

Gaussian estimates of order α and L^p -spectral independence of generators of C_0 -semigroups II

Shizuo Miyajima and Hisakazu Shindoh

(Received November 19, 2006)

Abstract. Without any assumptions on the space dimension or boundedness of the region, we prove L^p -spectral independence of generators of C_0 -semigroups estimated by the positive C_0 -semigroup $e^{-t(-\Delta)^\alpha}$ ($0 < \alpha \leq 1$). In particular, if the semigroup is self-adjoint in L^2 , it is shown that only the estimate by $e^{-t(-\Delta)^\alpha}$ is sufficient for L^p -spectral independence. The proof depends on the idea of considering the spectra of the operators $e^{-t(-A)^\beta}$ ($0 < \beta < 1$) and applying the spectral independence result of B.A. Barnes for integral operators, where A is the generator of the semigroup in question.

AMS 2000 Mathematics Subject Classification. 47A25, 47D03, 47B65, 45P05, 46H15, 47B15.

Key words and phrases. Gaussian estimates, L^p -spectrum, positive semigroups, integral kernels, Banach algebras, fractional powers of operators, spectral mapping theorem.

§1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be an open set, and suppose that a C_0 -semigroup $T_p = (T_p(t))_{t \geq 0}$ on $L^p(\Omega)$ with generator A_p is given for each $1 \leq p < \infty$. Assume further that T_p 's are consistent in the sense that

$$T_p(t) = T_q(t) \quad \text{on } L^p(\Omega) \cap L^q(\Omega)$$

for all $t \geq 0$. Under these assumptions, it is natural to expect L^p -spectral independence of generators, that is to say,

$$(1.1) \quad \sigma(A_p) = \sigma(A_2)$$

for all $1 \leq p < \infty$. However, W. Arendt [1, Section 3] revealed that this equality is not necessarily true. Nonetheless, there are important cases where

L^p -spectral independence (1.1) does hold. In fact, R. Hempel and J. Voigt [5, Theorem] proved that, for a potential V belonging to a large class including a Kato class, the spectrum of Schrödinger operator $-\Delta/2 + V$ acting in $L^p(\mathbb{R}^N)$ is independent of $p \in [1, \infty)$. They used the Feynman–Kac formula to obtain their result and so their method of proof is peculiar to the perturbation $-\Delta/2 + V$. However, Arendt [1] found that if a C_0 -semigroup $T = (T(t))_{t \geq 0}$ on $L^2(\Omega)$ is dominated by the heat semigroup $e^{t\Delta}$ (for details, see (1.2) below), then T naturally induces a C_0 -semigroup T_p on $L^p(\Omega)$ for each $p \in [1, \infty)$ and the spectrum of the generator A_p of T_p is independent of p provided $T(t)$ is self-adjoint. Roughly speaking, his proof relies on an subtle argument to obtain an estimate of the integral kernel of the resolvent of T . He also shows the p -independence of the connected component of the resolvent set of A_p containing a right half-plane for non-self-adjoint semigroups. We should note here that Arendt's result contains L^p -spectral independence for the case of $-\Delta/2 + V$ with a positive potential V . After the work of Arendt, P.C. Kunstmann [6] proved that a weaker estimate of the integral kernel of the resolvents implies L^p -spectral independence of the generators, and he generalized and completed, in a sense, the work of Arendt.

Arendt's results were generalized in a different direction in [8] and [9]. To state in more details, let $T = (T(t))_{t \geq 0}$ be a C_0 -semigroup on $L^2(\Omega)$ with generator A and $\alpha \in (0, 1]$. We say that T satisfies a Gaussian estimate of order α if there exist constants $M \geq 1$, $\omega \in \mathbb{R}$ and $b > 0$ such that

$$(1.2) \quad |T(t)f| \leq M e^{\omega t} e^{-bt(-A)^\alpha} |f|$$

for all $t \geq 0$ and $f \in L^2(\Omega)$. Here, Δ denotes the usual Laplacian in $L^2(\mathbb{R}^N)$ with domain $H^2(\mathbb{R}^N)$, and we identify $L^2(\Omega)$ with a subspace of $L^2(\mathbb{R}^N)$ by considering the elements of $L^2(\Omega)$ to have value 0 on $\mathbb{R}^N \setminus \Omega$. In the case of $\alpha = 1$, (1.2) is equivalent to an upper Gaussian estimate defined by Arendt [1, Definition 4.1]. If T satisfies the stronger estimate obtained by replacing $e^{-bt(-\Delta)^\alpha}$ in (1.2) with $e^{-bt(I-\Delta)^\alpha}$, then the resolvent of A satisfies an estimate assumed in [6, Theorem 1.1] and accordingly the spectrum of A_p is independent of $p \in [1, \infty)$, where A_p is the generator of a version of T on $L^p(\Omega)$ ([10, Theorem 3.17]). In the case of $\alpha = 1$, this result coincides with that of Arendt. On the other hand, as long as we assume only the estimate (1.2), we could not prove L^p -spectral independence except for the case of bounded Ω or of space dimension 1 ([9]).

It is the purpose of this paper to prove L^p -spectral independence without limitations mentioned above. A crucial tool for this purpose is the result of B.A. Barnes [3] which gives a sufficient condition for L^p -spectral independence of integral operators by using the theory of Banach algebras. More precisely, he gave an estimate for a measurable function $K: \Omega \times \Omega \rightarrow \mathbb{C}$ that guarantees that K defines a bounded linear operator K_p on $L^p(\Omega)$ for each $p \in [1, \infty)$ and

the spectrum of K_p is independent of $p \in [1, \infty)$ ([3, Theorem 3.8]). Suppose that a C_0 -semigroup $T = (T(t))_{t \geq 0}$ on $L^2(\Omega)$ with generator A satisfies the estimate (1.2). Then it can be verified that the integral kernel of $T(t)$ ($t > 0$) satisfies the condition of Barnes, while the resolvent of A does not in general. Therefore, by Barnes' theorem, we can prove that if a C_0 -semigroup $T = (T(t))_{t \geq 0} = (e^{tA})_{t \geq 0}$ on $L^2(\Omega)$ satisfies a Gaussian estimate of order α for an $\alpha \in (0, 1]$ and a resolvent of the generator of T is normal, then the spectrum of $T_p(t)$ is independent of $p \in [1, \infty)$, where T_p is a version of T on $L^p(\Omega)$. However, in general, L^p -spectral independence of semigroups does not imply that of their generators. But we can fill this gap by considering simultaneously the spectrum of the semigroups generated by fractional powers $(-A)^\beta$ ($\beta \in (0, 1)$) of the generator A in question (Lemma 2.9). Noting that $e^{-t(-A)^\beta}$ satisfies an Gaussian estimate of order $\alpha\beta$ and combining the observations above, we obtain the desired L^p -spectral independence of the generators of T_p (Theorem 2.11).

