SUT Journal of Mathematics
Vol. 44, No. 1 (2008), 1–22

The Parseval formula for wave equations with dissipative term of rank one

Mitsuteru Kadowaki*, **Hideo Nakazawa**† and **Kazuo Watanabe**

(Received May 21, 2007; Revised December 8, 2007)

Abstract. In this paper we show the Parseval formula (Theorem 2.5) for the generator of wave equations with some dissipative terms of rank one. Moreover we resolve the solutions for that equations into several modes by using the Parseval formula.

AMS 2000 *Mathematics Subject Classification.* 35P10, 35P25.

Key words and phrases. Wave equation with dissipative term, dissipative mode, scattering mode, the Parseval formula, wave operator, spectral singularity.

§1. Introduction

The resolution of modes for selfadjoint systems has been discussed and studied by many authors. However, it seems that such arguments for non-selfadjoint (dissipative) systems are not sufficient. In this paper we shall consider the problem above for a dissipative system (1.1) below.

Let $s \in \mathbb{R}$ and $L^2_s(\mathbb{R})$ be the weighted L^2 space on $\mathbb{R}$ defined by

$$L^2_s(\mathbb{R}) = \{u(x); \|u\|_s < \infty\},$$

where

$$\|u\|_s^2 = \int_{\mathbb{R}} (1 + |x|^2)^s |u(x)|^2 \, dx.$$

In the case $s = 0$, we denote $L^2_0(\mathbb{R})$ by $L^2(\mathbb{R})$.

*Supported by Grant-in-Aid for Scientific Research (No.19540189), Ministry of Education, Culture, Sports, Science and Technology, Japan.

†Supported by Grant-in-Aid for Scientific Research (No.17740082), Ministry of Education, Culture, Sports, Science and Technology, Japan

In this paper we deal with the following equation:

$$\partial_t^2 u(x,t) + \langle \partial_t u, \varphi \rangle_0 \varphi(x) - \partial_x^2 u(x,t) = 0, \quad (x,t) \in \mathbb{R} \times \mathbb{R}_+ \tag{1.1}$$

where $\partial_t = \partial/\partial t$, $\partial_x = \partial/\partial x$, $\varphi(x) \in L^2_s(\mathbb{R})$ for some $s > 1/2$ and $\langle \cdot, \cdot \rangle_0$ is the inner product of $L^2(\mathbb{R})$.

We deal with (1.1) as a perturbed system of

$$\partial_t^2 u(x,t) - \partial_x^2 u(x,t) = 0, \quad (x,t) \in \mathbb{R} \times \mathbb{R}. \tag{1.2}$$

Put $f(t) = {}^t(u(x,t), \partial_t u(x,t))$. Then (1.1) and (1.2) can be written as

$$\partial_t f(t) = -iAf(t) \quad \text{and} \quad \partial_t f(t) = -iA_0 f(t),$$

where

$$A = i \begin{pmatrix} 0 & 1 \\ \partial_x^2 & -\langle \cdot, \varphi \rangle_0 \varphi \end{pmatrix} \quad \text{and} \quad A_0 = i \begin{pmatrix} 0 & 1 \\ \partial_x^2 & 0 \end{pmatrix},$$

respectively.

To formulate A and A_0, we prepare the following notations.

Let $\mathcal{H}$ be Hilbert space with the inner product:

$$\langle f, g \rangle = \int_{\mathbb{R}} (\partial_x f_1(x) \overline{\partial_x g_1(x)} + f_2(x) \overline{g_2(x)})\, dx,$$

where $f = {}^t(f_1, f_2)$ and $g = {}^t(g_1, g_2)$. The norm of $\mathcal{H}$ is denoted by $\| \cdot \|$. We define the domain of A and A_0 as follows:

$$D(A) = D(A_0) = \left\{ f = {}^t(f_1, f_2) \in \mathcal{H}; \partial_x^2 f_1 \in L^2(\mathbb{R}), f_2 \in H^1(\mathbb{R}) \right\},$$

where $H^s(\mathbb{R})$ is the Sobolev space of order $s\,(\in \mathbb{R})$ on $\mathbb{R}$. Then [12] Theorem X-50 implies that A and A_0 generate contraction semi-group $\{e^{-itA}\}_{t \geqq 0}$ and unitary group $\{e^{-itA_0}\}_{t \in \mathbb{R}}$, respectively. We denote by $R(z)$ (resp. $R_0(z)$) the resolvent of A (resp. A_0), $(A - z)^{-1}$ (resp. $(A_0 - z)^{-1}$) for $z \in \rho(A)$ (resp. $\rho(A_0)$), where $\rho(A)$ (resp. $\rho(A_0)$) is the resolvent set of A (resp.A_0).

We denote by $\hat{u}$ the Fourier transformation of tempered distribution u. We use the notation $\mathcal{B}(E, F)$ as the space of bounded operators from E to F, where E and F are Banach spaces.

Applying the proof of [9] (or [4]), we obtain the following theorem.

Theorem 1.1. *The following holds.*

(1) *A has no real eigenvalues.*

(2) *The wave operator*

$$W = \operatorname*{s-lim}_{t \to \infty} e^{itA_0} e^{-itA}$$

exists as a non-trivial operator from $\mathcal{H}$ to $\mathcal{H}$.

Theorem 1.1 (2) implies that there exist non-trivial initial data f and f^+ ($= Wf$) such that

$$\lim_{t\to\infty} \|e^{-itA}f - e^{-itA_0}f^+\| = 0$$

and

$$\lim_{t\to\infty} \|e^{-itA}f\| \neq 0.$$

This means that the system from (1.1),

$$f(t) = e^{-itA}f, \; f(0) = f \in \mathcal{H} \tag{1.3}$$

has the scattering mode. Theorem 1.1(1) implies that (1.3) does not have the bound mode which is also

$$\lim_{t\to\infty} \|e^{-itA}f\| \neq 0$$

for some $f \in \mathcal{H}$, but not scattering mode. Then we have questions:

(Q1) Does (1.3) have the dissipative mode ? This mode means

$$\lim_{t\to\infty} \|e^{-itA}f\| = 0$$

for some $f\ (\neq 0) \in \mathcal{H}$.

