (2k-1)+4k = 6k-1$, a contradiction. □

In the rest of this section, we consider the case where $\mathscr{G}$ is connected.

Case 1. *$\mathscr{G}$ is connected.*

Claim 3.2. *Let $w' \in X_{a+1}$. Then there exists $x \in X_{a+1}$ such that $d_{\mathscr{G}}(x, w') \leq 1$, and $d_{\mathscr{G}}(x,w) \leq 2$ for every $w \in X_{a+1}$.*

Proof. If $d_{\mathscr{G}}(w', w) \leq 1$ for every $w \in X_{a+1}$, then the desired conclusion holds with $x = w'$. Thus we may assume that there exists $u \in X_{a+1}$ such that $d_{\mathscr{G}}(u, w') = 2$. Let v be a vertex adjacent to both u and w' in $\mathscr{G}$. Since $\alpha \leq 2$ by Claim 3.1, each vertex in $X_{a+1} - \{u, v, w'\}$ is adjacent to u or w' in $\mathscr{G}$. This means that $d_{\mathscr{G}}(v,w) \leq 2$ for every $w \in X_{a+1}$, and hence the desired conclusion holds with $x = v$. □

Claim 3.3. *Let $w' \in X_{a+1}$. Then there exists $x \in X_{a+1}$ such that $d_G(x, w') \leq 2$, and $d_G(x,w) \leq 4$ for every $w \in X_{a+1}$.*

Proof. Since $d_G(u,v) \leq 2d_{\mathscr{G}}(u,v)$ for every $u, v \in X_{a+1}$ by the definition of $\mathscr{G}$, this follows from Claim 3.2. □

Claim 3.4. *Suppose that $a = r - 3$. Then one of the following holds:*

(0) $|X_{r-4}| = 2k + 2$ *and* $|X_{r-3}| + |X_{r-2}| = 4k - 1$; *or*

(1) *there exists* $x \in X_{r-3}$ *such that* $d_G(x, y) \leq 5$ *for every* $y \in \cup_{r-3 \leq i \leq r} X_i$.

Proof. Note that $b = r - 1$. Hence

$$|X_{r-3}| + |X_{r-2}| \leq 4k - 1. \tag{3.1}$$

By the assumptions of the proposition, we also have

$$|X_{r-4}| \leq 2k + 2,\ |X_{r-1}| = 2k - 1,\ |X_r| = 1. \tag{3.2}$$

Suppose that (1) does not hold. Under this assumption, we first prove the following subclaims.

Subclaim 1. $d_G(w, v) \leq 3$ *for every* $w \in X_{r-2}$ *and every* $v \in X_{r-1} \cup X_r$.

Proof. Suppose that there exist $w \in X_{r-2}$ and $v \in X_{r-1} \cup X_r$ such that $d_G(w, v) \geq 4$. Then $(\{w\} \cup N(w)) \cap (\{v\} \cup N(v) \cup N_2(v)) = \phi$. Since $|N(w)| \geq 2k - 1$, $|N(v)| \geq 2k - 1$ and $|N_2(v)| \geq 2k - 1$ by the assumption that G is $(2k - 1)$-connected, we obtain $6k - 1 \leq |\{w, v\} \cup N(w) \cup N(v) \cup N_2(v)| \leq |\cup_{r-3 \leq i \leq r} X_i|$. In view of (3.1) and (3.2), this implies that $\cup_{r-3 \leq i \leq r} X_i = \{w, v\} \cup N(w) \cup N(v) \cup N_2(v)$. Since $\mathscr{G}$ is connected, it follows that there exist $w_1 \in (\{w\} \cup N(w)) \cap X_{r-2}$ and $v_1 \in (N(v) \cup N_2(v)) \cap X_{r-2}$ such that $d_G(w_1, v_1) \leq 2$. Now let $x \in N(v_1) \cap X_{r-3}$. Take $y \in \cup_{r-3 \leq i \leq r} X_i$. If $y \in \{w\} \cup N(w)$, $d_G(x, y) \leq d_G(x, v_1) + d_G(v_1, w_1) + d_G(w_1, w) + d_G(w, y) \leq 5$; if $y \in \{v\} \cup N(v) \cup N_2(v)$, $d_G(x, y) \leq d_G(x, v_1) + d_G(v_1, v) + d_G(v, y) \leq 5$. Thus (1) holds, which contradicts the assumption that (1) does not hold. □

Subclaim 2. $d_G(w, w') \leq 4$ *for every* $w, w' \in X_{r-2}$.

Proof. We may assume $w \neq w'$. Then since $|X_{r-2}| \leq 2k$, it follows from Lemma 1 that one of w or w', say w', satisfies $N(w') \cap X_{r-1} \neq \phi$. Take $v \in N(w') \cap X_{r-1}$. Then by Subclaim 1, $d_G(w, w') \leq d_G(w, v) + d_G(v, w') \leq 3 + 1 = 4$. □

Subclaim 3. *Let* $u \in X_{r-3}$. *Then the following hold.*

(1) *There exists* $u' \in X_{r-3}$ *such that* $d_G(u, u') \geq 6$.

(2) *There exists* $w' \in X_{r-2}$ *such that* $d_G(u, w') \geq 5$.

Proof. By Lemma 3, $N(u) \cap X_{r-2} \neq \phi$. Take $w \in N(u) \cap X_{r-2}$. By the assumption that (1) of the statement of Claim 3.4 does not hold, there exists $u' \in \cup_{r-3 \leq i \leq r} X_i$ such that $d_G(u, u') \geq 6$. If $u' \in \cup_{r-2 \leq i \leq r} X_i$, then by Subclaims 1 and 2, $d_G(u, u') \leq d_G(u, w) + d_G(w, u') \leq 1 + 4 = 5$, a contradiction. Thus $u' \in X_{r-3}$ and, if we let $w' \in N(u') \cap X_{r-2}$, we get $d_G(u, w') \geq d_G(u, u') - d_G(w', u') \geq 6 - 1 = 5$. □

Subclaim 4. *There exist $w, w' \in X_{r-2}$ such that $d_{\mathscr{G}}(w, w') = 2$ and $d_G(w, w') = 4$.*

Proof. Let $w \in X_{r-2}$ be a central vertex of $\mathscr{G}$. Take $u \in N(w) \cap X_{r-3}$. By Subclaim 3(2), there exists $w' \in X_{r-2}$ such that $d_G(u, w') \geq 5$. Then $d_G(w, w') \geq d_G(u, w') - d_G(u, w) \geq 4$. On the other hand, since Claim 3.2 implies that the radius of $\mathscr{G}$ is at most 2, we have $d_{\mathscr{G}}(w, w') \leq 2$. Since $d_G(w, w') \leq 2d_{\mathscr{G}}(w, w')$, this forces $d_{\mathscr{G}}(w, w') = 2$ and $d_G(w, w') = 4$. □

Subclaim 5. *Suppose that there exist $w, w' \in X_{r-2}$ and $u \in X_{r-3}$ such that $d_G(w, w') \geq 3$ and $d_G(w, u) = d_G(w', u) = 3$. Then (0) holds.*

Proof. By Subclaim 3(1), there exists $u' \in X_{r-3}$ such that $d_G(u, u') \geq 6$. Then $d_G(w, u') \geq d_G(u, u') - d_G(u, w) \geq 3$ and $d_G(w', u') \geq d_G(u, u') - d_G(u, w') \geq 3$. Since $|N(w)|, |N(w')|, |N(u)|, |N(u')| \geq 2k-1$, we obtain $8k \leq |\{w, w', u, u'\} \cup N(w) \cup N(w') \cup N(u) \cup N(u')| \leq |\cup_{r-4 \leq i \leq r-1} X_i| \leq 8k$ by (3.1) and (3.2). This implies that equality holds in (3.1) and (3.2), and hence (0) holds. □

