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Longest Cycles of a 3-Connected Graph Which Contain Four Contractible Edges

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Abstract. We classify all pairs (G,C) of a 3-connected graph G and a longest cycle C of G such that C contains precisely four contractible edges of G.

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§1. Introduction

In this paper, we consider only finite, simple, undirected graphs with no loops and no multiple edges.

A graph G is called 3-*connected* if $|V(G)| \geq 4$ and $G - S$ is connected for any subset S of $V(G)$ having cardinality 2. An edge e of a 3-connected graph G is called *contractible* if the graph which we obtain from G by contracting e (and replacing each of the resulting pairs of parallel edges by a simple edge) is 3-connected; otherwise e is called *noncontractible*. In [7], Tutte proved that all 3-connected graphs other than K_4 have a contractible edge. In [2], Dean, Hemminger and Ota proved that every longest cycle in a 3-connected graph other than K_4 or $K_2 \times K_3$ contains at least three contractible edges. In [3], Ellingham, Hemminger and Johnson proved that every longest cycle in a nonhamiltonian 3-connected graph has at least six contractible edges. In view of these results, it is likely and desirable that one should obtain a complete classification of those pairs (G, C) of a 3-connected graph G and a longest cycle C of G such that C contains at most five contractible edges. In fact, the case where C contains precisely three contractible edges was settled by Aldred, Hemminger and Ota in [1], and by Ota in [6]. Further in [4], Fujita classified all pairs (G, C) of a 3-connected graph G and a longest cycle C of G such that C contains precisely four contractible edges of G, under the assumption that

G has order at least 13. In this paper, we completely classify such pairs (G, C) without any assumption on the order of G.

Main Theorem. *Let G be a 3-connected graph, and let C be a longest cycle of G. Suppose that C contains precisely four contractible edges of G. Then the pair (G, C) belongs to one of the eight types, Types 1 through 8, which are defined in Section 2.*

In passing, we remarked that in [5], Fujita and Kotani classified all pairs (G, C) of a 3-connected graph G of order at least 16 and a longest cycle C of G such that C contains precisely five contractible edges of G.

The organization of this paper is as follows. In Section 2, we define the type of a pair (G, C) satisfying the assumptions of the Main Theorem. Section 3 contains fundamental results concerning noncontractible edges lying on a hamiltonian cycle of a 3-connected graph. In Section 4, we derive basic properties of a pair (G, C) satisfying the assumptions of the Main Theorem, and we complete the proof of the Main Theorem in Section 5.

Our notation and terminology are standard except possibly for the following. Let G be a graph. For $U \subseteq V(G)$, we let $\langle U \rangle = \langle U \rangle_G$ denote the graph induced by U in G. For U, $V \subseteq V(G)$, we let $E(U, V)$ denote the set of edges of G which join a vertex in U and a vertex in V; if $U = \{u\}$ $(u \in V(G))$, we write $E(u, V)$ for $E(\{u\}, V)$. A subset S of $V(G)$ is called a *cutset* if $G - S$ is disconnected; thus G is 3-connected if and only if $|V(G)| \geq 4$ and G has no cutset of cardinality 2. Now assume that G is 3-connected, and let $e = uv \in E(G)$. We let $K(e) = K(u, v)$ denote the set of vertices x of G such that $\{u, v, x\}$ is a cutset; thus e is contractible if and only if $K(e) = \phi$. If e is noncontractible, then for each $x \in K(e)$, $\{u, v, x\}$ is called a *cutset associated with e*.

§2. Definition of the Type of a Pair (G, C)

In this section, we define the type of a pair (G, C) of a 3-connected graph G and a hamiltonian cycle C of G such that C contains precisely four contractible edges of G. Throughout this section, we let n_0, n_1, n_2 and n_3 be nonnegative integers, and let G denote a graph of order $n_0 + n_1 + n_2 + n_3 + 4$ with vertex set

$$V(G) = \{a_i \,|\, 0 \leq i \leq n_0\} \cup \{b_i \,|\, 0 \leq i \leq n_1\} \cup \{c_i \,|\, 0 \leq i \leq n_2\} \cup \{d_i \,|\, 0 \leq i \leq n_3\}$$

such that G contains $C = a_0 a_1 \cdots a_{n_0} b_0 b_1 \cdots b_{n_1} c_0 c_1 \cdots c_{n_2} d_0 d_1 \cdots d_{n_3} a_0$ as a hamiltonian cycle. In the definition of each type, it is easy to verify that if

G satisfies the required conditions, then G is 3-connected, and $a_{n_0}b_0$, $b_{n_1}c_0$, $c_{n_2}d_0$, $d_{n_3}a_0$ are the only contractible edges of G that are on C.

Type 1. Let $n_0 = 0$ or 2, $n_1 \geq 1$, $n_2 = 0$ or 2, and $n_3 \geq 1$. Let r be an integer with

$$1 \leq r \leq \min\{n_1 + 1, n_3\}, \tag{2-1}$$

and let k_1, k_2,..., k_r, k_{r+1} and l_1, l_2,..., l_r, l_{r+1} be integers satisfying

$$\begin{gathered} 0 = k_1 < k_2 < \cdots < k_{r-1} < k_r \leq k_{r+1} = n_1 \\ \text{and } n_3 = l_1 > l_2 > \cdots > l_r > l_{r+1} = 0. \end{gathered} \tag{2-2}$$