(Q2) Is it true that (1.3) is equal to the linear combination of modes which are found ?

(Q2) asks the possibility of the resolution of modes for (1.3).

Our aim is to answer (Q1) and (Q2). We assume the following:

$$\varphi(x) \in L^2_{1+s}(\mathbb{R}) \tag{A1}$$

for some $s > 1/2$ and

$$\Phi(\lambda) \leqq \Phi(\mu), \quad (0 \leqq \mu \leqq \lambda), \tag{A2}$$

where

$$\Phi(\lambda) = |\hat{\varphi}(\lambda)|^2 + |\hat{\varphi}(-\lambda)|^2.$$

Remark 1.2. If we assume that $1 - 1/2 \left| \int_{\mathbb{R}} \varphi(x)\, dx \right|^2 \neq 0$, (A1) can be replaced by

$$\varphi(x) \in L^2_s(\mathbb{R})$$

for some $s > 1/2$ (cf. section 3∼section 5).

We give answers for (Q1) and (Q2). Denote the set of the spectrum of A by $\sigma(A)(=\mathbb{C}\backslash\rho(A))$. Since A is m-dissipative, $\sigma(A)\subset\overline{\mathbb{C}_-}$. Then we have

$$\sigma(A)\cap\mathbb{C}_- = \begin{cases} \emptyset, & \left(1-\frac{1}{2}\left|\int_{\mathbb{R}}\varphi(x)\right|^2 \geqq 0\right), \\ \{i\kappa_0\}, & \left(1-\frac{1}{2}\left|\int_{\mathbb{R}}\varphi(x)\right|^2 < 0\right) \end{cases} \tag{1.4}$$

for some $\kappa_0<0$. Moreover $i\kappa_0$ is an eigenvalue and its multiplicity is one. The details and proof for (1.4) are given in section 2 and 4, respectively.

In the case $1-1/2\left|\int_{\mathbb{R}}\varphi(x)\,dx\right|^2<0$, according to [13] Theorem XII.5, we define a projection P with respect to $i\kappa_0$ as follows: for $f,g\in\mathcal{H}$

$$\langle Pf,g\rangle = \frac{-1}{2\pi i}\int_{\Gamma}\langle R(z)f,g\rangle\,,dz, \tag{1.5}$$

where Γ $(\subset\mathbb{C}_-)$ is a closed curve enclosed $i\kappa_0$.

Then it is easy to show that

$$\text{Range } P\subset \text{Ker } W. \tag{1.6}$$

(1.6) gives an answer for (Q1).

The Parseval formula (Theorem 2.5) stated in section 2 implies

$$\text{Ker } W = \begin{cases} \{0\}, & \left(1-\frac{1}{2}\left|\int_{\mathbb{R}}\varphi(x)\right|^2 \geqq 0\right), \\ \text{Range } P, & \left(1-\frac{1}{2}\left|\int_{\mathbb{R}}\varphi(x)\right|^2 < 0\right). \end{cases} \tag{1.7}$$

The Parseval formula (Theorem 2.5) will be proven in section 4. The answer for (Q2) is affirmative by (1.7). Indeed, for the cases of $\Gamma(-i0)\geqq 0$, (1.3) is described by scattering mode only, i.e. $f\neq 0$ if and only if $Wf\neq 0$ and

$$\lim_{t\to\infty}\|e^{-itA}f-e^{-itA_0}Wf\|=0.$$

In the cases $\Gamma(-i0)<0$, (1.3) is described by linear combination of scattering and dissipative modes, i.e. for the decomposition

$$e^{-itA}f=e^{-itA}(f-Pf)+e^{-itA}Pf,$$

$f-Pf\neq 0$ if and only if $Wf\neq 0$ and

$$\lim_{t\to\infty}\|e^{-itA}f-e^{-itA_0}Wf\|=\lim_{t\to\infty}\|e^{-itA}(f-Pf)-e^{-itA_0}Wf\|=0.$$

We are interested in scattering theory and the resolution of modes for dissipative systems. In our previous paper [5] we have dealt with some Schrödinger equations with dissipative perturbation of rank one which are solvable models. [5] has also established the Parseval formula and answered the same questions.

In order to prove the Parseval formula, we have to characterize some singular points of $R(z)$ (cf. section 2). These points are non-real eigenvalue $i\kappa_0$ as in (1.4) and "spectral singularity" for A. Spectral singularites are on the continuous spectrum and not eigenvalues (cf. [11] or [14]). In the case $1-1/2\left|\int_{\mathbb{R}} \varphi(x)\,dx\right|^2 = 0$ only, "0" appears as a spectral singularity. In [5], we already has dealt with spectral singularity. Spectral singularity is also a singular (non-established) point of the "in-coming" limiting absorption principle (cf. (4.3)). According to the dissipative scattering theory (cf. [15]), we can show that a singular point of the "out-going" limiting absorption principle is a real eigenvalue. Dealing with the limiting absorption princple for non-selfadjoint operators, we should note that the singular points for the in-coming do not necessarily coincide with that for the out-going. In the case of no spectral singularities ($1-1/2\left|\int_{\mathbb{R}} \varphi(x)\,dx\right|^2 \neq 0$), we obtain (1.7) as a immediate result from the Parseval formula. However, if the spectral singularity exists, i.e., in the case $1-1/2\left|\int_{\mathbb{R}} \varphi(x)\,dx\right|^2 = 0$, we should deal with that effect carefully (cf. Lemma 2.4 and section 4).

We give brief comments for (A1) and (A2). These are technical assumptions. Especially, (A2) is artificial. However it is difficult to characterize spectral singularities to establish the resolution of modes for (1.3) with the usual short range condition: $\varphi(x) \in L^2_s(\mathbb{R})$ for some $s > 1/2$ only.