We return to the proof of the claim. By Subclaim 4, there exist $w_0, w_1, w_2 \in X_{r-2}$ such that $d_G(w_0, w_1) = d_G(w_1, w_2) = 2$ and $d_G(w_0, w_2) = 4$. Take $u \in N(w_1) \cap X_{r-3}$. If $d_G(w_0, u) = d_G(w_2, u) = 3$, then by Subclaim 5, (0) holds. Thus by symmetry, we may assume $d_G(w_0, u) \leq 2$. Take $y \in N(w_1) \cap N(w_2)$. Then $d_G(y, u) \leq 2$. Since $d_G(w_0, w_1) = 2$ and $d_G(w_0, w_2) = 4$, we also get $d_G(w_0, y) = 3$. By Subclaim 3(2), there exists $w_3 \in X_{r-2}$ such that $d_G(w_3, u) \geq 5$. Since $d_G(w_0, u) \leq 2$ and $d_G(y, u) \leq 2$, we have $d_G(w_3, w_0) \geq 3$ and $d_G(w_3, y) \geq 3$. Now if $y \in X_{r-2} \cup X_{r-1}$, then $N(w_0) \cup N(w_3) \cup N(y) \subseteq \cup_{r-3 \leq i \leq r} X_i$, and hence $6k \leq |\{w_0, w_3, y\} \cup N(w_0) \cup N(w_3) \cup N(y)| \leq |\cup_{r-3 \leq i \leq r} X_i| \leq 6k - 1$ by (3.1) and (3.2), a contradiction. Thus $y \in X_{r-3}$. On the other hand, since $d_G(w_0, w_2) = 4 \geq 3$ and $d_G(w_3, w_0) \geq 3$, we obtain $d_G(w_3, w_2) \leq 2$ by Claim 3.1. Since $y \in N(w_2)$ and $d_G(w_3, y) \geq 3$, this forces $d_G(w_3, y) = 3$. Therefore applying Subclaim 5 to w_0, w_3 and y, we see that (0) holds. ■

Having Claims 3.3 and 3.4 in mind, we divide the rest of the proof for Case 1 into four cases, Cases 1.1 through 1.4. Except in Case 1.3, we derive a contradiction by showing that there exists $u \in V(G)$ such that $d_G(u, v) < r$ for every $v \in V(G)$.

Case 1.1. *$a \geq 4$ and $b \leq r-3$.*

We have $a \leq r-5$. By Claim 3.3, there exists $x \in X_{a+1}$ such that $d_G(x,w) \leq 4$ for every $w \in X_{a+1}$. Let u be a vertex in X_5 which is on a shortest $z-x$ path. Take $v \in V(G)$, and let $v \in X_i$. If $0 \leq i \leq a-1$, $d_G(u,v) \leq d_G(u,z)+d_G(z,v) = 5+i \leq 5+(a-1) = a+4 \leq (r-5)+4 = r-1 < r$. Thus we may assume $a \leq i \leq r$. First assume $i = a$. By Lemma 3, $N(v) \cap X_{a+1} \neq \phi$. Take $w \in N(v) \cap X_{a+1}$. Then $d_G(x,w) \leq 4$, and hence $d_G(u,v) \leq d_G(u,x)+d_G(x,w)+d_G(w,v) \leq \{(a+1)-5\}+4+1 = a+1 \leq (r-5)+1 = r-4 < r$. Next assume $a+1 \leq i \leq r$. Let w be a vertex in X_{a+1} which is on a shortest $z-v$ path. Then $d_G(x,w) \leq 4$, and hence $d_G(u,v) \leq d_G(u,x)+d_G(x,w)+d_G(w,v) \leq \{(a+1)-5\}+4+\{i-(a+1)\} = i-1 \leq r-1 < r$.

Case 1.2. *$a \leq r-4$, and either $a = 3$ and $b \leq r-1$ or $a \geq 4$ and $r-2 \leq b \leq r-1$.*

By the assumptions of the proposition, we have $|X_{r-1} \cup X_r| \leq 2k+2$. By Lemma 5, there exists $v_0 \in X_{r-1}$ such that $d_G(v_0,v') \leq 2$ for every $v' \in X_r$. Let w' be a vertex in X_{a+1} which is on a shortest $z-v_0$ path. By Claim 3.3, there exists $x \in X_{a+1}$ such that $d_G(x,w') \leq 2$, and $d_G(x,w) \leq 4$ for every $w \in X_{a+1}$. Let u be a vertex in X_4 which is on a shortest $z-x$ path. Take $v \in V(G)$, and let $v \in X_i$. If $0 \leq i \leq a-1$, $d_G(u,v) \leq d_G(u,z)+d_G(z,v) = 4+i \leq 4+(a-1) = a+3 \leq (r-4)+3 = r-1 < r$. If $i = a$, then taking $w \in N(v) \cap X_{a+1}$ (see Lemma 3), we get $d_G(x,w) \leq 4$, and hence $d_G(u,v) \leq d_G(u,x)+d_G(x,w)+d_G(w,v) \leq \{(a+1)-4\}+4+1 = a+2 \leq (r-4)+2 = r-2 < r$. If $a+1 \leq i \leq r-1$, then letting w be a vertex in X_{a+1} which is on a shortest $z-v$ path, we get $d_G(x,w) \leq 4$, and hence $d_G(u,v) \leq d_G(u,x)+d_G(x,w)+d_G(w,v) \leq \{(a+1)-4\}+4+\{i-(a+1)\} = i \leq r-1 < r$. If $i = r$, then letting v_0, w' be as above, we obtain $d_G(u,v) \leq d_G(u,x)+d_G(x,w')+d_G(w',v_0)+d_G(v_0,v) \leq \{(a+1)-4\}+2+\{(r-1)-(a+1)\}+2 = r-1 < r$.

Case 1.3. *$a = r-3$ and Claim 3.4(0) holds.*

We have $b = r-1$, which means that the conditions in (1) and (2) of the proposition are not satisfied. Further $\sum_{i=a}^{b-1} |X_i| = |X_{r-3}|+|X_{r-2}| = 4k-1 = 2k(b-a)-1$, and therefore (3)(ii) holds.

Case 1.4. *$a = r-3$ and Claim 3.4(1) holds.*

Let x be as in Claim 3.4(1). Let u be a vertex in X_3 which is on a shortest $z-x$ path. Take $v \in V(G)$, and let $v \in X_i$. If $0 \leq i \leq r-4$, $d_G(u,v) \leq d_G(u,z)+d_G(z,v) = 3+i \leq 3+(r-4) = r-1 < r$. If $r-3 \leq i \leq r$, then $d_G(x,v) \leq 5$ by Claim 3.4(1), and hence $d_G(u,v) \leq d_G(u,x)+d_G(x,v) \leq \{(r-3)-3\}+5 = r-1 < r$.

This concludes the discussion for the case where $\mathscr{G}$ is connected.

§4. The case where $\mathscr{G}$ is disconnected

In this section and the next section, we consider the case where $\mathscr{G}$ is disconnected. The main results of this section are Claims 4.13 through 4.16, which correspond to Claims 3.3 and 3.4 in Case 1.

Case 2. *$\mathscr{G}$ is disconnected.*

By Claim 3.1, $\mathscr{G}$ consists of two components. Let S_{a+1} and T_{a+1} be the vertex sets of the components of $\mathscr{G}$. For j with $a \leq j \leq b$, set

$$S_j = (\cup_{v \in S_{a+1}} N_{|a+1-j|}(v)) \cap X_j,$$
$$T_j = (\cup_{w \in T_{a+1}} N_{|a+1-j|}(w)) \cap X_j.$$

Since $S_{a+1} \cup T_{a+1} = X_{a+1}$, it immediately follows from the definition of S_j and T_j that $S_j \cup T_j = X_j$ for each $a+1 \leq j \leq b$. Applying Lemma 3 with $i=a$, we also see that $S_a \cup T_a = X_a$. Thus $S_j \cup T_j = X_j$ for each $a \leq j \leq b$. Also since $d_G(v,w) \geq 3$ for every $v \in S_{a+1}$ and every $w \in T_{a+1}$, $S_j \cap T_j = \phi$ for each $a \leq j \leq a+2$.

Claim 4.1. *$|S_a| \geq k-1$ and $|T_a| \geq k-1$.*

Proof. By way of contradiction, suppose that $|S_a| \leq k-2$. Then since $G-(S_a \cup T_{a+2})$ is disconnected, $|T_{a+2}| \geq k+1$ by the assumption that G is $(2k-1)$-connected, and hence $|S_{a+2}| = |X_{a+2} - T_{a+2}| \leq (2k+1)-(k+1) = k$. But since $G-(S_a \cup S_{a+2})$ is also disconnected, this contradicts the assumption that G is $(2k-1)$-connected. Thus $|S_a| \geq k-1$. We can prove $|T_a| \geq k-1$ in exactly the same way. □

In view of Claim 4.1, we may assume $|S_a| = k-1$ and $|T_a| = k$.