§2. Gaussian estimates of order α and L^p -spectral independence

Hereafter Ω denotes an open subset of \mathbb{R}^N . In this section, we treat C_0 -semigroups that satisfy the following estimates.

Definition 2.1. Let $T = (T(t))_{t \geq 0}$ be a C_0 -semigroup on $L^2(\Omega)$ and $\alpha \in (0, 1]$. Then we say that T satisfies a Gaussian estimate of order α if there exist $M \geq 1, \omega \in \mathbb{R}$ and $b > 0$ such that

$$(2.1) \quad |T(t)f| \leq M e^{\omega t} e^{-bt(-\Delta)^\alpha} |f|$$

holds for all $t \geq 0$ and $f \in L^2(\Omega)$. Here, Δ denotes the usual Laplacian in $L^2(\mathbb{R}^N)$ with domain $H^2(\mathbb{R}^N)$, and we identify $L^2(\Omega)$ with a subspace of $L^2(\mathbb{R}^N)$ by considering the elements of $L^2(\Omega)$ to have value 0 on $\mathbb{R}^N \setminus \Omega$.

We collect some basic facts concerning the C_0 -semigroups which satisfy Gaussian estimates of order α .

Proposition 2.2. Let $T = (T(t))_{t \geq 0}$ be a C_0 -semigroup on $L^2(\Omega)$. Assume that T satisfies a Gaussian estimate of order α for an $\alpha \in (0, 1]$. Then the following assertions hold.

(i) For all $t > 0$, there exists a measurable function $K_t: \Omega \times \Omega \rightarrow \mathbb{C}$ such that for all $f \in L^2(\Omega)$,

$$(T(t)f)(x) = \int_{\Omega} K_t(x, y) f(y) dy$$

for a.e. $x \in \Omega$.

(ii) *There exists a constant $C > 0$ such that the function K_t in (i) satisfies the estimate*

$$|K_t(x, y)| \leq C e^{\omega t} \frac{bt}{((bt)^{\frac{1}{\alpha}} + |x - y|^2)^{\frac{N}{2} + \alpha}}$$

for all $t > 0$ and a.e. $(x, y) \in \Omega \times \Omega$.

(iii) *For each $p \in [1, \infty)$, there exists a unique C_0 -semigroup $T_p = (T_p(t))_{t \geq 0}$ on $L^p(\Omega)$ such that for all $t > 0$ and $f \in L^p(\Omega)$,*

$$(T_p(t)f)(x) = \int_{\Omega} K_t(x, y) f(y) dy$$

for a.e. $x \in \Omega$. (Note that K_t is independent of $p \in [1, \infty)$.)

Proof. (i) and (iii) are proved in Proposition 3.5 in [9].

(ii) follows from the estimates (3.5) and (3.4) in [9]. \square

In this paper, we use an abstract result by Barnes in [3]. To state his result, we define some function spaces and weight functions.

Definition 2.3 (cf. [3, pp. 122, 123]). (i) A_1 is defined as the space consisting of all measurable functions $K: \Omega \times \Omega \rightarrow \mathbb{C}$ such that

$$\|K\|_1 := \max \left\{ \operatorname{ess. sup}_{x \in \Omega} \int_{\Omega} |K(x, y)| dy, \operatorname{ess. sup}_{y \in \Omega} \int_{\Omega} |K(x, y)| dx \right\} < \infty.$$

Similarly, A_2 is defined as the linear space of all measurable functions $K: \Omega \times \Omega \rightarrow \mathbb{C}$ such that

$$\|K\|_2 := \max \left\{ \operatorname{ess. sup}_{x \in \Omega} \left(\int_{\Omega} |K(x, y)|^2 dy \right)^{\frac{1}{2}}, \operatorname{ess. sup}_{y \in \Omega} \left(\int_{\Omega} |K(x, y)|^2 dx \right)^{\frac{1}{2}} \right\} < \infty.$$

The space $(A_1, \|\cdot\|_1)$ and $(A_2, \|\cdot\|_2)$ are Banach spaces. Moreover, A_1 is a Banach $*$ -algebra with the following involution $K \mapsto K^*$ and multiplication:

$$\begin{aligned} K^*(x, y) &:= \overline{K(y, x)} \quad ((x, y) \in \Omega \times \Omega), \\ (K * J)(x, y) &:= \int_{\Omega} K(x, z) J(z, y) dz \quad (K, J \in A_1). \end{aligned}$$

(ii) The weight function w_{δ} is defined by

$$w_{\delta}(x, y) := (1 + |x - y|)^{\delta} \quad ((x, y) \in \mathbb{R}^N \times \mathbb{R}^N)$$

for each $\delta \in (0, 1]$. Let $A_{w_{\delta}}$ be the linear space of all measurable functions $K: \Omega \times \Omega \rightarrow \mathbb{C}$ such that $K w_{\delta} \in A_1$ and $\|\cdot\|_{w_{\delta}}$ be defined by $\|K\|_{w_{\delta}} := \|K w_{\delta}\|_1$ for each $\delta \in (0, 1]$, where $K w_{\delta}$ denotes the pointwise product of K and w_{δ} .

Then, A_{w_δ} is a $*$ -subalgebra of A_1 and $(A_{w_\delta}, \|\cdot\|_{w_\delta})$ is a Banach $*$ -algebra (cf. [3, Note 4.3]).