Let

$$X = \left(\bigcup_{t=2}^{r+1}\{b_ib_{i+2} \mid k_{t-1} \leq i \leq k_t - 2\}\right) \cup \left(\bigcup_{t=1}^{r}\{d_jd_{j-2} \mid l_t \geq j \geq l_{t+1} + 2\}\right),$$

$$Y_1 = \begin{cases} \{b_{k_t+1}d_{l_{t+1}+1} \mid 1 \leq t \leq r\} & (\text{if } k_r < n_1) \\ \{b_{k_t+1}d_{l_{t+1}+1} \mid 1 \leq t \leq r-1\} & (\text{if } k_r = n_1), \end{cases}$$

$$Y_2 = \{b_{k_t-1}d_{l_t-1} \mid 2 \leq t \leq r\},$$

$$Y_3 = \bigcup_{t=1}^{r}\{b_{k_t}d_j \mid l_t \geq j \geq l_{t+1}\},$$

$$Y_4 = \bigcup_{t=2}^{r+1}\{b_id_{l_t} \mid k_{t-1} \leq i \leq k_t\},$$

$$F_1 = \begin{cases} \{a_0d_{n_3-1}\} & (\text{if } n_0 = 0) \\ \{a_0a_2, a_1d_{n_3-1}\} & (\text{if } n_0 = 2), \end{cases}$$

$$F_1' = \begin{cases} \phi & (\text{if } n_0 = 0) \\ \{a_1b_0, a_1d_{n_3}\} & (\text{if } n_0 = 2), \end{cases}$$

$$F_2 = \begin{cases} \{c_0b_{n_1-1}\} & (\text{if } k_r < n_1 \text{ and } n_2 = 0) \\ \{c_0c_2, c_1b_{n_1-1}\} & (\text{if } k_r < n_1 \text{ and } n_2 = 2) \\ \{c_0d_1\} & (\text{if } k_r = n_1 \text{ and } n_2 = 0) \\ \{c_0c_2, c_1d_1\} & (\text{if } k_r = n_1 \text{ and } n_2 = 2), \end{cases}$$

$$F_2' = \begin{cases} \phi & (\text{if } n_2 = 0) \\ \{b_{n_1}c_1, c_1d_0\} & (\text{if } n_2 = 2), \end{cases}$$

$$H_1 = \begin{cases} Y_3 & (\text{if } n_0 = 0) \\ Y_3 \cup \{a_1 b_0\} & (\text{if } n_0 = 2), \end{cases}$$

$$H_2 = \begin{cases} Y_4 & (\text{if } n_2 = 0) \\ Y_4 \cup \{c_1 d_0\} & (\text{if } n_2 = 2). \end{cases}$$

Now G is said to be of Type 1, if there exists r satisfying (2-1) and there exist $k_1, k_2, \ldots, k_{r+1}$ and $l_1, l_2, \ldots, l_{r+1}$ satisfying (2-2), such that if we define X, Y_1, Y_2, Y_3, Y_4, F_1, F_1', F_2, F_2', H_1, H_2 as above, then G satisfies the following three conditions:

- $F_1 \cup F_2 \cup X \cup Y_1 \cup Y_2 \subseteq E(G) - E(C) \subseteq F_1 \cup F_2 \cup X \cup Y_1 \cup Y_2 \cup Y_3 \cup Y_4 \cup F_1' \cup F_2'$;
- if $n_1 = 1$, $r = 1$ and $n_2 = 0$, then $H_1 \cap E(G) \neq \phi$;
- if $n_3 = 1$ and $n_0 = 0$, then $H_2 \cap E(G) \neq \phi$.

Type 2. Let $n_0 = 0$ or 2, $n_1 = 0$ or 2, $n_2 \geq 1$, and $n_3 \geq 1$. Let

$$X = \{c_{n_2-1} d_1\} \cup \{c_i c_{i+2} \mid 0 \leq i \leq n_2 - 2\} \cup \{d_j d_{j+2} \mid 0 \leq j \leq n_3 - 2\},$$

$$X' = \{c_{n_2-1} d_0, c_{n_2} d_1\},$$

$$F_1 = \begin{cases} \{a_0 d_{n_3-1}\} & (\text{if } n_0 = 0) \\ \{a_0 a_2, a_1 d_{n_3-1}\} & (\text{if } n_0 = 2), \end{cases}$$

$$F_1' = \begin{cases} \phi & (\text{if } n_0 = 0) \\ \{a_1 d_{n_3}\} & (\text{if } n_0 = 2), \end{cases}$$

$$F_2 = \begin{cases} \{b_0 c_1\} & (\text{if } n_1 = 0) \\ \{b_0 b_2, b_1 c_1\} & (\text{if } n_1 = 2), \end{cases}$$

$$F_2' = \begin{cases} \phi & (\text{if } n_1 = 0) \\ \{b_1 c_0\} & (\text{if } n_1 = 2), \end{cases}$$

$$Y_1 = \begin{cases} \phi & (\text{if } n_1 = 2 \text{ or } n_2 \geq 2) \\ \{c_{n_2} d_1\} & (\text{if } n_1 = 0 \text{ and } n_2 = 1), \end{cases}$$

$$Y_2 = \begin{cases} \phi & (\text{if } n_0 = 2 \text{ or } n_3 \geq 2) \\ \{c_{n_2-1} d_0\} & (\text{if } n_0 = 0 \text{ and } n_3 = 1). \end{cases}$$

Under this notation, G is said to be of Type 2 if G satisfies

$$F_1 \cup F_2 \cup X \cup Y_1 \cup Y_2 \subseteq E(G) - E(C) \subseteq F_1 \cup F_2 \cup X \cup F_1' \cup F_2' \cup X'.$$