Claim 4.2. *Let $a+1 \leq i \leq b$, and suppose that for each h with $a+1 \leq h \leq i-1, d_G(v,w) \geq 3$ for every $v \in S_h$ and every $w \in T_h$. Then the following hold.*

(1) (a) $|S_i| \geq k-1$. (b) *If $i \geq a+2, |S_i| \geq k$.*

(2) $|T_i| \geq k$.

Proof. From the assumption of Claim 4.2, it follows that $G-(S_i \cup T_a)$ is disconnected, and hence (1)(a) follows from the assumption that G is $(2k-1)$-connected. Similarly, $G-(S_a \cup T_i)$ is disconnected and, in the case where $i \geq a+2, G-(S_a \cup S_i)$ is also disconnected, and hence (1)(b) and (2) also follow from the assumption that G is $(2k-1)$-connected. □

We define an integer c as follows. Set

$$Q := \{i | a+1 \leq i \leq b-1, \text{ there exists } w_1 \in S_i \text{ and there exists } w_2 \in T_i \text{ such that } d_G(w_1, w_2) \leq 2\}.$$

We have $Q \neq \phi$ because if $Q = \phi$, then $|X_b| = |S_b| + |T_b| \geq k + k = 2k$ by Claim 4.2, which contradicts the assumption that $|X_b| = 2k-1$. Now set

$$c = \min Q.$$

Then $a+2 \leq c \leq \max Q \leq b-1$ by the definition of S_{a+1} and T_{a+1}. The following remarks follow from the definition of c.

Remark. *For each $a \leq i \leq c$, we have $X_i - S_i = T_i$.*

Remark. *Let $a+1 \leq i \leq c-1$. Then $N(v) \subset S_{i-1} \cup S_i \cup S_{i+1}$ for every $v \in S_i$, and $N(w) \subset T_{i-1} \cup T_i \cup T_{i+1}$ for every $w \in T_i$.*

The following two claims immediately follow from Claim 4.2.

Claim 4.3.

(1) *If* (A) *holds, then $|S_{a+1}| = k-1$ or k, and $|S_i| = k$ for each $a+2 \leq i \leq c$.*

(2) *If* (B) *holds, then $|S_{a+1}| = k-1, |S_i| = k$ or $k+1$ for each $a+2 \leq i \leq c$, and the number of those indices i with $a+2 \leq i \leq c$ for which $|S_i| = k+1$ is at most one.*

Claim 4.4.

(1) *If* (A) *holds, then $|T_{a+1}| = k$ or $k+1$, and $|T_i| = k$ for each $a+2 \leq i \leq c$.*

(2) *If* (B) *holds, then $|T_i| = k$ or $k+1$ for each $a+1 \leq i \leq c$, and the number of those indices i with $a+1 \leq i \leq c$ for which $|T_i| = k+1$ is at most one.*

Claim 4.5. $|S_{i-1} \cup S_i \cup S_{i+1}| \leq 3k+1$ *for each $a+1 \leq i \leq c-1$.*

Proof. Since Claim 4.3 implies that $|S_i| \leq k+1$ for each $a \leq i \leq c$, and that the number of indices i with $a \leq i \leq c$ such that $|S_i| = k+1$ is at most one, the desired inequality follows immediately. □

Claim 4.6. $|T_{i-1} \cup T_i \cup T_{i+1}| \leq 3k+1$ *for each $a+1 \leq i \leq c-1$.*

Proof. Since Claim 4.4 implies that $|T_i| \le k+1$ for each $a \le i \le c$, and that the number of indices i with $a \le i \le c$ such that $|T_i| = k+1$ is at most one, the desired inequality follows immediately. □

Claim 4.7. *Let $a+1 \le i \le c-1$. Then the following hold.*

(1) $d_G(v, v') \le 2$ *for every* $v, v' \in S_i$.

(2) $d_G(w, w') \le 2$ *for every* $w, w' \in T_i$.

Proof. Suppose that there exist $v, v' \in S_i$ such that $d_G(v, v') \ge 3$. Then $(\{v\} \cup N(v)) \cap (\{v'\} \cup N(v')) = \phi$. Since $|N(v)| \ge 2k-1$ and $|N(v')| \ge 2k-1$ by the assumption that G is $(2k-1)$-connected, it follows from Claim 4.5 that $4k \le |\{v, v'\} \cup N(v) \cup N(v')| \le |S_{i-1} \cup S_i \cup S_{i+1}| \le 3k+1$, a contradiction. Thus (1) is proved. We can prove (2) in exactly the same way by using Claim 4.6 in place of Claim 4.5. □

Claim 4.8. *Let $a \le i < j \le c$. Then the following hold.*

(1) $d_G(w, v) \le j - i + 2$ *for every* $w \in S_j$ *and every* $v \in S_i$.

(2) $d_G(w, v) \le j - i + 2$ *for every* $w \in T_j$ *and every* $v \in T_i$.

Proof. Let $w \in T_j$ and $v \in T_i$. If $a+1 \le i \le c-1$, then letting v' be a vertex in T_i which is on a shortest $z-w$ path, we get $d_G(v', v) \le 2$ by Claim 4.7(2), and hence $d_G(w, v) \le d_G(w, v') + d_G(v', v) \le j - i + 2$. Thus we may assume $i = a$. Take $v'' \in N(v) \cap T_{a+1}$ (see Lemma 3). Let v''' be a vertex in T_{a+1} which is on a shortest $z-w$ path. Then $d_G(v''', v'') \le 2$ by Claim 4.7(2), and hence $d_G(w, v) \le d_G(w, v''') + d_G(v''', v'') + d_G(v'', v) \le \{j - (a+1)\} + 2 + 1 = j - a + 2$. This proves (2). Note that we did not make use of the assumption that $|S_a| = k-1$ and $|T_a| = k$ in the proof of (2). Thus (1) similarly follows from Claim 4.7(1). □

Letting $i = c-1$ and $j = c$ in Claim 4.8, we obtain the following claim.

Claim 4.9.

(1) $d_G(u, v) \le 3$ *for every* $u \in S_{c-1}$ *and every* $v \in S_c$.

(2) $d_G(u, v) \le 3$ *for every* $u \in T_{c-1}$ *and every* $v \in T_c$.

Claim 4.10.

(1) $d_G(w, w') \le 4$ *for every* $w, w' \in S_c$.

(2) $d_G(w, w') \le 4$ *for every* $w, w' \in T_c$.

Proof. Let $w, w' \in S_c$. Let u be a vertex in S_{c-1} which is on a shortest $z - w$ path (i.e., $u \in N(w) \cap S_{c-1}$). Then by Claim 4.9(1), $d_G(w, w') \leq 1 + d_G(u, w') \leq 4$. Thus (1) is proved, and (2) similarly follows from Claim 4.9(2). □

Claim 4.11. *Suppose that $c \geq a + 3$. Then $d_G(v, v') \leq 2$ for every $v \in S_{a+1}$ and every $v' \in S_{a+2}$.*

Proof. By way of contradiction, suppose that there exist $v \in S_{a+1}$ and $v' \in S_{a+2}$ such that $d_G(v, v') \geq 3$. Then $(\{v\} \cup N(v)) \cap (\{v'\} \cup N(v')) = \phi$. By Claim 4.3, we have $|\cup_{a \leq i \leq a+3} S_i| \leq 4k-1$. Since $|N(v)| \geq 2k-1$ and $|N(v')| \geq 2k-1$, we now get $4k \leq |\{v, v'\} \cup N(v) \cup N(v')| \leq |\cup_{a \leq i \leq a+3} S_i| \leq 4k-1$, a contradiction. □

Throughout the rest of the discussion for Case 2, we fix two vertices w_1, w_2 with $w_1 \in S_c$ and $w_2 \in T_c$ such that $d_G(w_1, w_2) \leq 2$. If possible, we choose w_1 and w_2 so that $d_G(w_1, w_2) = 1$. We prove one more auxiliary result.