(iii) Let $\Gamma[m]$ be the set

$$\Gamma[m] := \{(x, y) \in \Omega \times \Omega \mid |x - y| \leq m\}$$

for each $m \in \mathbb{N}$ and $\chi(\Gamma)$ be the characteristic function of $\Gamma \subset \mathbb{R}^N$. A_1^0 is defined as the linear subspace of all $K \in A_1$ such that

$$\lim_{m \rightarrow \infty} \|\chi(\Gamma[m]^c)K\|_1 = 0.$$

A_1^0 is a closed $*$ -subalgebra of A_1 . In addition, A_2^0 and $A_{w_\delta}^0$ are defined as subspaces of A_2 and A_{w_δ} by replacing $\|\cdot\|_1$ with $\|\cdot\|_2$ and $\|\cdot\|_{w_\delta}$, respectively in the definition of A_1^0 .

(iv) Let $A_{w_\delta,2} := A_{w_\delta} \cap A_2$, $A_{w_\delta,2}^{0,0} := A_{w_\delta}^0 \cap A_2^0$ for each $\delta \in (0, 1]$ and $\|K\|_{w_\delta,2} := \max\{\|K\|_{w_\delta}, \|K\|_2\}$. Then, $(A_{w_\delta,2}, \|\cdot\|_{w_\delta,2})$ is a Banach $*$ -algebra (cf. [3, Lemma 4.4]) and $A_{w_\delta,2}^{0,0}$ is a closed $*$ -subalgebra of A_{w_δ} .

Remark 2.4. As is stated in [3, p. 122], any $K \in A_1$ defines the bounded linear operator K_p on $L^p(\Omega)$ by

$$(K_p f)(x) := \int_{\Omega} K(x, y) f(y) dy \quad (f \in L^p(\Omega), x \in \Omega)$$

for each $p \in [1, \infty]$.

Now, we introduce a result by Barnes in [3]. For the reason described in Remark 2.6 below, we state it in a form where its “assumption part” is a little strengthened.

Theorem 2.5 (Barnes, cf. [3, Theorem 4.8]). *Assume that K is in $A_{w_\delta,2}^{0,0}$ for some $\delta \in (0, 1]$. Then the following assertions hold:*

(i) $\sigma_{w_\delta,2}(K) = \sigma(K_p)$ for all $p \in [1, \infty]$ when K is normal (i.e., $K^* * K = K * K^*$).

(ii) $\sigma_{w_\delta,2}(K) = \sigma(K_p) \cup \overline{\sigma((K^*)_p)}$ for all $p \in [1, \infty]$ in general.

In these assertions, $\sigma_{w_\delta,2}(K)$ denotes the spectrum of K in $A_{w_\delta,2}$ and K_p is as in Remark 2.4.

Remark 2.6. Let $A_{w_\delta,2}^0 := A_{w_\delta}^0 \cap A_2$. Theorem 4.8 in [3] states that the same conclusions (i), (ii) in Theorem 2.5 hold for all $K \in A_{w_\delta,2}^0$. Moreover, in the proof of Theorem 4.8 in [3], it is claimed that if $K = K^* \in A_{w_\delta,2}^0$, then we have

$$\|\chi(\Gamma[m])K - K\|_{w_\delta,2} \rightarrow 0$$

as $m \rightarrow \infty$, in other words, $K \in A_{w_\delta, 2}^{0,0}$. However, let K be defined by

$$K(x, y) := \begin{cases} \sqrt{y} & (y \geq 2, 2y \leq x \leq 2y + 1/y) \\ 0 & (\text{otherwise}). \end{cases}$$

Then, $K + K^*$ is hermitian and belongs to $A_{w_\delta, 2}^0$ for each $\delta \in (0, 1/2)$ but does not belong to $A_{w_\delta, 2}^{0,0}$ for any $\delta \in (0, 1/2)$. We will give a detailed proof of this fact in Section 3. For this reason, we replaced $A_{w_\delta, 2}^0$ in Theorem 4.8 in [3] with $A_{w_\delta, 2}^{0,0}$. Once this replacement is made, Theorem 2.5 can be proved in exactly the same way as in [3] except for the part concerning the assertion $K \in A_{w_\delta, 2}^{0,0}$.

Remark 2.7. It is easy to see that for all $K \in A_1$,

$$((K^*)_p)' f = \overline{K_{p'} f} \quad (f \in L^p(\Omega))$$

for each $p \in [1, \infty)$, where $((K^*)_p)'$ is the conjugate operator of $(K^*)_p$ and p' is the conjugate exponent of p . Hence, it follows from assertion (ii) that

$$\sigma_{w_\delta, 2}(K) = \sigma(K_p) \cup \sigma(K_{p'})$$

holds for each $p \in [1, \infty)$.

Here we would like to note the following relation between a C_0 -semigroup T on $L^2(\Omega)$ satisfying a Gaussian estimate of order α and the Banach $*$ -algebra $A_{w_\delta, 2}^{0,0}$ in Barnes' theorem.

Lemma 2.8. *Let $T = (T(t))_{t \geq 0}$ be a C_0 -semigroup on $L^2(\Omega)$ and $\alpha \in (0, 1]$ and suppose that T satisfies a Gaussian estimate of order α . Moreover, let $K_t: \Omega \times \Omega \rightarrow \mathbb{C}$ be the integral kernel of $T(t)$ for each $t > 0$ as in Proposition 2.2. Then, $K_t \in A_{w_\delta, 2}^{0,0}$ holds for each $\delta \in (0, 2\alpha)$.*

Proof. This assertion readily follows from Proposition 2.2 (ii). \square

Now, let $T = (T(t))_{t \geq 0}$ be a C_0 -semigroup on $L^2(\Omega)$ satisfying a Gaussian estimate of order α for an $\alpha \in (0, 1]$. Lemma 2.8 and Barnes' theorem imply that if T is normal, the spectrum of $T_p(t)$ is independent of $p \in [1, \infty)$. However, it is not evident that the spectrum of the generator A_p of T_p is independent of $p \in [1, \infty)$. The next lemma connects L^p -spectral independence of T_p 's to that of A_p 's, which is the key in this paper. The lemma depends heavily on the theory of fractional powers of a generator of a C_0 -semigroup and the spectral mapping theorem.