Remark 1.3. (1) If we deal with high dimensional cases ($n \geqq 2$) of (1.1) with an assumption:

$$|\varphi(x)| \leqq Ce^{-\alpha|x|} \quad (x \in \mathbb{R}^n)$$

for some $C > 0$ and $\alpha > 0$, we can also show the Parseval formula for that generator in a similar way as in this paper. However, since we do not characterize spectral singularities, we can not obtain the answer for (Q2).

(2) If we deal with high dimensional cases ($n \geqq 2$) of (A2), we may put

$$\Phi(\lambda) = \lambda^{n-1} \int_{\mathbb{S}^{n-1}} |\hat{\varphi}(\lambda\omega)|^2\,d\omega,$$

where $\mathbb{S}^{n-1}$ is the unit sphere in $\mathbb{R}^n$.

However that example is trivial case ($\varphi(x) \equiv 0$) only (cf. [8]).

The Parseval formula also means the spectral resolution. In order to show the Parseval formula, we use not only the spectral representation for A but also that for A^* (see section 4). It is natural to use the adjoint operator A^*. Indeed, let H be a self-adjoint operator in Hilbert space $\mathfrak{H}$ and $E(\lambda)$ be the spectral resolution. Then it is well-known that for $x,\ y \in \mathfrak{H}$, the following relation holds:

$$\left\langle \frac{d}{d\lambda}E(\lambda)x, y \right\rangle_{\mathfrak{H}} = \lim_{\kappa\downarrow 0} \frac{1}{2\pi i} \left\langle \{(H-\lambda-i\kappa)^{-1} - (H-\lambda+i\kappa)^{-1}\}x, y\right\rangle_{\mathfrak{H}}$$
$$= \lim_{\kappa\downarrow 0} \frac{\kappa}{\pi} \left\langle (H-\lambda-i\kappa)^{-1}x, (H-\lambda-i\kappa)^{-1}y\right\rangle_{\mathfrak{H}}.$$

So, for a dissipative operator D, it is natural to deal with

$$\lim_{\kappa\downarrow 0} \frac{1}{2\pi i} \left\langle \{(D-\lambda-i\kappa)^{-1} - (D-\lambda+i\kappa)^{-1}\}x, y\right\rangle_{\mathfrak{H}}$$
$$= \lim_{\kappa\downarrow 0} \frac{\kappa}{\pi} \left\langle (D-\lambda-i\kappa)^{-1}x, (D^*-\lambda-i\kappa)^{-1}y\right\rangle_{\mathfrak{H}}$$

as a spectral density (cf. Remark 4.2).

There is not so many literatures concerning scattering theory for dissipative systems. For instance, Mochizuki [9] has showed the existence of scattering mode for wave equations with dissipative terms. Simon [15] has also dealt with the same problem for Schrödinger equations with absorption. Kadowaki [4] has dealt with a similar problem as in [9]. We mention several results concerning the resolution of modes for dissipative systems. Kato [6] has already studied the resolution of modes for non-selfadjoint systems. However he has assumed a small perturbation which implies no eigenvalues nor the spectral singularites. So, as a consequence, his framework can be applied to the system of single mode which is scattering only. Mochizuki [10] has also applied Kato's theory to wave equation with dissipative terms and dealt with fixed energy inverse scattering problem. Gilliam and Schulenberger [2] have dealt with Maxwell's equations with dissipative boundary conditions in half space in $\mathbb{R}^3$, and established the Parseval formula and the resolution of modes (cf. Theorem 4.11 and 4.16 of [2]). However their equations are not perturbed systems of selfadjoint. [3] has studied similar equations and discussed the Parseval formula and the resolution of modes. In order to prove the Parseval formula, they have already combined and used the generalized Fourier transforms of the generator and the adjoint operator. [10] has used the spectral representation of the generator and the adjoint operator to obtain the representation of scattering matrix. Stepin [16] has informed results concerning (inverse) wave operators and the spectrum for dissipative Schrödinger operator and Boltzmann transport operator without assuming a small perturbation. He also has dealt with direct wave operators. As a consequence, his results (Theorem 3 and 4 of [16]) imply the resolution of modes. From the viewpoint of the proof of the resolution of modes, he has used direct wave operators instead of the adjoint operators of the generators. However he has assumed no spectral singularities. Finally, in present paper and in [5] we have dealt with the resolution of modes for dissipative systems with spectral singularities. However, there is not so many literatures concerning the resolution of modes for dissipative systems with spectral singularities.

This paper is organized as follows. In section 2 we state our results. Especially, our main results are Therorem 2.5 and Corollary 2.7. In section 3 we prepare some estimates for the resolvent of $-\partial^2/\partial x^2$. In section 4 and 5 we give the proof of results stated in section 2.

§2. Results

In order to state our main results (Theorem 2.5 and Corollary 2.7), we describe proposition (Proposition 2.1) and two lemmas (Lemma 2.2 and 2.4). Except for Lemma 2.4 and Corollary 2.7 these proof is given in section 4. Corollary 2.7 and Lemma 2.4 are showed in this section and section 5, respectively.

For $z \in \mathbb{C} \setminus \mathbb{R}$, we denote $(-\partial_x^2 - z^2)^{-1}$ by $r_0(z)$. We state the limiting absorption principle for $r_0(z)$. Let $s > 1/2$ and $u \in L_s^2(\mathbb{R})$. Since

$$r_0(\lambda \pm i\kappa)u(x) = \frac{\pm i}{2(\lambda \pm i\kappa)} \int_{\mathbb{R}} e^{\pm i(\lambda \pm i\kappa)|x-y|} u(y)\, dy \tag{2.1}$$

for $\lambda \in \mathbb{R}, \kappa \geqq 0$ and $\lambda\kappa \neq 0$, a straightfoward calculation implies the following.