Claim 4.12. *Let $a \leq i \leq c-1$, and let $v \in X_i$.*

(1) *If $v \in S_i$, then $d_G(w_1, v) \leq c - i + 2$ and $d_G(w_2, v) \leq c - i + 4$.*

(2) *If $v \in T_i$, then $d_G(w_1, v) \leq c - i + 4$ and $d_G(w_2, v) \leq c - i + 2$.*

(3) *If $c \geq a + 3$, $a \leq i \leq a + 1$ and $v \in S_i$, then $d_G(w_1, v) \leq c - i + 1$ and $d_G(w_2, v) \leq c - i + 3$.*

Proof. Since $d_G(w_1, w_2) \leq 2$, (1) and (2) follow from Claim 4.8. To prove (3), assume $c \geq a + 3$, and let $a \leq i \leq a + 1$ and $v \in S_i$. Let v' be a vertex in S_{a+2} which is on a shortest $z - w_1$ path. If $i = a + 1$, then $d_G(v', v) \leq 2 = (a + 2) - i + 1$ by Claim 4.11; if $i = a$, then taking $v'' \in N(v) \cap S_{a+1}$ (see Lemma 3), we get $d_G(v', v) \leq d_G(v', v'') + d_G(v'', v) \leq 2 + 1 = (a + 2) - i + 1$ by Claim 4.11. Thus in either case, $d_G(v', v) \leq (a + 2) - i + 1$. Consequently $d_G(w_1, v) \leq d_G(w_1, v') + d_G(v', v) \leq \{c - (a + 2)\} + \{(a + 2) - i + 1\} = c - i + 1$ and, since $d_G(w_1, w_2) \leq 2$, we also get $d_G(w_2, v) \leq c - i + 3$. □

We now prove the four main claims of this section.

Claim 4.13. *Suppose that $c = a + 2$ and $d_G(w_1, w_2) = 2$. Then one of the following holds:*

(1) *$d_G(w_2, w) \leq 4$ for every $w \in X_{a+2}$; or*

(2) *for each $x \in N(w_1) \cap S_{a+1}$, we have $d_G(x, y) \leq 7$ for every $y \in X_{a+1} \cup X_{a+2}$ and $d_G(x, y') \leq 4$ for every $y' \in X_{a+3}$.*

Proof. Suppose that (1) does not hold. In view of Claim 4.10(2), this means that there exists $w_3 \in S_{a+2}$ with $d_G(w_2, w_3) \geq 5$. Then since $d_G(w_1, w_2) = 2$, we have $d_G(w_1, w_3) \geq 3$, and hence $(\{w_1\} \cup N(w_1)) \cap (\{w_3\} \cup N(w_3)) = \phi$. Note that the assumption that $d_G(w_1, w_2) = 2$, together with the choice of w_1 and w_2 mentioned in the paragraph preceding Claim 4.12, implies that there is no edge between S_{a+2} and T_{a+2}. Hence $N(w_1) \cup N(w_3) \subseteq S_{a+1} \cup S_{a+2} \cup X_{a+3}$. On the other hand, whether (A) holds or (B) holds, $|X_{a+1}| + |X_{a+2}| + |X_{a+3}| \leq 6k$. Since $|T_{a+1}| + |T_{a+2}| \geq 2k$ by Claim 4.4, this implies $|S_{a+1} \cup S_{a+2} \cup X_{a+3}| \leq 4k$. Since $|N(w_1)| \geq 2k - 1$ and $|N(w_3)| \geq 2k - 1$, we now obtain $4k \leq |\{w_1, w_3\} \cup N(w_1) \cup N(w_3)| \leq |S_{a+1} \cup S_{a+2} \cup X_{a+3}| \leq 4k$, which implies that $X_{a+3} = (N(w_1) \cup N(w_3)) \cap X_{a+3}$. Let $x \in N(w_1) \cap S_{a+1}$. Since $X_{a+3} = (N(w_1) \cup N(w_3)) \cap X_{a+3}$, it follows from Claim 4.9(1) that $d_G(x, y') \leq 4$ for every $y' \in X_{a+3}$. By Claims 4.7(1) and 4.9(1), $d_G(x, y) \leq 3$ for every $y \in S_{a+1} \cup S_{a+2}$. Since $d_G(x, w_2) \leq 3$ and since $d_G(w_2, y) \leq 4$ for every $y \in T_{a+1} \cup T_{a+2}$ by Claims 4.9(2) and 4.10(2), we also get $d_G(x, y) \leq 7$ for every $y \in X_{a+1} \cup X_{a+2}$. Therefore (2) holds. □

Claim 4.14. *Suppose that $c = r - 2$ and $d_G(w_1, w_2) = 1$. Then for each $x \in (N(w_1) \cap S_{r-3}) \cup (N(w_2) \cap T_{r-3})$, we have $d_G(x, y) \leq 6$ for every $y \in \cup_{r-4 \leq i \leq r} X_i$.*

Proof. Note that from the assumption that $c = r - 2$, it follows that $b = r - 1$, and hence $|X_{r-1}| = 2k - 1$ and $|X_r| = 1$ by the assumptions of the proposition. Let $x \in (N(w_1) \cap S_{r-3}) \cup (N(w_2) \cap T_{r-3})$. By symmetry, we may assume $x \in N(w_1) \cap S_{r-3}$ (in the proof of this claim, we do not make use of the assumption that $|S_a| = k - 1$ and $|T_a| = k$). Then by Claims 4.7(1) and 4.8(1), $d_G(x, y) \leq 3$ for every $y \in \cup_{r-4 \leq i \leq r-2} S_i$. Since $d_G(w_2, y) \leq 4$ for every $y \in \cup_{r-4 \leq i \leq r-2} T_i$ by Claims 4.10(2) and 4.12(2), $d_G(x, y) \leq 6$ for every $y \in \cup_{r-4 \leq i \leq r-2} T_i$. Now if $(\cup_{w \in S_{r-2}} N(w)) \cap X_{r-1} = \phi$, then $G - T_{r-2}$ is disconnected, and hence $|T_{r-2}| \geq 2k - 1$ by the assumption that G is $(2k-1)$-connected, which contradicts Claim 4.4. Thus there exist $w_0 \in S_{r-2}$ and $y_0 \in X_{r-1}$ such that $w_0 y_0 \in E(G)$. Since $|X_{r-1}| = 2k - 1$ and $|X_r| = 1$, $d_G(y_0, y) \leq 2$ for every $y \in X_{r-1} \cup X_r$. Since $d_G(x, w_0) \leq 3$ by Claim 4.9(1), it now follows that $d_G(x, y) \leq 6$ for every $y \in X_{r-1} \cup X_r$. Thus the claim is proved. □

Claim 4.15. *Suppose that $c = r - 2$ and $d_G(w_1, w_2) = 2$. Then $d_G(w_2, y) \leq 4$ for every $y \in \cup_{r-2 \leq i \leq r} X_i$.*

Proof. Note that $|X_{r-1}| = 2k-1$ and $|X_r| = 1$, and hence $d_G(y, y') \leq 2$ for every $y, y' \in X_{r-1} \cup X_r$. Suppose that the claim is false. Then by Claim 4.10(2),

$$\text{there exists } v \in S_{r-2} \cup X_{r-1} \cup X_r \text{ such that } d_G(w_2, v) \geq 5. \tag{4.1}$$

By the assumption that $d_G(w_1, w_2) = 2$, there is no edge between S_{r-2} and T_{r-2}. Since $d_G(w_1, w_2) = 2$ and $N(w_1) \cap N(w_2) \cap X_{r-3} = \phi$, this implies $N(w_2) \cap X_{r-1} \neq \phi$, and hence

$$d_G(w_2, y) \leq 3 \text{ for every } y \in X_{r-1} \cup X_r. \tag{4.2}$$

By (4.1) and (4.2), there exists $w_3 \in S_{r-2}$ such that $d_G(w_2, w_3) \geq 5$. By (4.2), $N(w_3) \cap X_{r-1} = \phi$. Hence $N(w_3) \subseteq S_{r-3} \cup S_{r-2}$. On the other hand, since $d_G(w_1, w_3) \geq d_G(w_2, w_3) - d_G(w_2, w_1) \geq 3$, $(\{w_1\} \cup N(w_1)) \cap (\{w_3\} \cup N(w_3)) = \phi$. Consequently $N(w_3) \subseteq (S_{r-3} \cup S_{r-2}) - (\{w_1, w_3\} \cup (N(w_1) \cap S_{r-3}))$. Since $|S_{r-3} \cup S_{r-2}| \leq 2k+1$ by Claim 4.3 and $N(w_1) \cap S_{r-3} \neq \phi$ by the definition of S_{r-2}, we now obtain $|N(w_3)| \leq (2k+1) - 3$, which contradicts the assumption that G is $(2k-1)$-connected. □

Claim 4.16. *Suppose that $c = a + 2 = r - 2$ and $d_G(w_1, w_2) = 2$. Then one of the following holds:*

(1) *there exists $x \in X_{r-3}$ such that $d_G(x, y) \leq 5$ for every $y \in \cup_{r-3 \leq i \leq r} X_i$; or*