Type 3. Let $n_0 = 0$ or 2, $n_1 = 0$, $n_2 = 0$ or 2, and $n_3 \geq 1$. Let

$$
\begin{aligned}
X &= \{d_j d_{j+2} \mid 0 \leq j \leq n_3 - 2\},\\
X' &= \{b_0 d_j \mid 0 \leq j \leq n_3\},\\
F_1 &= \begin{cases} \{a_0 d_{n_3-1}\} & (\text{if } n_0 = 0)\\ \{a_0 a_2, a_1 d_{n_3-1}\} & (\text{if } n_0 = 2), \end{cases}\\
F_1' &= \begin{cases} \phi & (\text{if } n_0 = 0)\\ \{a_1 d_{n_3}, a_1 b_0\} & (\text{if } n_0 = 2), \end{cases}\\
F_2 &= \begin{cases} \{c_0 d_1\} & (\text{if } n_2 = 0)\\ \{c_0 c_2, c_1 d_1\} & (\text{if } n_2 = 2), \end{cases}\\
F_2' &= \begin{cases} \phi & (\text{if } n_2 = 0)\\ \{c_1 d_0, b_0 c_1\} & (\text{if } n_2 = 2), \end{cases}\\
Y_1' &= \begin{cases} \phi & (\text{if } n_0 = 0)\\ \{a_1 b_0\} & (\text{if } n_0 = 2), \end{cases}\\
Y_2' &= \begin{cases} \phi & (\text{if } n_2 = 0)\\ \{b_0 c_1\} & (\text{if } n_2 = 2), \end{cases}\\
Z_1' &= \begin{cases} \{b_0 d_0\} & (\text{if } n_2 = 0)\\ \phi & (\text{if } n_2 = 2), \end{cases}\\
Z_2' &= \begin{cases} \{b_0 d_{n_3}\} & (\text{if } n_0 = 0)\\ \phi & (\text{if } n_0 = 2), \end{cases}\\
H' &= \{b_0 d_0, c_1 d_0\},\\
J' &= \{a_1 d_1, b_0 d_1\}.
\end{aligned}
$$

Under this notation, G is said to be of Type 3 if G satisfies

$$
\begin{gathered}
F_1 \cup F_2 \cup X \subseteq E(G) - E(C) \subseteq F_1 \cup F_2 \cup X \cup F_1' \cup F_2' \cup X',\\
((X' \cup Y_1') - Z_1') \cap E(G) \neq \phi,\\
((X' \cup Y_2') - Z_2') \cap E(G) \neq \phi,\\
H' \cap E(G) \neq \phi \text{ in the case where } n_0 = 0, n_2 = 2 \text{ and } n_3 = 1,\\
J' \cap E(G) \neq \phi \text{ in the case where } n_0 = 2, n_2 = 0 \text{ and } n_3 = 1.
\end{gathered}
$$

Type 4. Let $n_0 = 0$ or 2, $n_1 = 2$, $n_2 = 0$ or 2, and $n_3 \geq 1$. Let

$$
X = \{b_0 b_2\} \cup \{d_j d_{j+2} \mid 0 \leq j \leq n_3 - 2\},
$$

$$F_1 = \begin{cases} \{a_0 d_{n_3-1}\} & (\text{if } n_0 = 0) \\ \{a_0 a_2, a_1 d_{n_3-1}\} & (\text{if } n_0 = 2), \end{cases}$$
$$F_1' = \begin{cases} \phi & (\text{if } n_0 = 0) \\ \{a_1 d_{n_3}\} & (\text{if } n_0 = 2), \end{cases}$$
$$F_2 = \begin{cases} \{c_0 d_1\} & (\text{if } n_2 = 0) \\ \{c_0 c_2, c_1 d_1\} & (\text{if } n_2 = 2), \end{cases}$$
$$F_2' = \begin{cases} \phi & (\text{if } n_2 = 0) \\ \{c_1 d_0\} & (\text{if } n_2 = 2), \end{cases}$$
$$Z_1' = \begin{cases} \{b_1 d_{n_3}\} & (\text{if } n_0 = 0) \\ \phi & (\text{if } n_0 = 2), \end{cases}$$
$$Z_2' = \begin{cases} \{b_1 d_0\} & (\text{if } n_2 = 0) \\ \phi & (\text{if } n_2 = 2), \end{cases}$$
$$X' = \{b_1 d_0, b_1 d_1\}.$$

In the case where $n_3 \geq 2$, for each $1 \leq p \leq n_3 - 1$, define X_p' by

$$X_p' = \{b_1 d_j \mid p-1 \leq j \leq p+1\}.$$

Under this notation, G is said to be of Type 4 if either $n_3 = 1$ and G satisfies

$$F_1 \cup F_2 \cup X \subseteq E(G) - E(C) \subseteq F_1 \cup F_2 \cup X \cup F_1' \cup F_2' \cup X',$$
$$(X' - Z_1') \cap E(G) \neq \phi,$$
$$(X' - Z_2') \cap E(G) \neq \phi,$$

or $n_3 \geq 2$ and there exists p with $1 \leq p \leq n_3 - 1$ such that G satisfies

$$F_1 \cup F_2 \cup X \subseteq E(G) - E(C) \subseteq F_1 \cup F_2 \cup X \cup F_1' \cup F_2' \cup X_p',$$
$$(X_p' - Z_1') \cap E(G) \neq \phi,$$
$$(X_p' - Z_2') \cap E(G) \neq \phi.$$

Type 5. Let $n_0 = 0$ or 2, $n_1 = 2$, $n_2 = 0$ or 2, and $n_3 \geq 1$. Let

$$X = \{b_0 b_2\} \cup \{d_j d_{j+2} \mid 0 \leq j \leq n_3 - 2\},$$
$$X' = \{b_1 d_{n_3-1}, b_1 d_{n_3}\},$$
$$F_1 = \begin{cases} \{a_0 b_1, a_0 d_{n_3-1}\} & (\text{if } n_0 = 0) \\ \{a_0 a_2, a_1 b_1, a_1 d_{n_3-1}\} & (\text{if } n_0 = 2), \end{cases}$$
$$F_1' = \begin{cases} \phi & (\text{if } n_0 = 0) \\ \{a_1 d_{n_3}\} & (\text{if } n_0 = 2), \end{cases}$$

$$F_2 = \begin{cases} \{c_0d_1\} & (\text{if } n_2 = 0) \\ \{c_0c_2, c_1d_1\} & (\text{if } n_2 = 2), \end{cases}$$

$$F_2' = \begin{cases} \phi & (\text{if } n_2 = 0) \\ \{c_1d_0\} & (\text{if } n_2 = 2), \end{cases}$$

$$Z' = \begin{cases} \{b_1d_{n_3}\} & (\text{if } n_0 = 0) \\ \phi & (\text{if } n_0 = 2), \end{cases}$$

$$H' = \begin{cases} \{b_1d_1\} & (\text{if } n_0 = 0) \\ \{a_1d_1, b_1d_1\} & (\text{if } n_0 = 2). \end{cases}$$