Lemma 2.9. *Let $T_p = (T_p(t))_{t \geq 0}$ be a bounded C_0 -semigroup on $L^p(\Omega)$ with generator A_p for each $p \in [1, \infty)$. Then the following assertions hold.*

(i) Assume that there exists a $t_0 > 0$ such that for all $\beta \in (0, 1)$ the spectrum of $e^{-t_0(-A_p)^\beta}$ is independent of $p \in [1, \infty)$. Then the spectrum of A_p is independent of $p \in [1, \infty)$.

(ii) Assume that there exists a $t_0 > 0$ such that for all $\beta \in (0, 1)$, the union $\sigma(e^{-t_0(-A_p)^\beta}) \cup \sigma(e^{-t_0(-A_{p'})^\beta})$ is independent of $p \in (1, \infty)$. Then $\sigma(A_p) \cup \sigma(A_{p'})$ is independent of $p \in (1, \infty)$.

Proof. (i) As is well-known, for each $p \in [1, \infty)$ and $\beta \in (0, 1)$, the fractional power $-(-A_p)^\beta$ generates a bounded analytic semigroup with angle $\pi(1 - \beta)/2$. Hence, $\sigma((-A_p)^\beta)$ is included in the sector $\{\lambda \in \mathbb{C} \mid |\arg \lambda| < \pi\beta/2\}$. Keeping this in mind, let p and q be in $[1, \infty)$ and $\lambda \in \sigma(A_p)$. We use the spectral mapping theorem

$$\sigma((-A_p)^\beta) = [\sigma(-A_p)]^\beta (= \{(-\lambda)^\beta \mid \lambda \in \sigma(A_p)\})$$

by Theorem 3.1 in [2] or Theorem 5.3.1 in [7], where β is an arbitrary number in $(0, 1)$ and $(-\lambda)^\beta$ denotes the principal value of $e^{\beta \log(-\lambda)}$ for $\lambda \neq 0$ and denotes 0 for $\lambda = 0$. This equality means that in the case of $0 \in \sigma(A_p)$, we have $0 \in \sigma((-A_p)^\beta)$. The spectral mapping theorem implies that

$$e^{-t_0(-\lambda)^\beta} \in e^{-t_0\sigma((-A_p)^\beta)}$$

for all $\beta \in (0, 1)$. In addition, since $e^{-t_0(-A_p)^\beta}$ is a bounded analytic semigroup as stated above, the spectral mapping theorem

$$(2.2) \quad e^{-t_0\sigma((-A_p)^\beta)} = \sigma(e^{-t_0(-A_p)^\beta}) \setminus \{0\}$$

holds for all $\beta \in (0, 1)$ (cf. Corollary 3.12 in [4]). Thus, we have

$$e^{-t_0(-\lambda)^\beta} \in \sigma(e^{-t_0(-A_p)^\beta}) \setminus \{0\}$$

for all $\beta \in (0, 1)$. Since $\sigma(e^{-t_0(-A_p)^\beta}) \setminus \{0\} = \sigma(e^{-t_0(-A_q)^\beta}) \setminus \{0\}$ by the assumption and (2.2) holds also in the case where p is replaced with q ,

$$e^{-t_0(-\lambda)^\beta} \in e^{-t_0\sigma((-A_q)^\beta)}$$

for all $\beta \in (0, 1)$.

Hence for all $\beta \in (0, 1)$ there exists an $n_\beta \in \mathbb{Z}$ such that

$$(-\lambda)^\beta + \frac{2n_\beta\pi i}{t_0} \in \sigma((-A_q)^\beta).$$

In the case of $\lambda = 0$, $n_\beta \neq 0$ implies $(-\lambda)^\beta + 2n_\beta\pi i/t_0 \in i\mathbb{R} \setminus \{0\}$, hence $(-\lambda)^\beta + 2n_\beta\pi i/t_0 \notin \sigma((-A_q)^\beta)$. Therefore $n_\beta = 0$ and hence $(-\lambda)^\beta \in \sigma((-A_q)^\beta)$

holds in this case. So let $\lambda \neq 0$ in what follows. Suppose that $\beta \in (0, 1)$ is sufficiently small so that

$$\operatorname{Re}(-\lambda)^\beta \tan\left(\frac{\pi}{2}\beta\right) < \frac{\pi}{t_0}.$$

If $n_\beta \neq 0$, then

$$\frac{\pi}{2} > \left| \arg\left((- \lambda)^\beta + \frac{2n_\beta \pi i}{t_0}\right) \right| > \frac{\pi}{2}\beta,$$

hence, $(-\lambda)^\beta + 2n_\beta \pi i/t_0 \notin \sigma((-A_q)^\beta)$. Therefore $n_\beta = 0$ and $(-\lambda)^\beta \in \sigma((-A_q)^\beta)$, hence $\lambda \in \sigma(A_q)$ by Theorem 3.1 in [2].

(ii) This assertion is proved in a similar way as in the proof of (i). \square

We need the next proposition to use Lemma 2.9.

Proposition 2.10. *Let $T = (T(t))_{t \geq 0}$ be a C_0 -semigroup on $L^2(\Omega)$ with generator A and suppose that T satisfies a Gaussian estimate of order α for an $\alpha \in (0, 1]$ with $\omega = 0$ in (2.1). Then the following assertions hold.*

(i) *For all $\beta \in (0, 1)$, the C_0 -semigroup $e^{-t(-A)^\beta}$ satisfies a Gaussian estimate of order $\alpha\beta$. In addition, for all $\beta \in (0, 1)$ and $t > 0$, $e^{-t(-A)^\beta}$ is an integral operator and its kernel $K_{t,\beta}(x, y)$ satisfies the following estimate: There exists a constant $C_\beta > 0$ such that for all $t > 0$*

$$(2.3) \quad |K_{t,\beta}(x, y)| \leq C_\beta \frac{b^\beta t}{(b^{\frac{1}{\alpha}} t^{\frac{1}{\alpha\beta}} + |x - y|^2)^{\frac{N}{2} + \alpha\beta}}$$

for a.e. $(x, y) \in \Omega \times \Omega$, where b is as in (2.1).