(2) *there exists $x \in X_{r-4}$ such that $d_G(x, y) \leq 4$ for every $y \in X_{r-1} \cup X_r$; or*

(3) *there exists $x \in X_{r-3}$ such that $d_G(x, y) \leq 4$ for $y \in \cup_{r-2 \leq i \leq r} X_i$.*

Proof. Note that $|X_{r-1}| = 2k - 1$ and $|X_r| = 1$. Write $X_r = \{y_0\}$. Then

$$N(y_0) = X_{r-1}, \text{ and } d_G(y, y') \leq 2 \text{ for every } y, y' \in X_{r-1}. \tag{4.3}$$

By the assumption that $d_G(w_1, w_2) = 2$, there is no edge between S_{r-2} and T_{r-2}, and $N(w_1) \cap X_{r-1} \neq \phi$. Hence by (4.3),

$$d_G(y, y_0) \leq 4 \text{ for every } y \in N_2(w_1) \cap S_{r-4}. \tag{4.4}$$

If $d_G(w_1, w) \leq 2$ for every $w \in S_{r-3}$, then it follows from Claims 4.7(2) and 4.15 that (1) holds with $x \in N(w_2) \cap X_{r-3}$. Thus we may assume that there exists $w' \in S_{r-3}$ such that $d_G(w_1, w') \geq 3$. If there exists $w'' \in X_{r-1}$ such that $d_G(w_1, w'') \geq 3$ and $d_G(w', w'') \geq 3$, then since $|T_{r-4} \cup T_{r-3}| \geq 2k$ by Claim 4.4 and since $|X_{r-3} \cup X_{r-2}| \leq 4k$ and $|X_{r-4}| = 2k - 1$, we get $6k \leq |\{w_1, w', w''\} \cup N(w_1) \cup N(w') \cup N(w'')| \leq |(S_{r-4} \cup S_{r-3}) \cup (\cup_{r-2 \leq i \leq r} X_i)| = |\cup_{r-4 \leq i \leq r} X_i| - |T_{r-4} \cup T_{r-3}| \leq (8k-1) - 2k$, a contradiction. Thus

$$d_G(w_1, w) \leq 2 \text{ or } d_G(w', w) \leq 2 \text{ for each } w \in X_{r-1}. \tag{4.5}$$

If $N_2(w_1) \cap N(w') \cap S_{r-4} \neq \phi$, then it follows from (4.4) and (4.5) that (2) holds with $x \in N_2(w_1) \cap N(w') \cap S_{r-4}$. Thus we may assume that $N_2(w_1) \cap N(w') \cap S_{r-4} = \phi$. Since $|S_{r-4}| = k-1$ and $N_2(w_1) \cap S_{r-4} \neq \phi$, this implies that $|N(w') \cap S_{r-4}| \leq k-2$. Since $|N(w')| \geq 2k-1$, it follows that $|N(w') \cap (S_{r-3} \cup$

$S_{r-2})| \geq k+1$. Since $d_G(w_1, w') \geq 3$ and $|S_{r-3}| + |S_{r-2}| \leq 2k$ by Claim 4.3, this in turn implies that $|N(w_1) \cap (S_{r-3} \cup S_{r-2})| \leq 2k-2-(k+1) = k-3$, and hence $|N(w_1) \cap X_{r-1}| \geq (2k-1)-(k-3) = k+2$. If $d_G(w_1, w) \leq 3$ for every $w \in T_{r-2}$, then it follows from (4.3) and Claim 4.9(1) that (3) holds with $x \in N(w_1) \cap X_{r-3}$. Thus we may assume that there exists $w_4 \in T_{r-2}$ such that $d_G(w_1, w_4) \geq 4$. Set $M = \cup_{v \in N(w_1) \cap X_{r-1}} (N(v) - (N(w_1) \cap X_{r-1}))$. Note that $w_1 \in M$ and $M \subseteq \{w_1\} \cup N(w_1) \cup N_2(w_1)$. Also $|M| \geq 2k-1$ by the assumption that G is $(2k-1)$-connected. Since $|S_{r-3}| \geq k-1$ by Claim 4.3 and $|X_{r-3} \cup X_{r-2}| \leq 4k$, we now obtain $5k+1 = 1+(2k-1)+(k+2)+(2k-1) \leq |\{w_4\} \cup N(w_4) \cup (N(w_1) \cap X_{r-1}) \cup M| \leq |T_{r-3} \cup (\cup_{r-2 \leq i \leq r} X_i)| = |\cup_{r-3 \leq i \leq r} X_i| - |S_{r-3}| \leq 6k-(k-1) = 5k+1$. This implies that

$$T_{r-3} \subseteq N(w_4), \tag{4.6}$$

$$\cup_{r-2 \leq i \leq r} X_i \subseteq \{w_1, w_4\} \cup N(w_4) \cup N(w_1) \cup N_2(w_1). \tag{4.7}$$

If $X_{r-1} \subseteq N(w_1) \cup N_2(w_1)$, then it follows from (4.4) that (2) holds with $x \in N_2(w_1) \cap S_{r-4}$. Thus we may assume that $X_{r-1} - (N(w_1) \cup N_2(w_1)) \neq \phi$. Let $u \in X_{r-1} - (N(w_1) \cup N_2(w_1))$. Then $d_G(w', u) \leq 2$ by (4.5), and $d_G(w_4, u) \leq 1$ by (4.7). Hence $d_G(w', w_4) \leq 3$. Since $d_G(w', w_1) \leq 3$ by Claim 4.9(1), this together with (4.7) implies that $d_G(w', y) \leq 5$ for every $y \in \cup_{r-2 \leq i \leq r} X_i$. Furthermore it follows from (4.6) that $d_G(w', y) \leq 4$ for every $y \in T_{r-3}$. Therefore it follows from Claim 4.7(1) that (1) holds with $x = w'$. This completes the proof of Claim 4.16. □

§5. Proof of Proposition 1

We continue with the notation of the preceding section, and complete the proof of Proposition 1.

We divide the rest of the proof for Case 2 into eight cases. In each case, we derive a contradiction by showing that there exists $u \in V(G)$ such that $d_G(u, v) < r$ for every $v \in V(G)$.

Case 2.1. $d_G(w_1, w_2) = 1$.

Subcase 2.1.1. $a \geq 4$ *and* $c \leq r-4$.

Let u be a vertex in X_6 which is on a shortest $z-w_2$ path. Take $v \in V(G)$, and let $v \in X_i$. If $0 \leq i \leq c-3$, $d_G(u, v) \leq d_G(u, z) + d_G(z, v) = 6+i \leq 6+(c-3) \leq 6+(r-7) = r-1 < r$. If $c-2 \leq i \leq c-1$, then since $c-2 \geq a$, it follows from (1) and (2) of Claim 4.12 that $d_G(w_2, v) \leq c-i+4 \leq c-(c-2)+4 = 6$, and hence $d_G(u, v) \leq d_G(u, w_2) + d_G(w_2, v) \leq (c-6)+6 \leq \{(r-4)-6\}+6 = r-4 < r$. If $c \leq i \leq r$, then letting w be a vertex in X_c which is on a shortest $z-v$ path, we get $d_G(w_2, w) \leq 5$ by Claim 4.10, and hence $d_G(u, v) \leq d_G(u, w_2) + d_G(w_2, w) + d_G(w, v) \leq (c-6)+5+(i-c) = i-1 \leq r-1 < r$.

Subcase 2.1.2. *$c \le r-3$, and either $a=3$ or $c=r-3$.*

Note that $b \ge r-2$ in the case where $c=r-3$. Thus we have $|X_{r-1} \cup X_r| \le 2k+2$ by the assumptions of the proposition.

