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Motion of charged particles in Sasakian manifolds

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Abstract. It is known that the image of a horizontal geodesic under a Riemannian submersion is a geodesic. However, in general the image of a geodesic under a Riemannian submersion is not a geodesic. In this paper, we define a Sasaki-Kähler submersion from a Sasakian manifold onto a Kähler manifold, and show that the image of the motion of a charged particle is the motion of a charged particle. In particular, the image of a geodesic is the motion of a charged particle under a Sasaki-Kähler submersion. A Sasaki-Kähler submersion is a kind of Riemannian submersion.

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§1. Charged particles and Okumura geodesics

Let (M, g) be an odd-dimensional Riemannian manifold with the Riemannian metric g. We denote by ∇ the Levi-Civita connection of M. A Sasakian structure on M is defined by a tensor field ϕ of type $(1, 1)$, a vector field ξ and 1-form η such that

$$\phi^2 = -1 + \eta \otimes \xi, \tag{1.1}$$

$$\eta(\xi) = 1, \tag{1.2}$$

$$g(X, \xi) = \eta(X), \tag{1.3}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{1.4}$$

$$d\eta(X, Y) = \frac{1}{2}(X(\eta(Y)) - Y(\eta(X)) - \eta([X, Y])) = g(X, \phi Y), \tag{1.5}$$

$$(\nabla_X \phi)(Y) = g(X, Y)\xi - \eta(Y)X. \tag{1.6}$$

(If we adopt the notation in [5], we must replace ϕ here by $-\phi$.)

A Riemannian manifold equipped with a Sasakian structure is called a Sasakian manifold.

In this section we assume that (M, g, ϕ, η, ξ) is a Sasakian manifold. Then by (1.5), ϕ is skew-symmetric with respect to g. Further, ξ is a Killing vector field, which satisfies $||\xi|| = 1$, $\phi\xi = 0$ and $\nabla_X \xi = -\phi X$. The integral curves of ξ are geodesics.

For a constant $r \in \boldsymbol{R}$, we define a tensor field A of type $(1, 2)$ by

$$A(X)Y = d\eta(X, Y)\xi + r\eta(X)\phi Y + \eta(Y)\phi X.$$

Then $A(X)$ is skew-symmetric with respect to g. The Okumura linear connection $\tilde{\nabla}$ is defined by $\tilde{\nabla}_X Y = \nabla_X Y + A(X)Y$, which satisfies $\tilde{\nabla} g = 0$ and $\tilde{\nabla}\xi = 0$ (See [5]). We have

$$\tilde{\nabla}_X X = \nabla_X X + (r + 1)\eta(X)\phi X. \tag{1.7}$$

A curve $x(t)$ in M is called the motion of a charged particle if $\nabla_{\dot{x}}\dot{x} = \kappa\phi(\dot{x})$ for a constant κ. The constant κ is the charge-to-mass ratio of $x(t)$ (See [1], [2] and [3] for related topics).

Proposition 1.1. (1) *If $x(t)$ is an Okumura geodesic, that is $\tilde{\nabla}_{\dot{x}}\dot{x} = 0$, then $\eta(\dot{x}(t))$ is a constant.*

(2) *If $x(t)$ is the motion of a charged particle, then $\eta(\dot{x}(t))$ is a constant.*

Proof. (1) Using (1.3), $\tilde{\nabla} g = 0$ and $\tilde{\nabla}\xi = 0$, we have

$$\frac{d}{dt}\eta(\dot{x}(t)) = \frac{d}{dt}g(\dot{x}(t), \xi) = g(\tilde{\nabla}_{\dot{x}}\dot{x}, \xi) + g(\dot{x}, \tilde{\nabla}_{\dot{x}}\xi) = 0.$$

(2) Using (1.3), $\nabla g = 0$, we have

$$\frac{d}{dt}\eta(\dot{x}(t)) = g(\nabla_{\dot{x}}\dot{x}, \xi) + g(\dot{x}, \nabla_{\dot{x}}\xi) = \kappa g(\phi(\dot{x}), \xi) - g(\dot{x}, \phi(\dot{x})) = 0.$$

□

Proposition 1.1 and (1.7) immediately imply the following:

Proposition 1.2. (1) *Let $x(t)$ be an Okumura geodesic. Set $c = \eta(\dot{x}(t))$, then $x(t)$ is the motion of a charged particle of the charge-to-mass ratio $\kappa = -(r + 1)c$.*