(ii) *For all $\beta \in (0, 1)$ and $p \in [1, \infty)$, there exists a C_0 -semigroup $T_{\beta,p} = (T_{\beta,p}(t))_{t \geq 0}$ on $L^p(\Omega)$ such that $T_{\beta,p}$ is consistent with $e^{-t(-A)^\beta}$ (i.e., $T_{\beta,p}(t) = e^{-t(-A)^\beta}$ on $L^p(\Omega) \cap L^2(\Omega)$ for all $t \geq 0$). Moreover, $T_{\beta,p}(t)$ coincides with $e^{-t(-A_p)^\beta}$ for all $\beta \in (0, 1)$, $t \geq 0$ and $p \in [1, \infty)$, where A_p is the generator of T_p in Proposition 2.2.*

Proof. By the formula (2) in [11, Chapter IX, Section 11], for all $\beta \in (0, 1)$, $t > 0$ and $f \in L^2(\Omega)$,

$$\begin{aligned} |e^{-t(-A)^\beta} f| &= \left| \int_0^\infty f_{t,\beta}(s) e^{sA} f \, ds \right| \\ &\leq M \int_0^\infty f_{t,\beta}(s) e^{-bs(-\Delta)^\alpha} |f| \, ds \\ &= M e^{-b^\beta t ((-\Delta)^\alpha)^\beta} |f| \\ &= M e^{-b^\beta t (-\Delta)^{\alpha\beta}} |f|. \end{aligned}$$

(The function $f_{t,\beta} \geq 0$ is defined in [11, Chapter IX, Section 11 (1)].) Thus, $e^{-t(-A)^\beta}$ satisfies a Gaussian estimate of order $\alpha\beta$ with $\omega = 0$.

The latter assertion of (i) readily follows from Proposition 2.2.

Now we prove (ii). By assertion (i) and Proposition 2.2, there exists a C_0 -semigroup $T_{\beta,p} = (T_{\beta,p}(t))_{t \geq 0}$ such that $T_{\beta,p}$ is consistent with $e^{-t(-A)^\beta}$. On the other hand, since e^{tA_p} is consistent with e^{tA} , the formula in [11, Chapter IX, Section 11] implies that $e^{-t(-A_p)^\beta}$ is consistent with $e^{-t(-A)^\beta}$. Since the C_0 -semigroup on $L^p(\Omega)$ that is consistent with $e^{-t(-A)^\beta}$ is unique, we have $e^{-t(-A_p)^\beta} = T_{\beta,p}(t)$. Thus the proof is completed. \square

Now we are in a position to prove our main result. The authors would like to emphasize that the following Theorem 2.11 considerably improves our former results (Theorem 3.18, 3.19 and 3.20 in [9]).

Theorem 2.11. *Let $T = (T(t))_{t \geq 0}$ be a C_0 -semigroup on $L^2(\Omega)$ with generator A and suppose that T satisfies a Gaussian estimate of order α for some $\alpha \in (0, 1]$. Moreover, let $T_p = (T_p(t))_{t \geq 0}$ be the C_0 -semigroup naturally defined by T on $L^p(\Omega)$ for each $p \in [1, \infty)$ as in Proposition 2.2. Then, for the generator A_p of T_p , the following assertions hold.*

(i) *Let ω be as in (2.1). Assume that there exists a $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda > \omega$ such that $(\lambda - A)^{-1}$ is normal. Then $\sigma(A_p)$ is independent of $p \in [1, \infty)$.*

(ii) *$\sigma(A_p) \cup \sigma(A_{p'})$ is independent of $p \in (1, \infty)$ in general.*

Proof. (i) We may assume $\omega = 0$ in (2.1) (if necessary, consider $A - \omega$). We first show that $e^{-t(-A)^\beta}$ is normal for each $\beta \in (0, 1)$. In fact, by the assumption, there exists a $\lambda \in \rho(A) = \overline{\rho(A^*)}$ with $\operatorname{Re} \lambda > 0$, where A^* is the adjoint operator of A , such that $(\lambda - A)^{-1}$ and $((\lambda - A)^{-1})^* (= (\bar{\lambda} - A^*)^{-1})$ are commutative. (Note that λ with $\operatorname{Re} \lambda > 0$ belongs to $\rho(A)$ since e^{tA} is a bounded C_0 -semigroup.) If $|\mu - \lambda|$ and $|\nu - \bar{\lambda}|$ are sufficiently small, then $(\mu - A)^{-1}$ and $(\nu - A^*)^{-1}$ can be expanded into the infinite series at λ and $\bar{\lambda}$, respectively. Hence, for such μ and ν , $(\mu - A)^{-1}$ and $(\nu - A^*)^{-1}$ are commutative:

$$(2.4) \quad (\mu - A)^{-1}(\nu - A^*)^{-1} = (\nu - A^*)^{-1}(\mu - A)^{-1}.$$

Since both sides of this equality are holomorphic in $\mu \in \rho(A)$ for each $\nu \in \rho(A^*)$, by unique continuation, (2.4) holds for each $\mu \in \rho_\infty(A)$ and ν in a neighborhood of $\bar{\lambda}$, where $\rho_\infty(A)$ is the connected component of $\rho(A)$ including the right half-plane $\{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda > 0\}$. Accordingly since both sides of (2.4) are holomorphic in $\nu \in \rho(A^*)$ for each $\mu \in \rho_\infty(A)$, by unique continuation, (2.4) holds for each $\mu \in \rho_\infty(A)$ and $\nu \in \rho_\infty(A^*)$. In particular, $(\mu - A)^{-1}$ and $(\nu - A^*)^{-1}$ are commutative for each $\mu, \nu > 0$. By using the well-known