Under this notation, G is said to be of Type 5 if G satisfies

$$F_1 \cup F_2 \cup X \subseteq E(G) - E(C) \subseteq F_1 \cup F_2 \cup X \cup F_1' \cup F_2' \cup X',$$
$$(X' - Z') \cap E(G) \neq \phi,$$
$$H' \cap E(G) \neq \phi \text{ in the case where } n_2 = 0 \text{ and } n_3 = 1.$$

Type 6. We say that G is of Type 6 if $n_0 = 2$ and $n_1 = n_2 = n_3 = 0$ and G satisfies

$$E(G) - E(C) = \{a_0a_2, a_1b_0, a_1c_0, a_1d_0, b_0d_0\}.$$

Type 7. We say that G is of Type 7 if $n_0 = n_1 = 2$ and $n_2 = n_3 = 0$ and G satisfies

$$\{a_0a_2, b_0b_2, a_1b_1, a_1c_0, b_1d_0\} \subseteq E(G) - E(C) \subseteq \{a_0a_2, b_0b_2, a_1b_1, a_1c_0, b_1d_0, a_1d_0, b_1c_0\}.$$

Type 8. Let $n_0 = 2$, $n_1 = 0$ or 2, $n_2 = 2$ and $n_3 = 0$ or 2. Let

$$F = \{a_0a_2, c_0c_2\},$$

$$F_1 = \begin{cases} \phi & (\text{if } n_1 = 0) \\ \{b_0b_2\} & (\text{if } n_1 = 2), \end{cases}$$

$$F_2 = \begin{cases} \phi & (\text{if } n_3 = 0) \\ \{d_0d_2\} & (\text{if } n_3 = 2), \end{cases}$$

$$\bar{F} = \begin{cases} \{a_1c_1\} & (\text{if } n_1 \neq 2 \text{ or } n_3 \neq 2) \\ \phi & (\text{if } n_1 = 2 \text{ and } n_3 = 2). \end{cases}$$

Let $b = \begin{cases} b_0 & (\text{if } n_1 = 0) \\ b_1 & (\text{if } n_1 = 2) \end{cases}$ and $d = \begin{cases} d_0 & (\text{if } n_3 = 0) \\ d_1 & (\text{if } n_3 = 2) \end{cases}$ and, let

$$F' = \{a_1b, a_1c_1, a_1d, bc_1, bd, c_1d\}.$$

Under this notation, G is said to be of Type 8 if G satisfies

$$F \cup F_1 \cup F_2 \cup \bar{F} \subseteq E(G) - E(C) \subseteq F \cup F_1 \cup F_2 \cup F',$$
$$bd \in E(G) \text{ or } \{a_1b, bc_1, c_1d, da_1\} \subset E(G),$$
$$a_1c_1 \in E(G) \text{ or } \{a_1b_1, b_1c_1, c_1d_1, d_1a_1\} \subset E(G)$$
$$\text{in the case where } n_1 = n_3 = 2.$$

§3. Preliminaries

In this section, we prove fundamental results concerning noncontractible edges lying on a hamiltonian cycle of a 3-connected graph. All of the results in this section are already proved in [4] (and most of them also in Ota [6]), and we omit their proofs.

Throughout this section, we let G denote a 3-connected graph of order $n+1$ ($n \geq 4$), and let $C = v_0v_1 \cdots v_nv_0$ denote a hamiltonian cycle of G. Moreover, throughout this section, we assume that the edge v_nv_0 is noncontractible, and let $\{v_n, v_0, v_i\}$ be a cutset associated with it.

Lemma 3.1. *For v_i, $2 \leq i \leq n-2$. And $\langle\{v_k \mid 1 \leq k \leq i-1\}\rangle$ and $\langle\{v_k \mid i+1 \leq k \leq n-1\}\rangle$ are the two components of $G - \{v_n, v_0, v_i\}$.*

Lemma 3.2. (i) *No edge of G joins a vertex in $\{v_k \mid 1 \leq k \leq i-1\}$ and a vertex in $\{v_k \mid i+1 \leq k \leq n-1\}$.*

(ii) *For some k with $1 \leq k \leq i-1$, $v_nv_k \in E(G)$.*

Lemma 3.3. *If i=2, then $E(v_1, V(G)) - E(C) = \{v_1v_n\}$.*

Lemma 3.4. *Suppose that v_0v_1 is noncontractible and $v_i \in K(v_0, v_1)$. Then $v_nv_1 \in E(G)$.*

Lemma 3.5. *Suppose that v_iv_{i+1} is noncontractible, and let $\{v_i, v_{i+1}, v_j\}$ be a cutset associated with it. Then $i+3 \leq j \leq n$ (and hence $i \leq n-3$). Further, if $j = n$, then $v_0v_{i+1} \in E(G)$.*

Lemma 3.6. *Let $1 \leq j \leq i-2$. Suppose that v_jv_{j+1} is noncontractible, and let $\{v_j, v_{j+1}, v_l\}$ be a cutset associated with it, and suppose that $i+1 \leq l \leq n-1$. Then $l = i+1$, v_iv_l is contractible and, unless $l = n-1$, we have $v_l \in K(v_n, v_0)$.*

Lemma 3.7. *Suppose that v_0v_1 is noncontractible, and let $\{v_0, v_1, v_j\}$ be a cutset associated with it, and suppose that $i+1 \leq j \leq n-2$. Then $v_j \in K(v_n, v_0)$.*

Lemma 3.8. *Suppose that $K(v_n, v_0) = \{v_2\}$, and that v_0v_1 is noncontractible. Then $K(v_0, v_1) = \{v_{n-1}\}$.*

Lemma 3.9. (i) *If $i = 2$, then v_1v_2 is contractible.*

(ii) *If $i \geq 2$, then there exists j with $0 \leq j \leq i-1$ such that v_jv_{j+1} is contractible.*

(iii) *If $i \geq 3$ and there exists only one j with $0 \leq j \leq i-1$ such that v_jv_{j+1} is contractible, then v_iv_{i+1} is contractible.*