Let $s > 1/2$. Then for every $\lambda \in \mathbb{R}$, two limits

$$\lambda^l \partial_x^j r_0(\lambda \pm i0) = \lim_{\kappa \downarrow 0} (\lambda \pm i\kappa)^l \partial_x^j r_0(\lambda \pm i\kappa) \tag{2.2}$$

exist in the uniformly operator topology of $\mathcal{B}(L_s^2(\mathbb{R}), L_{-s}^2(\mathbb{R}))$, where $(l, j) = (0, 1)$ or $(1, 0)$.

We denote by $\langle \cdot, \cdot \rangle_0$ as the dual coupling of $L_s^2(\mathbb{R})$ and $L_{-s}^2(\mathbb{R})$, and put

$$\Sigma_\pm = \{z \in \mathbb{C}_\pm; \mathit{\Gamma}(z) = 0\} \quad \text{and} \quad \Sigma_\pm^0 = \{\lambda \in \mathbb{R}; \mathit{\Gamma}(\lambda \pm i0) = 0\},$$

where

$$\mathit{\Gamma}(z) = 1 - iz \langle r_0(z)\varphi, \varphi \rangle_0, \quad \mathit{\Gamma}(\lambda \pm i0) = 1 - i\lambda \langle r_0(\lambda \pm i0)\varphi, \varphi \rangle_0.$$

Noting that for $z \in \mathbb{C} \setminus \mathbb{R}$ and $f = {}^t(f_1, f_2) \in \mathcal{H}$,

$$R_0(z)f = {}^t(r_0(z)(zf_1 + if_2), i\partial_x r_0(z)\partial_x f_1 + zr_0(z)f_2) \tag{2.3}$$

and using the arguments similar to those used by [1] Theorem 1.1.1, we have the following:

$z \notin \Sigma_+$ (resp. Σ_-) if and only if $z \in \rho(A) \cap \mathbb{C}_+$ (resp. $z \in \rho(A) \cap \mathbb{C}_-$) and

$$R(z)f = R_0(z)f + \frac{i\,\langle f, v(\overline{z})\rangle}{\Gamma(z)} v(z) \tag{2.4}$$

for any $f = {}^t(f_1, f_2) \in \mathcal{H}$, where

$$v(z) = \begin{pmatrix} ir_0(z)\varphi \\ zr_0(z)\varphi \end{pmatrix}.$$

In order to analyze the spectrum of A, we have to characterize Σ_+, Σ_-, Σ_+^0 and Σ_-^0.

We can obtain the result concerning Σ_+ and Σ_+^0:

$$\Sigma_+ = \Sigma_+^0 = \emptyset. \tag{2.5}$$

(2.5) shall be proven in section 4.

Note that

$$\Gamma(-i0) = 1 - \frac{1}{2}\left|\int_{\mathbb{R}} \varphi(x)\,dx\right|^2.$$

We use $\Gamma(-i0)$ to state our results.

For Σ_- and Σ_-^0 we can obtain the following proposition.

Proposition 2.1. *Assume* (A1) *and* (A2)*. Then*

$$\Sigma_- = \begin{cases} \emptyset, & (\Gamma(-i0) \geqq 0)\,, \\ \{i\kappa_0\}, & (\Gamma(-i0) < 0) \end{cases} \tag{2.6}$$

for some $\kappa_0 < 0$ *and*

$$\Sigma_-^0 = \begin{cases} \emptyset, & (\Gamma(-i0) \neq 0)\,, \\ \{0\}, & (\Gamma(-i0) = 0)\,. \end{cases} \tag{2.7}$$

Moreover, in the case $\Gamma(-i0) < 0$ *and* $\Gamma(-i0) = 0$*, it holds* $\Gamma'(i\kappa_0) \neq 0$ *and* $\Gamma'(-i0) \neq 0$*,respectively.*

Noting (2.4), we have (1.4) by Proposition 2.1. Especially, in the case of $\Gamma(-i0) < 0$, $i\kappa_0$ is a simple pole of $R(z)$. Hence the residual theorem together with (1.5) gives

$$Pf = \frac{-1}{2\pi} \frac{\langle f, v(-i\kappa_0)\rangle}{\Gamma'(i\kappa_0)} v(i\kappa_0)$$

for $f \in \mathcal{H}$.

In order to state our main theorem (Theorem 2.5, the Parseval formula), we define a subspace of $\mathcal{H}$ and some operators.

Let $s > 1/2$ and $f = {}^t(f_1, f_2) \in \mathcal{H}_s$, where

$$\mathcal{H}_s = \left\{ f = {}^t(f_1, f_2); \int_{\mathbb{R}} (1+|x|^2)^s (|\partial_x f_1(x)|^2 + |f_2(x)|^2)\, dx < \infty \right\}.$$

Define operators $\mathfrak{F}_0$ and $\mathfrak{F}$, $\mathfrak{G}$ as

$$(\mathfrak{F}_0 f)(\lambda) = \begin{cases} {}^t\left(\dfrac{\lambda \hat{f}_1(\lambda) + i\hat{f}_2(\lambda)}{\sqrt{2}}, \dfrac{\lambda \hat{f}_1(-\lambda) + i\hat{f}_2(-\lambda)}{\sqrt{2}} \right), & (\lambda > 0), \\ {}^t\left(\dfrac{-\lambda \hat{f}_1(-\lambda) - i\hat{f}_2(-\lambda)}{\sqrt{2}}, \dfrac{-\lambda \hat{f}_1(\lambda) - i\hat{f}_2(\lambda)}{\sqrt{2}} \right), & (\lambda < 0) \end{cases}$$

and

$$\begin{cases} (\mathfrak{F} f)(\lambda) = (\mathfrak{F}_0 f)(\lambda) + \dfrac{i\langle f, v(\lambda - i0)\rangle}{\Gamma(\lambda + i0)} (\mathfrak{F}_0 \begin{pmatrix} 0 \\ \varphi \end{pmatrix})(\lambda), \\ (\mathfrak{G} f)(\lambda) = (\mathfrak{F}_0 f)(\lambda) - \dfrac{i\langle f, v(\lambda - i0)\rangle}{\overline{\Gamma(\lambda - i0)}} (\mathfrak{F}_0 \begin{pmatrix} 0 \\ \varphi \end{pmatrix})(\lambda). \end{cases}$$

It is well-known that

(1) $\mathfrak{F}_0$ is extended to a unitary operator from $\mathcal{H}$ onto $L^2(\mathbb{R}; \mathbb{C}^2)$.