formula

$$(\lambda_1 + (-A)^\beta)^{-1} = \frac{\sin(\pi\beta)}{\pi} \int_0^\infty \frac{\mu^\beta(\mu - A)^{-1}}{\mu^{2\beta} + 2\lambda_1\mu^\beta \cos(\pi\beta) + \lambda_1^2} d\mu$$

for all $\lambda_1 > 0$ (cf. [7, (5.24)]) and the resulting equality

$$\begin{aligned} [(\lambda_2 + (-A)^\beta)^{-1}]^* &= \frac{\sin(\pi\beta)}{\pi} \left[\int_0^\infty \frac{\nu^\beta(\nu - A)^{-1}}{\nu^{2\beta} + 2\lambda_2\nu^\beta \cos(\pi\beta) + \lambda_2^2} d\nu \right]^* \\ &= \frac{\sin(\pi\beta)}{\pi} \int_0^\infty \frac{\nu^\beta(\nu - A^*)^{-1}}{\nu^{2\beta} + 2\lambda_2\nu^\beta \cos(\pi\beta) + \lambda_2^2} d\nu \end{aligned}$$

for all $\lambda_2 > 0$, we obtain that $(\lambda_1 + (-A)^\beta)^{-1}$ and $[(\lambda_2 + (-A)^\beta)^{-1}]^*$ are commutative for all $\lambda_1, \lambda_2 > 0$. Since $(\frac{n}{t}(\frac{n}{t} + (-A)^\beta)^{-1})^n$ strongly converges to $e^{-t(-A)^\beta}$ as $n \rightarrow \infty$ and $[(\frac{m}{t}(\frac{m}{t} + (-A)^\beta)^{-1})^m]^*$ strongly converges to $(e^{-t(-A)^\beta})^*$ as $m \rightarrow \infty$, we conclude that $e^{-t(-A)^\beta}$ and $(e^{-t(-A)^\beta})^*$ are commutative. i.e., $e^{-t(-A)^\beta}$ is normal.

Next fix an arbitrary $t_0 > 0$. Let $\beta \in (0, 1)$ and $p \in [1, \infty)$. Then, by Proposition 2.10, there exists a C_0 -semigroup $T_{\beta,p} = (T_{\beta,p}(t))_{t \geq 0}$ on $L^p(\Omega)$ such that $T_{\beta,p}$ is consistent with $e^{-t(-A)^\beta}$. In addition, $T_{\beta,p}(t_0)$ is an integral operator, and its kernel $K_{t_0,\beta}$ is independent of $p \in [1, \infty)$ and $K_{t_0,\beta} \in A_{w_\delta,2}^{0,0}$ for each $\delta \in (0, 2\alpha\beta)$ by Lemma 2.8. Since $e^{-t_0(-A)^\beta}$ is normal and so is $K_{t_0,\beta}$, by applying Barnes' theorem, $\sigma(T_{\beta,p}(t_0))$ is proved to be independent of $p \in [1, \infty)$. Hence, $\sigma(e^{-t_0(-A)^\beta})$ is independent of $p \in [1, \infty)$ (cf. Proposition 2.10 (ii)). By Lemma 2.9 (i), $\sigma(A_p)$ is independent of $p \in [1, \infty)$.

(ii) is proved by using Lemma 2.9 (ii) instead of Lemma 2.9 (i) in the proof of assertion (i). \square

Now, we give a corollary to Theorem 2.11, which partly improves Theorem 4.2 in [10]. For each $\alpha \in (0, 1]$, H_α and $U_\alpha(t)$ denotes $(-\Delta)^\alpha$ and e^{-tH_α} , respectively, and let $V: \mathbb{R}^N \rightarrow \mathbb{R}$ be a bounded non-negative measurable function. We verify L^p -spectral independence of a version $H_\alpha + V$ in $L^p(\mathbb{R}^N)$, where we used the same symbol for the function V and also for the associated maximal multiplication operator in $L^p(\mathbb{R}^N)$ defined by V .

Corollary 2.12. *The operator sum $-(H_\alpha + V)$ generates a C_0 -semigroup $U_{\alpha,V} = (U_{\alpha,V}(t))_{t \geq 0}$ on $L^2(\mathbb{R}^N)$ and there exists a C_0 -semigroup $U_{\alpha,V,p} = (U_{\alpha,V,p}(t))_{t \geq 0}$ on $L^p(\mathbb{R}^N)$ such that $U_{\alpha,V,p}$ is consistent with $U_{\alpha,V}$ for each $p \in [1, \infty)$. The generator $-H_{\alpha,V,p}$ of $U_{\alpha,V,p}$ coincides with $-(H_{\alpha,p} + V)$ for each $p \in [1, \infty)$, where $-H_{\alpha,p}$ is the generator of the C_0 -semigroup naturally*

defined by U_α on $L^p(\mathbb{R}^N)$ for each $p \in [1, \infty)$ as in Proposition 2.2. Moreover, the spectrum

$$\sigma(H_{\alpha,p} + V)$$

is independent of $p \in [1, \infty)$.

Proof. It is clear that $-(H_\alpha + V)$ generates a positive C_0 -semigroup $U_{\alpha,V} = (U_{\alpha,V}(t))_{t \geq 0}$ on $L^2(\mathbb{R}^N)$ and $U_{\alpha,V}$ satisfies a Gaussian estimate of order α . More precisely,

$$0 \leq U_{\alpha,V}(t) \leq U_\alpha(t)$$

is obtained for all $t \geq 0$ by using Trotter product formula. Hence, by Proposition 2.2, there exists a C_0 -semigroup $U_{\alpha,V,p} = (U_{\alpha,V,p}(t))_{t \geq 0}$ on $L^p(\mathbb{R}^N)$ such that $U_{\alpha,V,p}$ is consistent with $U_{\alpha,V}$ for each $p \in [1, \infty)$. Since Trotter product formula implies that the C_0 -semigroup $\exp(-(H_{\alpha,p} + V))$ is consistent with $U_{\alpha,V}$, we have $U_{\alpha,V,p}$ coincides with $\exp(-(H_{\alpha,p} + V))$. Hence, $H_{\alpha,V,p} = H_{\alpha,p} + V$, where $H_{\alpha,V,p}$ is the generator of $U_{\alpha,V,p}$.