Subcase 2.1.2.1. *Lemma* 5(1) *holds.*

Let v_0 be as in Lemma 5(1), and let w' be a vertex in X_c which is on a shortest $z-v_0$ path. By symmetry, we may assume $w' \in T_c$ (in this subcase, we do not make use of the assumption that $|S_a| = k-1$ and $|T_a| = k$). Then $d_G(w_2, w') \le 4$ by Claim 4.10(2). Let u be a vertex in X_5 which is on a shortest $z-w_2$ path. Take $v \in V(G)$, and let $v \in X_i$. If $0 \le i \le c-3$, $d_G(u,v) \le d_G(u,z)+d_G(z,v) = 5+i \le 5+(c-3) \le 5+\{(r-3)-3\} = r-1 < r$. If $c-2 \le i \le c-1$, then $d_G(w_2,v) \le c-i+4 \le 6$ by Claim 4.12, and hence $d_G(u,v) \le d_G(u,w_2)+d_G(w_2,v) \le (c-5)+6 = c+1 \le (r-3)+1 = r-2 < r$. If $c \le i \le r-1$, then letting w be a vertex in X_c which is on a shortest $z-v$ path, we get $d_G(w_2,w) \le 5$ by Claim 4.10, and hence $d_G(u,v) \le d_G(u,w_2)+d_G(w_2,w)+d_G(w,v) \le (c-5)+5+(i-c) = i \le r-1 < r$. If $i=r$, then letting v_0, w' be as above, we get $d_G(v_0,v)=1$ by Lemma 5(1), and hence $d_G(u,v) \le d_G(u,w_2)+d_G(w_2,w')+d_G(w',v_0)+d_G(v_0,v) \le (c-5)+4+\{(r-1)-c\}+1 = r-1 < r$.

Subcase 2.1.2.2. *Lemma* 5(2) *holds.*

Let v_0 be as in Lemma 5(2), and let w' be a vertex in X_c which is on a shortest $z-v_0$ path. By symmetry, we may assume $w' \in T_c$. Let u be a vertex in X_4 which is on a shortest $z-w_2$ path. Take $v \in V(G)$, and let $v \in X_i$. If $0 \le i \le c-2$, $d_G(u,v) \le d_G(u,z)+d_G(z,v) = 4+i \le 4+(c-2) \le 4+\{(r-3)-2\} = r-1 < r$. If $i=c-1$, then $d_G(w_2,v) \le c-(c-1)+4 \le 5$ by Claim 4.12, and hence $d_G(u,v) \le d_G(u,w_2)+d_G(w_2,v) \le (c-4)+5 = c+1 \le (r-3)+1 = r-2 < r$. If $c \le i \le r-2$, then letting w be a vertex in X_c which is on a shortest $z-v$ path, we get $d_G(w_2,w) \le 5$ by Claim 4.10, and hence $d_G(u,v) \le d_G(u,w_2)+d_G(w_2,w)+d_G(w,v) \le (c-4)+5+(i-c) = i+1 \le r-1 < r$. Thus we may assume $r-1 \le i \le r$. Let v_0, w' be as above, and let w'' be a vertex in X_{c-1} which is on a shortest $u-w_2$ path. Then $d_G(v_0,v) \le 2$ by Lemma 5(2), and $d_G(w'',w') \le 3$ by Claim 4.9(2). Hence $d_G(u,v) \le d_G(u,w'')+d_G(w'',w')+d_G(w',v_0)+d_G(v_0,v) \le \{(c-1)-4\}+3+\{(r-1)-c\}+2 = r-1 < r$.

Subcase 2.1.3. *$c=r-2$.*

Fix $x \in N(w_1) \cap S_{r-3}$. Let u be a vertex in X_4 which is on a shortest $z-x$ path. Take $v \in V(G)$, and let $v \in X_i$. If $0 \le i \le r-5$, $d_G(u,v) \le d_G(u,z)+d_G(z,v) = 4+i \le 4+(r-5) = r-1 < r$. If $r-4 \le i \le r$, then $d_G(x,v) \le 6$ by Claim 4.14, and hence $d_G(u,v) \le d_G(u,x)+d_G(x,v) \le \{(r-3)-4\}+6 = r-1 < r$.

Case 2.2. *$a+3 \leq c \leq r-3$ and $d_G(w_1,w_2)=2$.*

Subcase 2.2.1. *$a \geq 4$ and $c \leq r-4$.*

Let u be a vertex in X_7 which is on a shortest $z-w_2$ path. Take $v \in V(G)$, and let $v \in X_i$. If $0 \leq i \leq c-4$, $d_G(u,v) \leq d_G(u,z)+d_G(z,v) = 7+i \leq 7+(c-4) \leq 7+\{(r-4)-4\} = r-1 < r$. If $c-3 \leq i \leq c-1$, then since $c-3 \geq a$ by the assumption of Case 2.2, it follows from Claim 4.12 that $d_G(w_2,v) \leq c-i+4 \leq 7$, and hence $d_G(u,v) \leq d_G(u,w_2)+d_G(w_2,v) \leq (c-7)+7 = c \leq r-4 < r$. If $c \leq i \leq r$, then letting w be a vertex in X_c which is on a shortest $z-v$ path, we get $d_G(w_2,w) \leq 6$ by Claim 4.10, and hence $d_G(u,v) \leq d_G(u,w_2)+d_G(w_2,w)+d_G(w,v) \leq (c-7)+6+(i-c) = i-1 \leq r-1 < r$.

Subcase 2.2.2. *$a=3$ or $c=r-3$.*

We have $|X_{r-1} \cup X_r| \leq 2k+2$ by the assumptions of the proposition.

Subcase 2.2.2.1. *Lemma 5(1) holds.*

Let v_0 be as in Lemma 5(1), and let w' be a vertex in X_c which is on a shortest z–v_0 path. By symmetry, we may assume $w' \in T_c$. Then $d_G(w_2,w') \leq 4$ by Claim 4.10(2). Let u be a vertex in X_6 which is on a shortest $z-w_2$ path. Take $v \in V(G)$, and let $v \in X_i$. If $0 \leq i \leq c-4$, $d_G(u,v) \leq d_G(u,z)+d_G(z,v) = 6+i \leq 6+(c-4) \leq 6+\{(r-3)-4\} = r-1 < r$. If $c-3 \leq i \leq c-1$, then $d_G(w_2,v) \leq c-i+4 \leq 7$ by Claim 4.12, and hence $d_G(u,v) \leq d_G(u,w_2)+d_G(w_2,v) \leq (c-6)+7 = c+1 \leq (r-3)+1 = r-2 < r$. If $c \leq i \leq r-1$, then letting w be a vertex in X_c which is on a shortest $z-v$ path, we get $d_G(w_2,w) \leq 6$ by Claim 4.10, and hence $d_G(u,v) \leq d_G(u,w_2)+d_G(w_2,w)+d_G(w,v) \leq (c-6)+6+(i-c) = i \leq r-1 < r$. If $i=r$, then letting v_0, w' be as above, we get $d_G(v_0,v)=1$ by Lemma 5(1), and hence $d_G(u,v) \leq d_G(u,w_2)+d_G(w_2,w')+d_G(w',v_0)+d_G(v_0,v) \leq (c-6)+4+\{(r-1)-c\}+1 = r-2 < r$.

Subcase 2.2.2.2. *Lemma 5(2) holds.*

Let v_0 be as in Lemma 5(2), and let w' be a vertex in X_c which is on a shortest $z-v_0$ path. By symmetry, we may assume $w' \in T_c$. Let u be a vertex in X_5 which is on a shortest $z-w_2$ path. Take $v \in V(G)$, and let $v \in X_i$. If $0 \leq i \leq c-3$, $d_G(u,v) \leq d_G(u,z)+d_G(z,v) = 5+i \leq 5+(c-3) \leq 5+\{(r-3)-3\} = r-1 < r$. If $c-2 \leq i \leq c-1$, then $d_G(w_2,v) \leq c-i+4 \leq 6$ by Claim 4.12, and hence $d_G(u,v) \leq d_G(u,w_2)+d_G(w_2,v) \leq (c-5)+6 = c+1 \leq (r-3)+1 = r-2 < r$. If $c \leq i \leq r-2$, then letting w be a vertex in X_c which is on a shortest $z-v$ path, we get $d_G(w_2,w) \leq 6$ by Claim 4.10, and hence $d_G(u,v) \leq d_G(u,w_2)+d_G(w_2,w)+d_G(w,v) \leq (c-5)+6+(i-c) = i+1 \leq r-1 <$

r. If $r-1 \leq i \leq r$, then letting v_0, w' be as above and w'' be a vertex in X_{c-1} which is on a shortest $u-w_2$ path, we get $d_G(v_0, v) \leq 2$ by Lemma 5(2) and $d_G(w'', w') \leq 3$ by Claim 4.9(2), and hence $d_G(u, v) \leq d_G(u, w'') + d_G(w'', w') + d_G(w', v_0) + d_G(v_0, v) \leq \{(c-1)-5\} + 3 + \{(r-1)-c\} + 2 = r-2 < r$.