(2) It holds that for any $f \in D(A_0)$ and $g \in \mathcal{H}$,

$$\langle A_0 f, g\rangle = \int_{-\infty}^{\infty} \lambda \langle (\mathfrak{F}_0 f)(\lambda), (\mathfrak{F}_0 g)(\lambda)\rangle_{\mathbb{C}^2}\, d\lambda,$$

where $\langle \cdot, \cdot \rangle_{\mathbb{C}^2}$ is usual inner-product of $\mathbb{C}^2$.

We call $\mathfrak{F}_0$ the spectral representation for A_0. Concerning $\mathfrak{F}$, we have the following proposition.

Lemma 2.2. *$\mathfrak{F}$ is extended to a bounded operator from $\mathcal{H}$ to $L^2(\mathbb{R}; \mathbb{C}^2)$ and satisfies $\mathfrak{F} = \mathfrak{F}_0 W$. Moreover we have*

$$\int_{-\infty}^{\infty} \langle (\mathfrak{F} A f)(\lambda), \tilde{g}(\lambda)\rangle_{\mathbb{C}^2}\, d\lambda = \int_{-\infty}^{\infty} \lambda \langle (\mathfrak{F} f)(\lambda), \tilde{g}(\lambda)\rangle_{\mathbb{C}^2}\, d\lambda \tag{2.8}$$

for any $f \in D(A)$ and $\tilde{g} \in L^2(\mathbb{R}; \mathbb{C}^2)$.

By (2.8) we call the operator $\mathfrak{F}$ the spectral representation for A.

Remark 2.3. $\mathfrak{G}$ is the formal spectral representation for A^*. In the case $\Gamma(-i0) \neq 0$, it is not difficult to see that $\mathfrak{G}$ is extended a bounded operator from $\mathcal{H}$ to $L^2(\mathbb{R}; \mathbb{C}^2)$. The proof is done by using Lemma 3.1 and (3.1) of Lemma 3.2 (cf. that of Lemma 2.4 in section 5). However, in the case $\Gamma(-i0) = 0$, $\mathfrak{G}$ is not bounded(cf. Lemma 2.4). In order to state and prove the Parseval formula (Theorem 2.5), we have to consider the pair of $\mathfrak{F}$ and $\mathfrak{G}$.

In order to deal with the case $\mathit{\Gamma}(-i0) = 0$, we take a subspace of $\mathcal{H}$:

$$\mathcal{E} = \{g \in \mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R}); \langle v(-i0), g\rangle = 0\}$$

as the domain of $\mathfrak{G}$, where $\mathcal{S}(\mathbb{R})$ is the Schwartz space on $\mathbb{R}$.

Then we have the following lemma.

Lemma 2.4. *Assume* (A1), (A2) *and* $\mathit{\Gamma}(-i0) = 0$. *Then the following holds.*

(1) $\mathfrak{G}g$ *belongs to* $L^2(\mathbb{R}; \mathbb{C}^2)$ *for* $g \in \mathcal{E}$.

(2) $\mathcal{E}$ *is dense in* $\mathcal{H}$.

Using the above preparation, we state main theorem (the Parseval formula).

Theorem 2.5. *Assume* (A1) *and* (A2). *Then*

(1) *in the cases* $\mathit{\Gamma}(-i0) \neq 0$, *it holds that*

$$\langle f, g\rangle = \begin{cases} \displaystyle\int_{-\infty}^{\infty} \langle(\mathfrak{F}f)(\lambda), (\mathfrak{G}g)(\lambda)\rangle_{\mathbb{C}^2}\, d\lambda, & (\mathit{\Gamma}(-i0) > 0) \\ \displaystyle\int_{-\infty}^{\infty} \langle(\mathfrak{F}f)(\lambda), (\mathfrak{G}g)(\lambda)\rangle_{\mathbb{C}^2}\, d\lambda + \langle Pf, g\rangle, & (\mathit{\Gamma}(-i0) < 0) \end{cases}$$

for any f, $g \in \mathcal{H}$.

(2) *in the cases* $\mathit{\Gamma}(-i0) = 0$, *it holds that*

$$\langle f, g\rangle = \int_{-\infty}^{\infty} \langle(\mathfrak{F}f)(\lambda), (\mathfrak{G}g)(\lambda)\rangle_{\mathbb{C}^2}\, d\lambda$$

for any $f \in \mathcal{H}$, $g \in \mathcal{E}$.

Remark 2.6. We may consider Theorem 2.5 as the spectral resolution theorem for the dissipative operator A. For instance, in the case $\mathit{\Gamma}(-i0) < 0$, by (2.8) it holds that

$$\langle Af, g\rangle = \int_{-\infty}^{\infty} \lambda \langle(\mathfrak{F}f)(\lambda), (\mathfrak{G}g)(\lambda)\rangle_{\mathbb{C}^2}\, d\lambda + i\kappa_0 \langle Pf, g\rangle$$

for any $f \in D(A)$ and $g \in \mathcal{H}$.

As a corollary of Theorem 2.5 we have the following.

Corollary 2.7. *Assume* (A1) *and* (A2). *Then* (1.7) *is true.*

Proof. We give the proof in the case $\Gamma(-i0) < 0$ and $\Gamma(-i0) = 0$ only. (In the case $\Gamma(-i0) > 0$, we can prove the assertion by the similar way.)