§4. Initial Reduction

Throughout the rest of this paper, we let G and C be as in the Main Theorem, and write $C = a_0a_1 \cdots a_{n_0}b_0b_1 \cdots b_{n_1}c_0c_1 \cdots c_{n_2}d_0d_1 \cdots d_{n_3}a_0$, where $a_{n_0}b_0$, $b_{n_1}c_0$, $c_{n_2}d_0$ and $d_{n_3}a_0$ are the four contractible edges contained in C. Note that C is a hamiltonian cycle by the result of Ellingham, Hemminger and Johnson [3] mentioned in Section 1; thus $|V(G)| = n_0 + n_1 + n_2 + n_3 + 4$. Let $C_0 = \{a_0, a_1, \ldots, a_{n_0}\}$, $C_1 = \{b_0, b_1, \ldots, b_{n_1}\}$, $C_2 = \{c_0, c_1, \ldots, c_{n_2}\}$ and $C_3 = \{d_0, d_1, \ldots, d_{n_3}\}$.

In this section, we derive some basic properties of (G, C). All of the results in this section are already proved in [4], and we omit their proofs.

The first three lemmas are concerned with the structure of $K(e)$, where e is a noncontractible edge lying on C.

Lemma 4.1. *Suppose that $n_1 = 2$. Then one of the following holds:*

(i) *$K(b_0, b_1) = \{c_0\}$ and $K(b_1, b_2) = \{a_{n_0}\}$; or*

(ii) *$K(b_0, b_1) \neq \{c_0\}$ and $K(b_1, b_2) \neq \{a_{n_0}\}$.*

Lemma 4.2. *Suppose that $n_1 \geq 1$.*

(i) *If $n_1 \neq 2$, then $K(b_0, b_1) \subseteq C_3 \cup \{c_{n_2}, a_0\}$.*

(ii) *If $n_1 = 2$, then $K(b_0, b_1) \subseteq C_3 \cup \{c_0, c_{n_2}, a_0\}$.*

Lemma 4.3. *One of the following holds:*

(i) *$n_1 = 0$;*

(ii) *$n_1 = 2$ and $K(b_0, b_1) = \{c_0\}$ and $K(b_1, b_2) = \{a_{n_0}\}$; or*

(iii) *$n_1 \geq 1$ and $K(b_i, b_{i+1}) \cap C_3 \neq \phi$ for all $0 \leq i \leq n_1 - 1$.*

With Lemma 4.3 in mind, we define the terms *degenerate* and *nondegenerate* as follows: for each $0 \leq l \leq 3$, C_l is said to be nondegenerate if $n_l \geq 1$ and $K(e) \cap C_{l+2} \neq \phi$ for all $e \in E(\langle C_l \rangle_C)$ (we take $C_{l+2} = C_{l-2}$ if $l = 2$ or 3); otherwise C_l is said to be degenerate. Thus, for example, C_1 is nondegenerate if and only if (iii) of Lemma 4.3 holds, and it is degenerate if and only if (i) or (ii) of Lemma 4.3 holds.

Lemma 4.4. *At most two of the* C_l $(0 \le l \le 3)$ *are nondegenerate.*

We now turn our attention to the distribution of edges of G.

Lemma 4.5. *Suppose that* C_0 *is degenerate and* $n_0 = 2$. *Then the following hold.*

(i) $E(a_0, V(G)) - E(C) = \{a_0a_2\}$, *and* $E(a_2, V(G)) - E(C) = \{a_0a_2\}$.

(ii) $E(\{a_0, a_2\}, V(G)) - E(C) = \{a_0a_2\}$.

Lemma 4.6. *Suppose that* C_0 *is degenerate, and that* C_3 *is nondegenerate and* $b_0 \in K(d_{n_3-1}, d_{n_3})$.

(I) *If* $n_0 = 0$, *then* $E(C_0, V(G)) - E(C) = \{a_0d_{n_3-1}\}$.

(II) *Suppose that* $n_0 = 2$. *Then the following hold.*

(i) $\{a_0a_2, a_1d_{n_3-1}\} \subseteq E(C_0, V(G)) - E(C) \subseteq \{a_0a_2, a_1b_0, a_1d_{n_3-1}, a_1d_{n_3}\}$.

(ii) *Suppose further that* C_1 *is degenerate, and that either* $n_1 = 2$, *or* $n_1 = 0$ *and* $n_2 \ge 1$ *and* $a_2 \in K(c_0, c_1)$. *Then*

$$\{a_0a_2, a_1d_{n_3-1}\} \subseteq E(C_0, V(G)) - E(C) \subseteq \{a_0a_2, a_1d_{n_3-1}, a_1d_{n_3}\}.$$

Lemma 4.7. *Suppose that* C_3 *is nondegenerate. Then* $d_id_j \notin E(G)$ *for any* i, j *with* $i + 3 \le j$.

§5. Proof of the Main Theorem

We continue with the notation of the preceding section, and complete the proof of the Main Theorem.

By Lemma 4.4 and by symmetry, it suffices to consider the following four cases:

- C_1 and C_3 are nondegenerate, and C_0 and C_2 are degenerate;
- C_2 and C_3 are nondegenerate, and C_0 and C_1 are degenerate;
- C_3 is nondegenerate, and C_0, C_1 and C_2 are degenerate; or
- C_0, C_1, C_2 and C_3 are degenerate.

We consider these four cases separately in four propositions, Propositions 1 through 4. Propositions 1 and 2 are already proved in [4], and thus we omit their proofs.