Case 2.3. *$a+3 \leq c = r-2$ and $d_G(w_1, w_2) = 2$.*

Let u be a vertex in X_{r-a-1} which is on a shortest $z-w_2$ path. Take $v \in V(G)$, and let $v \in X_i$. If $0 \leq i \leq a$, $d_G(u, v) \leq d_G(u, z) + d_G(z, v) = (r-a-1) + i \leq (r-a-1) + a = r-1 < r$. If $r-2 \leq i \leq r$, then $d_G(w_2, v) \leq 4$ by Claim 4.15, and hence $d_G(u, v) \leq d_G(u, w_2) + d_G(w_2, v) \leq \{(r-2) - (r-a-1)\} + 4 = a+3 \leq (r-5) + 3 = r-2 < r$. Thus we may assume that $a+1 \leq i \leq r-3$. If $v \in T_i$, then $d_G(w_2, v) \leq (r-2) - i + 2 \leq (r-2) - (a+1) + 2 = r-a-1$ by Claim 4.12(2); if $v \in S_i$ and $a+2 \leq i \leq r-3$, then $d_G(w_2, v) \leq (r-2) - i + 4 \leq (r-2) - (a+2) + 4 = r-a$ by Claim 4.12(1); if $v \in S_{a+1}$, then $d_G(w_2, v) \leq (r-2) - (a+1) + 3 = r-a$ by Claim 4.12(3). Hence $d_G(w_2, v) \leq r-a$. Therefore $d_G(u, v) \leq d_G(u, w_2) + d_G(w_2, v) \leq \{(r-2) - (r-a-1)\} + (r-a) = r-1 < r$.

Case 2.4. *$a+2 = c \leq r-3$, $d_G(w_1, w_2) = 2$ and Claim* 4.13(1) *holds.*

Note that $a \leq r-5$. Let u be a vertex in X_5 which is on a shortest $z-w_2$ path. Take $v \in V(G)$, and let $v \in X_i$. If $0 \leq i \leq a-1$, $d_G(u, v) \leq d_G(u, z) + d_G(z, v) = 5 + i \leq 5 + (a-1) = a+4 \leq (r-5) + 4 = r-1 < r$. If $a \leq i \leq a+1$, then $d_G(w_2, v) \leq (a+2) - a + 4 = 6$ by Claim 4.12, and hence $d_G(u, v) \leq d_G(u, w_2) + d_G(w_2, v) \leq \{(a+2) - 5\} + 6 = a+3 \leq (r-5) + 3 = r-2 < r$. If $a+2 \leq i \leq r$, then letting w be a vertex in X_{a+2} which is on a shortest $z-v$ path, we get $d_G(w_2, w) \leq 4$ by Claim 4.13(1), and hence $d_G(u, v) \leq d_G(u, w_2) + d_G(w_2, w) + d_G(w, v) \leq \{(a+2) - 5\} + 4 + \{i - (a+2)\} = i-1 \leq r-1 < r$.

Case 2.5. *$a+2 = c \leq r-3$, $d_G(w_1, w_2) = 2$ and Claim* 4.13(2) *holds.*

Note that $a \leq r-5$. Fix $x \in N(w_1) \cap X_{a+1}$. Let u be a vertex in X_4 which is on a shortest $z-x$ path. Take $v \in V(G)$, and let $v \in X_i$. If $0 \leq i \leq a$, $d_G(u, v) \leq d_G(u, z) + d_G(z, v) = 4 + i \leq 4 + a \leq 4 + (r-5) = r-1 < r$. If $a+1 \leq i \leq a+2$, then $d_G(x, v) \leq 7$ by Claim 4.13(2), and hence $d_G(u, v) \leq d_G(u, x) + d_G(x, v) \leq \{(a+1) - 4\} + 7 = a+4 \leq (r-5) + 4 = r-1 < r$. If $a+3 \leq i \leq r$, then letting y' be a vertex in X_{a+3} which is on a shortest $z-v$ path, we get $d_G(x, y') \leq 4$ by Claim 4.13(2), and hence $d_G(u, v) \leq d_G(u, x) + d_G(x, y') + d_G(y', v) \leq \{(a+1) - 4\} + 4 + \{i - (a+3)\} = i-2 \leq r-2 < r$.

Case 2.6. *$c = a+2 = r-2$, $d_G(w_1, w_2) = 2$ and Claim* 4.16(1) *holds.*

Let x be as in Claim 4.16(1). Let u be a vertex in X_3 which is on a shortest $z-x$ path. Take $v \in V(G)$, and let $v \in X_i$. If $0 \le i \le r-4$, $d_G(u,v) \le d_G(u,z)+d_G(z,v) = 3+i \le 3+(r-4) = r-1 < r$. If $r-3 \le i \le r$, then $d_G(x,v) \le 5$ by Claim 4.16(1), and hence $d_G(u,v) \le d_G(u,x)+d_G(x,v) \le \{(r-3)-3\}+5 = r-1 < r$.

Case 2.7. *$c = a+2 = r-2$, $d_G(w_1,w_2) = 2$ and Claim 4.16(2) holds.*

Let x be as in Claim 4.16(2). Let u be a vertex in X_1 which is on a shortest $z-x$ path. Take $v \in V(G)$, and let $v \in X_i$. If $0 \le i \le r-2$, $d_G(u,v) \le d_G(u,z)+d_G(z,v) = 1+i \le 1+(r-2) = r-1 < r$. If $r-1 \le i \le r$, then $d_G(x,v) \le 4$ by Claim 4.16(2), and hence $d_G(u,v) \le d_G(u,x)+d_G(x,v) \le \{(r-4)-1\}+4 = r-1 < r$.

Case 2.8. *$c = a+2 = r-2$, $d_G(w_1,w_2) = 2$ and Claim 4.16(3) holds.*

Let x be as in Claim 4.16(3). Let u be a vertex in X_2 which is on a shortest $z-x$ path. Take $v \in V(G)$, and let $v \in X_i$. If $0 \le i \le r-3$, $d_G(u,v) \le d_G(u,z)+d_G(z,v) = 2+i \le 2+(r-3) = r-1 < r$. If $r-2 \le i \le r$, then $d_G(x,v) \le 4$ by Claim 4.16(3), and hence $d_G(u,v) \le d_G(u,x)+d_G(x,v) \le \{(r-3)-2\}+4 = r-1 < r$.

This concludes the discussion for the case where $\mathscr{G}$ is disconnected, and completes the proof of Proposition 1.

§6. Proof of the Theorem

We continue with the notation of Section 2, and complete the proof of the Theorem. We first prove three propositions.

Proposition 2. *Suppose that $r \ge 9$. Then $\sum_{i=4}^{r-3} |X_i| \ge 2k(r-6)-2$.*

Proof. Let $I := \{i | 4 \le i \le r-3, |X_i| = 2k-1\}$. We may assume $|I| \ge 3$. Write $I = \{i_1, i_2, \ldots, i_{|I|}\}$ with $i_1 < i_2 < \cdots < i_{|I|}$. From I, we define a new sequence $j_1 < j_2 < \cdots < j_s$ inductively as follows. Set $j_1 = i_1$. For $l \ge 2$, set $j_l = \min\{i | i \in I, i \ge j_{l-1}+2\}$ (if $\{i | i \in I, i \ge j_{l-1}+2\} = \phi$, then we set $s = l-1$ and terminate this procedure). We have $j_s = i_{|I|}$ or $i_{|I|-1}$ by definition. By Proposition 1(1), $\sum_{i=j_{h-1}}^{j_h-1} |X_i| \ge 2k(j_h - j_{h-1})$ for all $2 \le h \le s$. Taking the sum of these inequalities, we get

$$\begin{aligned} \sum_{i=j_1}^{j_s-1} |X_i| &= \sum_{h=2}^{s} \sum_{i=j_{h-1}}^{j_h-1} |X_i| \\ &\ge 2k(j_s - j_1). \end{aligned}$$

Consequently

$$\begin{aligned}\sum_{i=4}^{r-3}|X_i| &= \sum_{i=4}^{j_1-1}|X_i| + \sum_{i=j_1}^{j_s-1}|X_i| + \sum_{i=j_s}^{r-3}|X_i| \\ &\geq 2k(j_1-4) + 2k(j_s-j_1) + 2k(r-2-j_s) - 2 \\ &= 2k(r-6) - 2,\end{aligned}$$

as desired. □

We can prove the following proposition by arguing as in the proof of Proposition 2.