Let $\Gamma(-i0) < 0$. Suppose $Wf = 0$. Then by (1.6) it suffices to show $f = Pf$. Since $\mathfrak{F}_0$ is unitary, we have $\mathfrak{F}f = 0$. By Theorem 2.5 (1), for any $g \in \mathcal{H}$ we have

$$\langle f, g \rangle = \langle Pf, g \rangle .$$

Hence we obtain $f = Pf$.

Let $\Gamma(-i0) = 0$. Suppose $Wf = 0$. Then we have $\mathfrak{F}f = 0$. Applying Theorem 2.5 (2), for any $g \in \mathcal{E}$ we have

$$\langle f, g \rangle = 0.$$

Hence using Lemma 2.4 (2), we have $f = 0$.

□

§3. Preliminaries

In this section we prepare two lemmas for $v(z)$ and $\Gamma(z)$.

Lemma 3.1. *Let $s > 1/2$ and $g \in \mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R})$. Then, for $z \in \mathbb{C}$ one has*

$$\langle v(z), g \rangle = \begin{cases} O(|z|^{-1}), & (|z| \to \infty), \\ O(1), & (|z| \to 0). \end{cases}$$

Proof. Let $g = {}^t(g_1, g_2)$. Then it follows from [8] Corollary 4.4.5 or a straightforward calculation by using (2.1) that

$$\begin{aligned} &\langle v(z), g \rangle = \left\langle \varphi, r_0(\overline{z})(i\partial_x^2 g_1 + \overline{z} g_2) \right\rangle_0 \\ &= \begin{cases} \left\langle \varphi, r_0(\overline{z}) i\partial_x^2 g_1 \right\rangle_0 - \dfrac{1}{z} \left\langle \varphi, r_0(\overline{z})\partial_x^2 g_2 \right\rangle_0 - \dfrac{1}{z} \langle \varphi, r_0(\overline{z}) g_2 \rangle_0 = O(|z|^{-1}), \\ \hfill (|z| \to \infty) \\ -i \langle \varphi, g_1 \rangle_0 - iz^2 \langle \varphi, r_0(\overline{z}) g_1 \rangle_0 + z \langle \varphi, r_0(\overline{z}) g_2 \rangle_0 = O(1), \qquad (|z| \to 0) \end{cases} \end{aligned}$$

The proof is completed. □

Lemma 3.2. *The following holds.*

$$\inf_{\operatorname{Im} z \geqq 0} \operatorname{Re} \Gamma(z) \geqq 1 \tag{3.1}$$

and there exists a positive constant C such that

$$\liminf_{|z| \to \infty,\, \operatorname{Im} z \leqq 0} |\operatorname{Re} \Gamma(z)| \geqq C. \tag{3.2}$$

Proof. Let $z = \lambda + i\kappa \in \mathbb{C}_+$. (3.1) immediately follows from

$$(3.3)\quad \mathit{\Gamma}(z) = 1 + \frac{1}{2}\int_{-\infty}^{\infty} \frac{\kappa}{(r-\lambda)^2+\kappa^2}\Phi(r)\,dr - \frac{i}{2}\int_{-\infty}^{\infty} \frac{r-\lambda}{(r-\lambda)^2+\kappa^2}\Phi(r)\,dr.$$

We shall prove that there exist $\eta > 0$ and $C > 0$ such that

$$\liminf_{|z|\to\infty,\,-\eta\leqq \operatorname{Im} z\leqq 0} |\operatorname{Re}\mathit{\Gamma}(z)| \geqq C \tag{3.4}$$

and

$$\liminf_{|z|\to\infty,\,\operatorname{Im} z\leqq -\eta} |\operatorname{Re}\mathit{\Gamma}(z)| \geqq C. \tag{3.5}$$

(3.2) follows from (3.4) and (3.5). We continue to prove (3.4). By the Riemann-Lebesgue Theorem, we have

$$\lim_{|\lambda|\to\infty} \Phi(\lambda) = 0,$$

uniformly in λ, $|\lambda| > 1$. By the property of the Possion integral on the lower half-plane, we have

$$\lim_{\kappa\uparrow 0} \operatorname{Re}\mathit{\Gamma}(z) = 1 - \frac{\pi}{2}\Phi(\lambda)$$

uniformly in λ, $|\lambda| > 1$, where $z = \lambda + i\kappa \in \mathbb{C}_-$.

Hence we have (3.4).

Next we give the proof of (3.5). For any fixed compact interval $J = (-b, -a)$ $(0 < a < b)$, it is not difficult to show that

$$|\operatorname{Re}\mathit{\Gamma}(z)| \geqq 1 - \frac{1}{2}\int_{-\infty}^{\infty} \frac{b}{(r-\lambda)^2+a^2}\Phi(r)\,dr$$

uniformly in $\kappa \in J$.

Thus we have

$$\lim_{\lambda\to\infty} |\operatorname{Re}\mathit{\Gamma}(z)| \geqq 1 \tag{3.6}$$

uniformly in $\kappa \in J$.

Moreover since

$$|\operatorname{Re}\mathit{\Gamma}(z)| \geqq 1 - \frac{1}{|\kappa|}\|\varphi\|_0^2,$$

we have

$$\liminf_{\kappa\to-\infty} |\operatorname{Re}\mathit{\Gamma}(z)| \geqq 1 \tag{3.7}$$

uniformly in $\lambda \in \mathbb{R}$.

Therefore (3.6) and (3.7) imply (3.5). □

§4. The proof of results

In this section we give the proof of results stated in section 2. (2.5) follows from (3.1) immediately. Next, we prove Proposition 2.1.