Proposition 1. *Suppose that C_1 and C_3 are nondegenerate, and C_0 and C_2 are degenerate. Then (G, C) is of Type 1.*

Proposition 2. *Suppose that C_2 and C_3 are nondegenerate, and C_0 and C_1 are degenerate. Then (G, C) is of Type 2.*

Proposition 3. *Suppose that C_3 is nondegenerate, and C_0, C_1 and C_2 are degenerate. Then (G, C) is of Type 3, 4 or 5.*

Proof. Note that for each $0 \leq l \leq 2$, $n_l = 0$ or 2 because C_l is degenerate. Since C_3 is nondegenerate,

$$K(d_j, d_{j+1}) \cap C_1 \neq \phi \text{ for all } 0 \leq j \leq n_3 - 1. \tag{5-1}$$

We first consider the case where $n_1 = 0$. In this case, we have $C_1 = \{b_0\}$. Set

$$X' = \{b_0 d_j \mid 0 \leq j \leq n_3\},$$
$$Y_1' = \begin{cases} \phi & (\text{if } n_0 = 0) \\ \{a_1 b_0\} & (\text{if } n_0 = 2), \end{cases}$$
$$Y_2' = \begin{cases} \phi & (\text{if } n_2 = 0) \\ \{b_0 c_1\} & (\text{if } n_2 = 2). \end{cases}$$

The following claim is already proved in [4], and we omit the proof.

Claim 5.1. $E(b_0, V(G)) - E(C) \subseteq X' \cup Y_1' \cup Y_2'$.

We now prove the following claim.

Claim 5.2. (i) If $n_0 = 0$, $n_2 = 2$ and $n_3 = 1$, then $\{b_0 d_0, c_1 d_0\} \cap E(G) \neq \phi$.

(ii) If $n_0 = 2$, $n_2 = 0$ and $n_3 = 1$, then $\{a_1 d_1, b_0 d_1\} \cap E(G) \neq \phi$.

Proof. To prove (i), suppose that $n_0 = 0$, $n_2 = 2$ and $n_3 = 1$. Sinse $d_1 a_0$ is contractible, $\{d_1, a_0, c_2\}$ is not a cutset, and hence $E(C_3 - \{d_1\}, \{b_0\} \cup C_2 - \{c_2\}) \neq \phi$ by the assumption that $n_0 = 0$. Since $C_3 - \{d_1\} = \{d_0\}$ by the assumption that $n_3 = 1$, this means $E(d_0, \{b_0\} \cup C_2 - \{c_2\}) \neq \phi$. In view of Claim 5.1, we obtain $\{b_0 d_0, c_1 d_0\} \cap E(G) \neq \phi$ by applying Lemma 4.5 (ii) to C_2. Thus (i) is proved, and (ii) can be verified in a similar way.

Now by Claim 5.2, we can prove (G, C) is of Type 3 by arguing exactly as in the case where $n_1 = 0$ of the proof of Proposition 3 in Section 5 of [4]. This completes the proof of the proposition for the case where $n_1 = 0$.

We henceforth assume that $n_1 = 2$. Applying Lemma 4.5 (ii) to C_1, we get

$$E(\{b_0, b_2\}, V(G)) - E(C) = \{b_0 b_2\}. \tag{5-2}$$

Claim 5.3. Let $0 \le k \le n_3 - 1$.

(i) If $b_2 \in K(d_k, d_{k+1})$, then $b_2 \in K(d_j, d_{j+1})$ for all $0 \le j \le k$.

(ii) If $b_0 \in K(d_k, d_{k+1})$, then $b_0 \in K(d_j, d_{j+1})$ for all $k \le j \le n_3 - 1$.

Proof. To prove (i), assume that $b_2 \in K(d_k, d_{k+1})$, and let $0 \le j \le k-1$. By (5-1), take $b_i \in K(d_j, d_{j+1}) \cap C_1$. We may assume $i \ne 2$. But then, applying Lemma 3.6 or 3.7 according to whether $j \le k-2$ or $j = k-1$, we obtain $b_2 \in K(d_j, d_{j+1})$. Thus (i) is proved, and (ii) can be verified in a similar way.

Claim 5.4. One of the following holds:

(i) either $n_3 = 1$ and $b_0, b_2 \in K(d_0, d_1)$, or $n_3 \ge 2$ and there exists p with $1 \le p \le n_3 - 1$ such that $b_2 \in K(d_j, d_{j+1})$ for all $0 \le j \le p-1$ and $b_0 \in K(d_j, d_{j+1})$ for all $p \le j \le n_3 - 1$;

(ii) $K(d_j, d_{j+1}) \cap C_1 = \{b_2\}$ for all $0 \le j \le n_3 - 1$; or

(iii) $K(d_j, d_{j+1}) \cap C_1 = \{b_0\}$ for all $0 \le j \le n_3 - 1$.

Proof. Let $0 \le j \le n_3 - 1$. If $b_1 \in K(d_j, d_{j+1})$, then since we have $a_{n_0} \in K(b_1, b_2)$ from the assumption that C_1 is degenerate, we get a contradiction by Lemma 3.5. Thus

$$b_1 \notin K(d_j, d_{j+1}) \text{ for all } 0 \le j \le n_3 - 1. \tag{5-3}$$

Now suppose that neither (ii) nor (iii) holds. Then there exists l such that $b_2 \in K(d_l, d_{l+1})$, and hence

$$b_2 \in K(d_0, d_1) \tag{5-4}$$

by Claim 5.3 (i). Similarly, we get $b_0 \in K(d_{n_3-1}, d_{n_3})$. If $n_3 = 1$, then (i) holds, as desired. Thus we may assume $n_3 \ge 2$. Let $p = \min\{1 \le j \le n_3 - 1 \mid b_0 \in K(d_j, d_{j+1})\}$. Then by Claim 5.3 (ii), $b_0 \in K(d_j, d_{j+1})$ for all $p \le j \le n_3 - 1$. Also by the minimality of p, it follows from (5-1) and (5-3) that $b_2 \in K(d_j, d_{j+1})$ for all $1 \le j \le p-1$, and this together with (5-4) implies that $b_2 \in K(d_j, d_{j+1})$ for all $0 \le j \le p-1$. Thus (i) holds, as desired.

By symmetry, we may assume that (i) or (ii) of Claim 5.4 holds. We now divide the proof into two cases according to whether (i) or (ii) of Claim 5.4 holds.

Case 1. Claim 5.4 (i) holds.