Since the generator of $U_{\alpha,V}$ is self-adjoint, Theorem 2.11 implies that the spectrum of $H_{\alpha,V,p}$ is independent of $p \in [1, \infty)$. Thus, the proof is completed. \square

§3. Appendix

We prove the statement in Remark 2.6. We first recall what we should prove.

Proposition 3.1. *Let K be defined by*

$$K(x, y) := \begin{cases} \sqrt{y} & (y \geq 2, 2y \leq x \leq 2y + 1/y) \\ 0 & (\text{otherwise}). \end{cases}$$

Then, $K + K^$ is hermitian and belongs to $A_{w_\delta, 2}^0$ for each $\delta \in (0, 1/2)$ but does not belong to $A_{w_\delta, 2}^{0,0}$ for any $\delta \in (0, 1/2)$.*

Proof. Let $\delta \in (0, 1/2)$. We prove that $K \in A_{w_\delta, 2}^0$ and $K \notin A_{w_\delta, 2}^{0,0}$, from which the assertion of this proposition follows. In fact, since $(w_\delta K)^* = w_\delta K^*$, $(\chi(\Gamma[m])K)^* = \chi(\Gamma[m])K^*$ and the involution is isometric in each of the norms of A_1, A_{w_δ} and A_2 , we obtain that K^* hence $K + K^*$ belongs to $A_{w_\delta, 2}^0$. On the other hand, since $\text{supp } K \cap \text{supp } K^* = \emptyset$, the inequality $\|\chi(\Gamma[m]^c)(K + K^*)\|_2 \geq \|\chi(\Gamma[m]^c)K\|_2$ holds for all $m \in \mathbb{N}$. Hence, by $K \notin A_2^0$, $\|\chi(\Gamma[m]^c)(K + K^*)\|_2$ does not converge to 0 as $m \rightarrow \infty$. i.e., $K + K^* \notin A_2^{0,0}$. Since it is clear that $K + K^*$ is hermitian, we see that the desired assertion concerning K leads to the assertion of this proposition.

Now, we prove that $K \in A_{w_\delta, 2}^0$. We first estimate $w_\delta(x, y)$ for all $(x, y) \in \text{supp } K$. Since each $(x, y) \in \text{supp } K$ satisfies the estimate $2y \leq x \leq 2y + 1/2$, we have

$$(3.1) \quad |x - y| \leq y + \frac{1}{2} \leq \frac{1}{2}(x + 1).$$

Hence, $w_\delta(x, y) \leq (3/2 + y)^\delta$ and $w_\delta(x, y) \leq 2^{-\delta}(3 + x)^\delta$ for all $(x, y) \in \text{supp } K$.

Next, we estimate the integrals $\int_{\mathbb{R}} K(x, y) dx$ and $\int_{\mathbb{R}} K(x, y) dy$. It is easy to see that

$$\int_{\mathbb{R}} K(x, y) dx = \begin{cases} 0 & (y < 2) \\ \frac{1}{\sqrt{y}} & (y \geq 2), \end{cases}$$

and there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}} K(x, y) dy \begin{cases} = 0 & (x < 4) \\ \leq C & (4 \leq x \leq 9/2). \end{cases}$$

In the case of $x > 9/2$, we have by using the trivial inequality $\sqrt{x^2 - 8} < x$

$$\begin{aligned} \int_{\mathbb{R}} K(x, y) dy &= \int_{(x+\sqrt{x^2-8})/4}^{x/2} \sqrt{y} dy \\ &= \frac{2}{3} \left\{ \left(\frac{x}{2} \right)^{\frac{3}{2}} - \left(\frac{x + \sqrt{x^2 - 8}}{4} \right)^{\frac{3}{2}} \right\} \\ &= \frac{2}{3} \left\{ \left(\frac{x}{2} \right)^{\frac{1}{2}} - \left(\frac{x + \sqrt{x^2 - 8}}{4} \right)^{\frac{1}{2}} \right\} \\ &\quad \times \left\{ \frac{x}{2} + \left(\frac{x}{2} \right)^{\frac{1}{2}} \left(\frac{x + \sqrt{x^2 - 8}}{4} \right)^{\frac{1}{2}} + \frac{x + \sqrt{x^2 - 8}}{4} \right\} \\ &< \frac{2}{3} \cdot \frac{x - \sqrt{x^2 - 8}}{2(\sqrt{2x} + (x + \sqrt{x^2 - 8})^{\frac{1}{2}})} \cdot \frac{3x}{2} \\ &< \frac{\sqrt{x}}{2\sqrt{2}} (x - \sqrt{x^2 - 8}) \\ &= \frac{\sqrt{x}}{2\sqrt{2}} \cdot \frac{8}{x + \sqrt{x^2 - 8}} < \frac{2\sqrt{2}}{\sqrt{x}}. \end{aligned}$$

Hence, $K \in A_{w_\delta}$ is shown for each $\delta \in (0, 1/2)$ by the following estimate:

$$\begin{aligned} \text{ess.sup}_{y \in \mathbb{R}} \int_{\mathbb{R}} w_\delta(x, y) K(x, y) dx &\leq \text{ess.sup}_{y \geq 2} \left(\frac{3}{2} + y \right)^\delta \cdot \frac{1}{\sqrt{y}} < \infty, \\ \text{ess.sup}_{x \leq 9/2} \int_{\mathbb{R}} w_\delta(x, y) K(x, y) dy &\leq 2^{-\delta} C \text{ess.sup}_{4 \leq x \leq 9/2} (3 + x)^\delta = \left(\frac{15}{4} \right)^\delta C < \infty, \\ \text{ess.sup}_{x > 9/2} \int_{\mathbb{R}} w_\delta(x, y) K(x, y) dy &\leq 2^{-\delta} \text{ess.sup}_{x > 9/2} (3 + x)^\delta \cdot \frac{2\sqrt{2}}{\sqrt{x}} < \infty. \end{aligned}$$