Proof of Proposition 2.1. Let $z = \lambda + i\kappa \in \mathbb{C}_-$. First of all, we give the proof of (2.6). By (3.3), we can easily show that

$$\operatorname{Im} \mathit{\Gamma}(z) = \frac{1}{2} \int_0^\infty \frac{r}{r^2 + \kappa^2} (\Phi(r - \lambda) - \Phi(r + \lambda))\, dr.$$

Then we prove that

$$\operatorname{Im} \mathit{\Gamma}(z) \begin{cases} > 0, & (\lambda > 0), \\ = 0, & (\lambda = 0), \\ < 0, & (\lambda < 0). \end{cases} \tag{4.1}$$

First, for $\lambda > 0$ we show (4.1). Let $\lambda > 0$ and $r > 0$. Noting $r + \lambda > r - \lambda$, $\lambda - r$, by (A2) we have

$$\Phi(r - \lambda) - \Phi(r + \lambda) > 0, \quad (\lambda < r)$$

and

$$\Phi(r - \lambda) - \Phi(r + \lambda) = \Phi(\lambda - r) - \Phi(r + \lambda) > 0, \quad (0 < r < \lambda).$$

Therefore we obtain

$$\operatorname{Im} \mathit{\Gamma}(z) = \frac{1}{2} \left(\int_0^\lambda + \int_\lambda^\infty \right) \frac{r}{r^2 + \kappa^2} (\Phi(r - \lambda) - \Phi(r + \lambda))\, dr > 0.$$

In the case $\lambda < 0$, we can prove (4.1) in the same way. In the case $\lambda = 0$, (4.1) is clear. Hence, since Σ_- consists of, at most, purely imaginary numbers, it is sufficient to deal with $\mathit{\Gamma}(i\kappa) = 0$.

It follows from (3.3) with $z = i\kappa$ ($\kappa < 0$) that

$$\operatorname{Re} \mathit{\Gamma}(z) = 1 - \frac{1}{2} \int_{-\infty}^\infty \tan^{-1} \frac{r}{\kappa} \Phi'(r)\, dr.$$

Then by (A2) we have

$$\frac{\partial}{\partial \kappa} \operatorname{Re} \mathit{\Gamma}(i\kappa) = \frac{1}{2} \int_{-\infty}^\infty \frac{r}{r^2 + \kappa^2} \Phi'(r)\, dr < 0. \tag{4.2}$$

Since

$$\lim_{\kappa \uparrow 0} \operatorname{Re} \mathit{\Gamma}(i\kappa) = \mathit{\Gamma}(-i0) = 1 - \frac{1}{2} \left| \int_{\mathbb{R}} \varphi(x)\, dx \right|^2,$$

we have (2.6) by (4.2). Moreover, suppose $\Gamma(-i0) < 0$. Then we have $\Gamma'(-i\kappa_0) \neq 0$ by (4.2).

(2.7) follows from (3.3) that

$$\begin{aligned}\operatorname{Im} \Gamma(\lambda - i0) =& \frac{1}{2}\text{p.v.} \int_{-\infty}^{\infty} \frac{1}{r-\lambda} \Phi(r) dr \\ =& \lim_{\delta \downarrow 0} \int_{\delta}^{\infty} \frac{\Phi(r+\lambda) - \Phi(r-\lambda)}{r} dr.\end{aligned}$$

Indeed, by the same argument as in above, we have

$$\operatorname{Im} \Gamma(\lambda - i0) \begin{cases} > 0, & (\lambda > 0), \\ = 0, & (\lambda = 0), \\ < 0, & (\lambda < 0). \end{cases}$$

Hence we obtain (2.7).

Suppose $\Gamma(-i0) = 0$. Since $\Gamma(z)$ is C^1-class in $\overline{\mathbb{C}_-}$ by (A1), we have $\Gamma'(-i0) \neq 0$ by (4.2). □

In order to prove Lemma 2.2, we state the limiting absorption principle of A_0 and A.

(2.1) implies the limiting absorption principle of A_0 as follows. Let $s > 1/2$. Then for every $\lambda \in \mathbb{R}$, two limits

$$R_0(\lambda \pm i0) = \operatorname*{s-lim}_{\kappa \downarrow 0} R_0(\lambda \pm i\kappa)$$

exist as operators in $\mathcal{B}(\mathcal{H}_s, \mathcal{H}_{-s})$, where $\mathcal{H}_s$ is the same as in section 2.

It follows from (2.2), (2.3) and Proposition 2.1 that the limiting absorption principle of A as follows. Let $s > 1/2$. Then for every $\lambda \in \mathbb{R}$,

$$R(\lambda + i0) = \operatorname*{s-lim}_{\kappa \downarrow 0} R(\lambda + i\kappa)$$

and for every $\lambda \in \begin{cases} \mathbb{R} \setminus \{0\}, & (\Gamma(-i0) = 0) \\ \mathbb{R}, & (\Gamma(-i0) \neq 0), \end{cases}$

$$R(\lambda - i0) = \operatorname*{s-lim}_{\kappa \downarrow 0} R(\lambda - i\kappa)$$

exist as operators in $\mathcal{B}(\mathcal{H}_s, \mathcal{H}_{-s})$ and

$$R(\lambda \pm i0) f = R_0(\lambda \pm i0) f + \frac{i \langle f, v(\lambda \mp i0) \rangle}{\Gamma(\lambda \pm i0)} v(\lambda \pm i0) \tag{4.3}$$

for any $f \in \mathcal{H}_s$.

proof of Lemma 2.2. Let $s > 1/2$ and let $f,\ g \in \mathcal{H}_s$. By the well-known relations in the stationary scattering theory (cf. [7]) we have

$$\langle Wf, g\rangle = \lim_{\kappa\downarrow 0} \frac{\kappa}{\pi} \int_{-\infty}^{\infty} \langle R(\lambda + i\kappa)f, R_0(\lambda + i\kappa)g\rangle \ d\lambda.$$