Let

$$Y = \begin{cases} \{b_1 d_0, b_1 d_1\} & (\text{if } n_3 = 1) \\ \{b_1 d_j \mid p-1 \le j \le p+1\} & (\text{if } n_3 \ge 2), \end{cases}$$

where p is as in Claim 5.4 (i). Then we can prove (G, C) is of Type 4 by arguing exactly as in Case 1 of the proof of Proposition 3 in Section 5 of [4].
Case 2. Claim 5.4 (ii) holds.

Applying Lemma 4.5 (ii) to C_0, we see that

$$\text{(5-5)} \qquad \text{if } n_0 = 2,\ E(\{a_0, a_2\},\ V(G)) - E(C) = \{a_0a_2\}.$$

For convenience, let $a = a_1$ if $n_0 = 2$, and let $a = a_0$ if $n_0 = 0$. Applying Lemma 3.2 (i) to $\{d_{n_3-1}, d_{n_3}, b_2\}$, we get

$$\text{(5-6)} \qquad E(\{a, b_1\},\ C_2 \cup (C_3 - \{d_{n_3-1}, d_{n_3}\})) = \phi.$$

Combining (5-6) and (5-5), we obtain

$$\text{(5-7)} \qquad E(b_1,\ V(G)) - E(C) \subseteq \{b_1a, b_1d_{n_3-1}, b_1d_{n_3}\},$$

and combining (5-6) and (5-2), we obtain

$$\text{(5-8)} \qquad E(a, V(G)) - E(C) \subseteq \begin{cases} \{ab_1, ad_{n_3-1}\} & (\text{if } n_0 = 0) \\ \{ab_1, ad_{n_3-1}, ad_{n_3}\} & (\text{if } n_0 = 2). \end{cases}$$

Set

$$H' = \begin{cases} \{b_1d_1\} & (\text{if } n_0 = 0) \\ \{a_1d_1, b_1d_1\} & (\text{if } n_0 = 2). \end{cases}$$

Claim 5.5. If $n_2 = 0$ and $n_3 = 1$, then $H' \cap E(G) \neq \phi$.

Proof. Suppose that $n_2 = 0$ and $n_3 = 1$. Then since c_0d_0 is contractible, $\{c_0, d_0, a_0\}$ is not a cutset, and hence $E(C_3 - \{d_0\}, (C_0 - \{a_0\}) \cup C_1) \neq \phi$ by the assumption that $n_2 = 0$. Since $C_3 - \{d_0\} = \{d_1\}$ by the assumption that $n_3 = 1$, this means $E(d_1, (C_0 - \{a_0\}) \cup C_1) \neq \phi$. In view of (5-2), (5-5), (5-7) and (5-8), we obtain $H' \cap E(G) \neq \phi$, as desired.

Now by Claim 5.5, we can prove (G, C) is of Type 5 by arguing exactly as in Case 2 of the proof of Proposition 3 in Section 5 of [4].

Proposition 4. *Suppose that C_0, C_1, C_2 and C_3 are degenerate. Then (G, C) is of Type 6, 7 or 8.*

Proof. Note that for each $0 \le l \le 3$, $n_l = 0$ or 2 because C_l is degenerate. If $n_0 = n_1 = n_2 = n_3 = 0$, then $|V(G)| = 4$, and hence no edge of G is contractible, a contradiction. Thus at least one of n_0, n_1, n_2 and n_3 is 2. By symmetry, we may assume that $n_0 = 2$. Then by Lemma 4.5 (ii), we have

$$\text{(5-9)} \qquad E(\{a_0, a_2\}, V(G)) - E(C) = \{a_0a_2\}.$$

Further by symmetry, it suffices to consider the following three cases:

- $n_1 = n_2 = n_3 = 0$;
- $n_1 = 2$ and $n_2 = n_3 = 0$; or
- $n_1 = 0$ or 2, $n_2 = 2$, and $n_3 = 0$ or 2.

We consider these three cases separately.

Case 1. $n_1 = n_2 = n_3 = 0$.

In this case, we have $C_1 = \{b_0\}$, $C_2 = \{c_0\}$ and $C_3 = \{d_0\}$. We prove the following three claims.

Claim 5.6. *a_1b_0, $a_1d_0 \in E(G)$.*

Proof. Since c_0d_0 is contractible by the assumption that $n_2 = 0$, $\{c_0, d_0, a_2\}$ is not a cutset, and hence $E((C_0 - \{a_2\}), \{b_0\}) \neq \phi$ by the assumption that $n_1 = n_3 = 0$. Consequently $a_1b_0 \in E(G)$ by (5-9). In view of the symmetry of the roles of c_0d_0 and b_0c_0, we similarly obtain $a_1d_0 \in E(G)$.

Claim 5.7. *$b_0d_0 \in E(G)$.*

Proof. Since C_0 is degenerate by the assumption of Proposition 4, $\{a_1, a_2, c_0\}$ is not a cutset, and hence $E((C_0 - \{a_1, a_2\}) \cup \{d_0\}, \{b_0\}) \neq \phi$ by the assumption of Case 1. In view of (5-9), this implies $b_0d_0 \in E(G)$.

Claim 5.8. *$a_1c_0 \in E(G)$.*

Proof. Since $\deg(c_0) \geq 3$ by the assumption that G is 3-connected, and since $n_1 = n_3 = 0$, the desired conclusion follows immediately from (5-9).

Now combining (5-9) and Claims 5.6 through 5.8, we see that (G, C) is of Type 6.

Case 2. $n_1 = 2$ and $n_2 = n_3 = 0$.

In this case, we have $C_1 = \{b_0, b_1, b_2\}$, $C_2 = \{c_0\}$ and $C_3 = \{d_0\}$. Applying Lemma 4.5 (ii) to C_1, we see that

(5-10) $$E(\{b_0, b_2\}, V(G)) - E(C) = \{b_0b_2\}.$$

By (5-10), we get

(5-11) $$E(a_1, V(G)) - E(C) \subseteq \{a_1b_1, a_1c_0, a_1d_0\},$$

and by (5-9), we also get

(5-12) $$E(b_1, V(G)) - E(C) \subseteq \{a_1b_1, b_1c_0, b_1d_0\}.$$

We now prove the following two claims.