(2) *Let $x(t)$ be the motion of a charged particle. Set $c = \eta(\dot{x}(t))$.*

(2-1) *When $c \neq 0$, then $x(t)$ is an Okumura geodesic for $r = -(\frac{\kappa}{c} + 1)$.*

(2-2) *When $c = 0$, then $\tilde{\nabla}_{\dot{x}}\dot{x} = \kappa\phi(\dot{x})$.*

Corollary 1.3. *A curve $x(t)$ is a geodesic with respect to the Levi-Civita connection if and only if*

(1) *$x(t)$ is an Okumura geodesic for $r = -1$ when $\eta(\dot{x}) \neq 0$,*

(2) *$x(t)$ is an Okumura geodesic for any r when $\eta(\dot{x}) = 0$.*

§2. Sasaki-Kähler submersion

Let $\pi : \bar{M} \to M$ be a Riemannian submersion from a Sasakian manifold $(\bar{M}, g, \phi, \eta, \xi)$ of dimension $2n+1$ onto a Kähler manifold (M, g, J) of real dimension $2n$. We call π a Sasaki-Kähler submersion if

(1) $\pi^{-1}(y)$ $(y \in M)$ is the image of an integral curve of ξ,

(2) $d\pi\phi X = Jd\pi X$ for any horizonal vector X, that is $\eta(X) = 0$.

For instance, we can construct a Sasaki-Kähler submersion from any Hermitian symmetric space M.

Theorem 2.1. *Let $\pi : \bar{M} \to M$ be a Sasaki-Kähler submersion. Assume that $x(t) \in \bar{M}$ is the motion of a charged particle of the charge-to-mass ratio κ. Define a constant c by $c = \eta(\dot{x})$. Then $y(t) = \pi(x(t))$ is the motion of a charged particle of the charge-to-mass ratio $\kappa + 2c$, that is $\nabla_{\dot{y}}\dot{y} = (\kappa + 2c)J\dot{y}$, where ∇ is the Levi-Civita connection of M. In particular, if $x(t)$ is a geodesic, then $y(t)$ is the motion of a charged particle of the charge-to-mass ratio $2c$.*

Proof. Since $||\dot{x}||$ is a constant, $\dot{x}(t) = 0$ for some t if and only if $\dot{x}(t) = 0$ for any t. In this case, $x(t)$ is a single point. Hence we may assume $\dot{x}(t) \neq 0$ for any t. If $\dot{x}(t)$ is proportional to ξ for some t, then $x(t)$ is an integral curve of ξ. In this case, $y(t)$ is a single point. Hence we may assume that $\dot{x}$ is not proportional to ξ for any t. In other words, we may assume $\dot{y}(t) \neq 0$ for any t. Hence there exists a (local) vector field X of M such that $X = \dot{y}$. If we denote by $\bar{X}$ the horizontal lift of X, then we have $\dot{x} = \bar{X} + \eta(\dot{x})\xi = \bar{X} + c\xi$. Since $x(t)$ is the motion of a charged particle, we get

$$\kappa\phi\bar{X} = \kappa\phi\dot{x} = \bar{\nabla}_{\dot{x}}\dot{x} = \bar{\nabla}_{\bar{X}+c\xi}(\bar{X} + c\xi) = \bar{\nabla}_{\bar{X}}\bar{X} + c(-2\phi\bar{X} + [\xi, \bar{X}]),$$

where $\bar{\nabla}$ is the Levi-Civita connection of $\bar{M}$. Since ξ and 0 are π-related, and $\bar{X}$ and X are π-related, we have $\pi[\xi, \bar{X}] = [\pi\xi, \pi\bar{X}] = 0$. Hence $[\xi, \bar{X}]$ is vertical. Since ξ is a Killing vector field and $\bar{X}$ is perpendicular to ξ, we have $\eta([\xi, \bar{X}]) = g(\xi, [\xi, \bar{X}]) = \xi(g(\xi, \bar{X})) = 0$. Hence $[\xi, \bar{X}] = 0$, which implies that $\kappa\phi\bar{X} = \bar{\nabla}_{\bar{X}}\bar{X} - 2c\phi\bar{X}$. Using [4, p. 212, Lemma 45, (3)], we obtain $\nabla_{\dot{y}}\dot{y} = \nabla_X X = d\pi(\bar{\nabla}_{\bar{X}}\bar{X}) = (\kappa + 2c)\pi\phi\bar{X} = (\kappa + 2c)J\dot{y}$. □

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