By a similar manner, we can prove that $K \in A_{w_\delta}^0$ for each $\delta \in (0, 1/2)$. In fact, if $(x, y) \in \text{supp } K$ satisfies $|x - y| > m$ for an $m \in \mathbb{N}$, then $y \geq m - 1/2$ and $x \geq 2m - 1$ by (3.1). Hence, we have for $m \geq 3$,

$$\begin{aligned} & \text{ess.sup}_{y \in \mathbb{R}} \int_{\mathbb{R}} \chi(\Gamma[m]^c)(x, y) w_\delta(x, y) K(x, y) dx \\ & \leq \text{ess.sup}_{y \geq m-1/2} \int_{\mathbb{R}} w_\delta(x, y) K(x, y) dx \\ & \leq \text{ess.sup}_{y \geq m-1/2} \left(\frac{3}{2} + y \right)^\delta \cdot \frac{1}{\sqrt{y}}, \\ & \text{ess.sup}_{x \in \mathbb{R}} \int_{\mathbb{R}} \chi(\Gamma[m]^c)(x, y) w_\delta(x, y) K(x, y) dy \\ & \leq \text{ess.sup}_{x \geq 2m-1} \int_{\mathbb{R}} w_\delta(x, y) K(x, y) dy \\ & \leq 2^{-\delta} \text{ess.sup}_{x \geq 2m-1} (3 + x)^\delta \cdot \frac{2\sqrt{2}}{\sqrt{x}}. \end{aligned}$$

Since the rightmost side of each inequality above converges to 0 as $m \rightarrow \infty$, the norm $\|\chi(\Gamma[m]^c)K\|_{w_\delta}$ converges to 0 as $m \rightarrow \infty$. i.e., $K \in A_{w_\delta}^0$.

Next, we verify that $K \in A_2$ for the completeness of the proof. It is easy to see that

$$\int_{\mathbb{R}} K(x, y)^2 dx = \begin{cases} 0 & (y < 2), \\ 1 & (y \geq 2), \end{cases}$$

and there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}} K(x, y)^2 dy \begin{cases} = 0 & (x < 4), \\ \leq C & (4 \leq x \leq 9/2). \end{cases}$$

In the case of $x > 9/2$, we have

$$\begin{aligned} \int_{\mathbb{R}} K(x, y)^2 dy &= \int_{(x+\sqrt{x^2-8})/4}^{x/2} y dy \\ &= \frac{1}{32} \{4x^2 - (x + \sqrt{x^2-8})^2\} \\ &= \frac{1}{32} (x - \sqrt{x^2-8})(3x + \sqrt{x^2-8}) \\ &\leq \frac{x}{8} \cdot \frac{8}{x + \sqrt{x^2-8}} \leq 1. \end{aligned}$$

Thus, $\|K\|_2 \leq \max\{1, \sqrt{C}\}$, hence, $K \in A_2$.

The remaining assertion is that $K \notin A_2^0$. To prove this assertion, note that if $(x, y) \in \text{supp } K$ satisfies $y > m$, then $x - y \geq y > m$, i.e., $(x, y) \notin \Gamma[m]$. Hence, we have

$$\text{ess.sup}_{y \in \mathbb{R}} \int_{\mathbb{R}} \chi(\Gamma[m]^c) K(x, y)^2 dx \geq \text{ess.sup}_{y \geq m+1} \int_{\mathbb{R}} K(x, y)^2 dx = 1$$

for all $m \in \mathbb{N}$. Thus, we conclude $K \notin A_2^0$. \square

Acknowledgments

The authors would like to thank the referee for useful comments.

References

- [1] Arendt, W., *Gaussian estimates and interpolation of the spectrum in L^p* , Diff. Int. Equations **7**(5) (1994), 1153–1168.
- [2] Balakrishnan, A.V., *Fractional powers of closed operators and the semigroups generated by them*, Pacific Journal of Mathematics **10**(1960), 419–437.
- [3] Barnes, B.A., *The spectrum of integral operators on Lebesgue spaces*, J. Operator Theory **18**(1987), 115–132.
- [4] Engel, K.J. and Nagel, R., “One-parameter semigroups for linear evolution equations”, Graduate texts in mathematics (no. 194), Springer-Verlag, New York, 2000.
- [5] Hempel, R. and Voigt, J., *The spectrum of a Schrödinger operator in $L_p(\mathbb{R}^\nu)$ is p -independent*, Comm. Math. Phys. **104** (1986), 243–250.
- [6] Kunstmann, P.C., *Kernel estimates and L^p -spectral independence of differential and integral operators*, Operator theoretical methods (Timișoara, 1998), 197–211, The Theta Foundation, Bucharest, 2000.
- [7] Martínez, C.C. and Sanz, M.A., “The theory of fractional powers of operators”, North-Holland Mathematics Studies, 187, North-Holland Publishing Co., Amsterdam, 2001.
- [8] Miyajima, S. and Ishikawa, M., *Generalization of Gaussian estimates and interpolation of the spectrum in L^p* , SUT J. Math. **31**(2) (1995), 161–176.
- [9] Miyajima, S. and Shindoh, H., *Gaussian estimates of order α and L^p -spectral independence of generators of C_0 -semigroups*, Positivity, to appear.
- [10] Shindoh, H., *L^p -spectral independence of fractional Laplacians perturbed by potentials*, SUT J. Math., to appear.

- [11] Yosida, K., “Functional analysis (6th edition)”, Springer-Verlag, Berlin, 1995.

Shizuo Miyajima

Department of Mathematics, Faculty of Science, Tokyo University of Science
26 Wakamiya-cho, Shinjuku-ku, Tokyo 162-0827, JAPAN

E-mail: miyajima@rs.kagu.tus.ac.jp

Hisakazu Shindoh

Department of Mathematics, Faculty of Science, Tokyo University of Science
26 Wakamiya-cho, Shinjuku-ku, Tokyo 162-0827, JAPAN

E-mail: j1104704@ed.kagu.tus.ac.jp