Claim 5.9. $a_1b_1 \in E(G)$.

Proof. Since c_0d_0 is contractible by the assumption that $n_2 = 0$, $\{c_0, d_0, a_2\}$ is not a cutset, and hence $E((C_0 - \{a_2\}), C_1) \neq \phi$ by the assumption that $n_3 = 0$. In view of (5-10) and (5-12), this implies $a_1b_1 \in E(G)$.

Claim 5.10. a_1c_0, $b_1d_0 \in E(G)$.

Proof. Since C_1 is degenerate by the assumption of Proposition 4, $\{b_1, b_2, d_0\}$ is not a cutset, and hence $E(C_0 \cup (C_1 - \{b_1, b_2\}), \{c_0\}) \neq \phi$ by the assumption of Case 2. By (5-9) and (5-10), we obtain $a_1c_0 \in E(G)$. In view of the symmetry of the roles of C_1 and C_0, and of C_3 and C_2, respectively, we similarly obtain $b_1d_0 \in E(G)$.

Now combining (5-9) through (5-12), and Claims 5.9 and 5.10, we see that (G, C) is of Type 7.

Case 3. $n_1 = 0$ or 2, $n_2 = 2$, and $n_3 = 0$ or 2.

Applying Lemma 4.5 (ii) to C_2, we see that

(5-13) $$E(\{c_0, c_2\}, V(G)) - E(C) = \{c_0c_2\}.$$

Further applying Lemma 4.5 (ii) to C_1 and C_3, we also get

(5-14) $$\text{if } n_1 = 2, \text{ then } E(\{b_0, b_2\}, V(G)) - E(C) = \{b_0b_2\}$$

and

(5-15) $$\text{if } n_3 = 2, \text{ then } E(\{d_0, d_2\}, V(G)) - E(C) = \{d_0d_2\}.$$

Set
$$b = \begin{cases} b_0 & (\text{if } n_1 = 0) \\ b_1 & (\text{if } n_1 = 2) \end{cases}$$
and
$$d = \begin{cases} d_0 & (\text{if } n_3 = 0) \\ d_1 & (\text{if } n_3 = 2). \end{cases}$$
Then by (5-9), (5-13), (5-14) and (5-15), we obtain the following claim.

Claim 5.11. (i) $E(a_1, V(G)) - E(C) \subseteq \{a_1b, a_1c_1, a_1d\}$.

(ii) $E(b, V(G)) - E(C) \subseteq \{a_1b, bc_1, bd\}$.

(iii) $E(c_1, V(G)) - E(C) \subseteq \{a_1c_1, bc_1, c_1d\}$.

(iv) $E(d, V(G)) - E(C) \subseteq \{a_1d, bd, c_1d\}$.

Set $H'_1 = \{a_1b, bd\}$, $H'_2 = \{bc_1, bd\}$, $H'_3 = \{a_1d, bd\}$ and $H'_4 = \{bd, c_1d\}$.

Claim 5.12. $H'_i \cap E(G) \neq \phi$ for all $1 \leq i \leq 4$.

Proof. Since C_2 is degenerate by the assumption of Proposition 4, $\{c_0, c_1, a_2\}$ is not a cutset, and hence $E(C_1, (C_0 - \{a_2\}) \cup (C_2 - \{c_0, c_1\}) \cup C_3) \neq \phi$. This together with (5-14) and Claim 5.11 (ii) implies $H'_1 \cap E(G) \neq \phi$. By symmetry, it can be verified in a similar way that $H'_i \cap E(G) \neq \phi$ ($i = 2$, 3 and 4).

Set $J'_1 = \{a_1b_1, a_1c_1\}$, $J'_2 = \{a_1c_1, a_1d_1\}$, $J'_3 = \{a_1c_1, b_1c_1\}$ and $J'_4 = \{a_1c_1, c_1d_1\}$.

Claim 5.13. If $n_1 = 2$ and $n_3 = 2$, then $J'_i \cap E(G) \neq \phi$ for all $1 \leq i \leq 4$.

Proof. Assume that $n_1 = 2$ and $n_3 = 2$. Then since C_3 is degenerate by the assumption of Proposition 4, $\{d_1, d_2, b_0\}$ is not a cutset, and hence $E(C_0, (C_1 - \{b_0\}) \cup C_2 \cup (C_3 - \{d_1, d_2\})) \neq \phi$ which, in view of (5-9) and Claim 5.11 (i), implies $J'_1 \cap E(G) \neq \phi$. By symmetry, it can be verified in a similar way that $J'_i \cap E(G) \neq \phi$ ($i = 2$, 3 and 4).

Claim 5.14. If $n_1 \neq 2$ or $n_3 \neq 2$, then $a_1c_1 \in E(G)$.

Proof. First assume that $n_1 = n_3 = 0$ (so $C_1 = \{b_0\}$ and $C_3 = \{d_0\}$). Then by $n_3 = 0$, it follows that d_0a_0 is contractible. Thus $\{d_0, a_0, b_0\}$ is not a cutset, and hence $E((C_0 - \{a_0\}), C_2) \neq \phi$ by the assumption that $n_1 = 0$. In view of (5-9) and (5-13), this implies $a_1c_1 \in E(G)$, as desired. Next assume that $n_1 = 2$ and $n_3 = 0$ (so $C_1 = \{b_0, b_1, b_2\}$ and $C_3 = \{d_0\}$). Then since C_1 is degenerate by the assumption of Proposition 4, $\{b_1, b_2, d_0\}$ is not a cutset, and hence $E(C_2, C_0 \cup (C_1 - \{b_1, b_2\})) \neq \phi$ by the assumption that $n_3 = 0$. In view of (5-9), (5-13) and (5-14), this implies $a_1c_1 \in E(G)$, as desired. In the case where $n_1 = 0$ and $n_3 = 2$, we similarly obtain $a_1c_1 \in E(G)$, replacing the roles of C_1 and C_3 by each other.

Now combining (5-9), (5-13), (5-14), (5-15), and Claims 5.11 through 5.14, we see that (G, C) is of Type 8.

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