Proposition 3. *Suppose that $r \geq 7$ and $2k+1 \leq |X_{r-1} \cup X_r| \leq 2k+2$. Then $\sum_{i=3}^{r-2}|X_i| \geq 2k(r-4)-2$.*

Proposition 4. *Suppose that $r \geq 6$ and $|X_{r-1} \cup X_r| = 2k$. Then $\sum_{i=2}^{r-1}|X_i| \geq 2k(r-2)-3$.*

Proof. Note that $|X_{r-1}| = 2k-1$ and $|X_r| = 1$ by Lemma 2. Let $I := \{i | 3 \leq i \leq r-1, |X_i| = 2k-1\}$. Then $r-1 \in I$. If $|I| \leq 2$, then it follows from Lemma 2 that

$$\begin{aligned}\sum_{i=2}^{r-1}|X_i| &\geq 2k(r-2) - (|I|+1) \\ &\geq 2k(r-2) - 3.\end{aligned}$$

Thus we may assume $|I| \geq 3$. Write $I = \{i_1, i_2, \ldots, i_{|I|}\}$ with $i_1 < i_2 < \cdots < i_{|I|}$. Then $i_{|I|} = r-1$. From I, we define a new sequence $j_1 < j_2 < \cdots < j_s$ inductively as follows. Set $j_1 = i_1$. For $l \geq 2$, set $j_l = \min\{i | i \in I, i \geq j_{l-1}+2\}$ (if $\{i | i \in I, i \geq j_{l-1}+2\} = \phi$, then we set $s = l-1$ and terminate this procedure). We have $j_s = i_{|I|}$ or $i_{|I|-1}$ by definition. Also $\frac{|I|}{2} \leq s \leq |I|$. Assume for the moment that $s = 2$. Then $|I| \leq 4$. If $|X_{j_1-1}| \leq 2k+1$, then $\sum_{i=j_1}^{j_2-1}|X_i| \geq 2k(j_2-j_1)$ by Proposition 1(3), and hence

$$\begin{aligned}\sum_{i=2}^{r-1}|X_i| &= |X_2| + \sum_{i=3}^{j_1-1}|X_i| + \sum_{i=j_1}^{j_2-1}|X_i| + \sum_{i=j_2}^{r-1}|X_i| \\ &\geq (2k-1) + 2k(j_1-3) + 2k(j_2-j_1) + 2k(r-j_2) - 2 \\ &= 2k(r-2) - 3;\end{aligned}$$

if $|X_{j_1-1}| \geq 2k+2$, then

$$\begin{aligned}\sum_{i=2}^{r-1}|X_i| &\geq 2k(r-2) - (|I|+1) + (|X_{j_1-1}| - 2k) \\ &\geq 2k(r-2) - 3.\end{aligned}$$

Thus we may assume $s \geq 3$. Then $r-1 \geq j_s \geq j_1 + 4 \geq 7$; i.e., $r \geq 8$. We first consider the case where $j_s = i_{|I|-1}$. In this case $j_s \leq r-2$. Hence by Proposition 1(2), $\sum_{i=j_{h-1}}^{j_h-1} |X_i| \geq 2k(j_h - j_{h-1})$ for all $2 \leq h \leq s$. Taking the sum of these inequalities, we get

$$\begin{aligned}\sum_{i=j_1}^{j_s-1} |X_i| &= \sum_{h=2}^{s} \sum_{i=j_{h-1}}^{j_h-1} |X_i| \\ &\geq 2k(j_s - j_1).\end{aligned}$$

Consequently

$$\begin{aligned}\sum_{i=2}^{r-1} |X_i| &= |X_2| + \sum_{i=3}^{j_1-1} |X_i| + \sum_{i=j_1}^{j_s-1} |X_i| + \sum_{i=j_s}^{r-1} |X_i| \\ &\geq (2k-1) + 2k(j_1-3) + 2k(j_s-j_1) + 2k(r-j_s) - 2 \\ &= 2k(r-2) - 3.\end{aligned}$$

We are now left with the case where $j_s = i_{|I|}$, i.e., $j_s = r-1$. We show that $\sum_{i=j_{s-2}}^{r-2} |X_i| \geq 2k(r - j_{s-2} - 1) - 1$. First assume $|X_{j_{s-1}-1}| \leq 2k+2$. Then by Proposition 1(3), $\sum_{i=j_{s-1}}^{r-2} |X_i| \geq 2k(r - j_{s-1} - 1) - 1$. Since $\sum_{i=j_{s-2}}^{j_{s-1}-1} |X_i| \geq 2k(j_{s-1} - j_{s-2})$ by Proposition 1(2), this implies $\sum_{i=j_{s-2}}^{r-2} |X_i| \geq 2k(r - j_{s-2} - 1) - 1$. Next assume $|X_{j_{s-1}-1}| \geq 2k+3$. Then

$$\begin{aligned}\sum_{i=j_{s-2}}^{j_{s-1}-1} |X_i| &\geq 2k(j_{s-1} - j_{s-2}) - 2 + (|X_{j_{s-1}-1}| - 2k) \\ &\geq 2k(j_{s-1} - j_{s-2}) + 1.\end{aligned}$$

Since we clearly have $\sum_{i=j_{s-1}}^{r-2} |X_i| \geq 2k(r - j_{s-1} - 1) - 2$, this implies $\sum_{i=j_{s-2}}^{r-2} |X_i| \geq 2k(r - j_{s-2} - 1) - 1$. Therefore $\sum_{i=j_{s-2}}^{r-2} |X_i| \geq 2k(r - j_{s-2} - 1) - 1$ in either case. By Proposition 1(2), $\sum_{i=j_{h-1}}^{j_h-1} |X_i| \geq 2k(j_h - j_{h-1})$ for all $2 \leq h \leq s-2$. Taking the sum of these inequalities, we get

$$\begin{aligned}\sum_{i=j_1}^{r-2} |X_i| &= \Big(\sum_{h=2}^{s-2} \sum_{i=j_{h-1}}^{j_h-1} |X_i|\Big) + \Big(\sum_{i=j_{s-2}}^{r-2} |X_i|\Big) \\ &\geq 2k(r - j_1 - 1) - 1.\end{aligned}$$

Consequently

$$\begin{aligned}\sum_{i=2}^{r-1}|X_i| &= |X_2| + \sum_{i=3}^{j_1-1}|X_i| + \sum_{i=j_1}^{r-2}|X_i| + |X_{r-1}| \\ &\geq (2k-1) + 2k(j_1-3) + \{2k(r-j_1-1)-1\} + (2k-1) \\ &= 2k(r-2)-3,\end{aligned}$$

as desired. □

We are now in a position to complete the proof of the Theorem. First we consider the case where $|X_{r-1} \cup X_r| \geq 2k+3$. If $r \leq 8$, the desired conclusion follows from Lemma 4. Thus we may assume $r \geq 9$. Note that $|X_0| = 1$, and $|X_i| \geq 2k-1$ for each $i \in \{1, 2, 3, r-2\}$ by Lemma 2. By Proposition 2, $\sum_{i=4}^{r-3}|X_i| \geq 2k(r-6)-2$. Therefore we obtain

$$\begin{aligned}|V(G)| &= \sum_{i=0}^{r}|X_i| \\ &\geq 1 + 3(2k-1) + \{2k(r-6)-2\} + (2k-1) + (2k+3) \\ &= 2k(r-1)-2.\end{aligned}$$

Next we consider the case where $2k+1 \leq |X_{r-1} \cup X_r| \leq 2k+2$. If $r \leq 6$, the desired conclusion follows from Lemma 4. Thus we may assume $r \geq 7$. By Proposition 3, $\sum_{i=3}^{r-2}|X_i| \geq 2k(r-4)-2$. Therefore we obtain

$$\begin{aligned}|V(G)| &= \sum_{i=0}^{r}|X_i| \\ &\geq 1 + 2(2k-1) + \{2k(r-4)-2\} + (2k+1) \\ &= 2k(r-1)-2.\end{aligned}$$

Finally we consider the case where $|X_{r-1} \cup X_r| = 2k$. If $r \leq 5$, the desired conclusion follows from Lemma 4. Thus we may assume $r \geq 6$. By Proposition 4, $\sum_{i=2}^{r-1}|X_i| \geq 2k(r-2)-3$. Therefore we obtain

$$\begin{aligned}|V(G)| &= \sum_{i=0}^{r}|X_i| \\ &\geq 1 + (2k-1) + \{2k(r-2)-3\} + 1 \\ &= 2k(r-1)-2.\end{aligned}$$

This completes the proof of the Theorem.

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