Note that

$$\lim_{\kappa\downarrow 0} \frac{\kappa}{\pi} \langle R_0(\lambda + i\kappa)f, R_0(\lambda + i\kappa)g\rangle = \frac{1}{2\pi i} \langle (R_0(\lambda + i0) - R_0(\lambda - i0))f, g\rangle$$

and

$$\frac{1}{2\pi i} \langle (R_0(\lambda + i0) - R_0(\lambda - i0))f, g\rangle = \langle (\mathfrak{F}_0 f)(\lambda), (\mathfrak{F}_0 g)(\lambda)\rangle_{\mathbb{C}^2} . \tag{4.4}$$

Hence by (3.1), Lemma 3.1 and (4.3), we obtain

$$\langle Wf, g\rangle = \int_{-\infty}^{\infty} \langle (\mathfrak{F} f)(\lambda), (\mathfrak{F}_0 g)(\lambda)\rangle_{\mathbb{C}^2} \ d\lambda.$$

Thus noting that $\mathfrak{F}_0$ is unitary, we have the first and the second assertions. (2.8) follows from the interwine property of W. □

In order to prove Theorem 2.5, we show the key formula (4.5) for $\mathfrak{F}$ and $\mathfrak{G}$.

Lemma 4.1. *Let $s > 1/2$. Then one has*

$$\begin{aligned} &\langle (\mathfrak{F} f)(\lambda), (\mathfrak{G} g)(\lambda)\rangle_{\mathbb{C}^2} \\ &= \langle (\mathfrak{F}_0 f)(\lambda), (\mathfrak{F}_0 g)(\lambda)\rangle_{\mathbb{C}^2} + \frac{1}{2\pi} \frac{\langle f, v(\lambda - i0)\rangle \langle v(\lambda + i0), g\rangle}{\Gamma(\lambda + i0)} \\ &\quad - \frac{1}{2\pi} \frac{\langle f, v(\lambda + i0)\rangle \langle v(\lambda - i0), g\rangle}{\Gamma(\lambda - i0)} \end{aligned} \tag{4.5}$$

for $f,\ g \in \mathcal{H}_s$ and $\lambda \notin \Sigma_-^0$.

Proof. We decompose $\langle (\mathfrak{F} f)(\lambda), (\mathfrak{G} g)(\lambda)\rangle_{\mathbb{C}^2}$ as follows:

$$\langle (\mathfrak{F} f)(\lambda), (\mathfrak{G} g)(\lambda)\rangle_{\mathbb{C}^2} = \sum_{j=1}^{4} J_j(\lambda),$$

where

$$\begin{aligned} J_1(\lambda) &= \langle (\mathfrak{F}_0 f)(\lambda), (\mathfrak{F}_0 g)(\lambda)\rangle_{\mathbb{C}^2} , \\ J_2(\lambda) &= \frac{i \langle f, v(\lambda - i0)\rangle}{\Gamma(\lambda + i0)} \left\langle (\mathfrak{F}_0 \begin{pmatrix} 0 \\ \varphi \end{pmatrix})(\lambda), (\mathfrak{F}_0 g)(\lambda) \right\rangle_{\mathbb{C}^2} , \\ J_3(\lambda) &= \frac{i \langle v(\lambda - i0), g\rangle}{\Gamma(\lambda - i0)} \left\langle (\mathfrak{F}_0 f)(\lambda), (\mathfrak{F}_0 \begin{pmatrix} 0 \\ \varphi \end{pmatrix})(\lambda) \right\rangle_{\mathbb{C}^2} \end{aligned}$$

and

$$J_4(\lambda) = \frac{i\,\langle f, v(\lambda - i0)\rangle}{\Gamma(\lambda + i0)} \frac{i\,\langle v(\lambda - i0), g\rangle}{\Gamma(\lambda - i0)} \times \\ \times \left\langle (\mathfrak{F}_0 \begin{pmatrix} 0 \\ \varphi \end{pmatrix})(\lambda), (\mathfrak{F}_0 \begin{pmatrix} 0 \\ \varphi \end{pmatrix})(\lambda) \right\rangle_{\mathbb{C}^2}.$$

Using (2.3) and (4.4), we have

$$J_2(\lambda) = \frac{1}{2\pi} \frac{\langle f, v(\lambda - i0)\rangle \langle v(\lambda + i0) - v(\lambda - i0), g\rangle}{\Gamma(\lambda + i0)},$$
$$J_3(\lambda) = \frac{1}{2\pi} \frac{\langle f, v(\lambda - i0) - v(\lambda + i0)\rangle \langle v(\lambda - i0), g\rangle}{\Gamma(\lambda - i0)}$$

and

$$\begin{aligned} J_4(\lambda) &= \frac{1}{2\pi} \frac{\langle f, v(\lambda - i0)\rangle}{\Gamma(\lambda + i0)} \frac{\langle v(\lambda - i0), g\rangle}{\Gamma(\lambda - i0)} \{\Gamma(\lambda - i0) - \Gamma(\lambda + i0)\} \\ &= \frac{1}{2\pi} \frac{\langle f, v(\lambda - i0)\rangle \langle v(\lambda - i0), g\rangle}{\Gamma(\lambda + i0)} - \frac{1}{2\pi} \frac{\langle f, v(\lambda - i0)\rangle \langle v(\lambda - i0), g\rangle}{\Gamma(\lambda - i0)}. \end{aligned}$$

Hence we obtain (4.5). □

Remark 4.2. Lemma 4.1 also means

$$\langle (\mathfrak{F}f)(\lambda), (\mathfrak{G}g)(\lambda)\rangle_{\mathbb{C}^2} = \frac{1}{2\pi i} \langle (R(\lambda + i0) - R(\lambda - i0))f, g\rangle$$

for $f,\, g \in \mathcal{H}_s (s > 1/2)$ and $\lambda \notin \Sigma_-^0$.

Proof of Theorem 2.5. Theorem 2.5 follows from

$$\int_{-\infty}^{\infty} \frac{1}{2\pi} \frac{\langle f, v(\lambda + i0)\rangle \langle v(\lambda - i0), g\rangle}{\Gamma(\lambda - i0)}\, d\lambda = \begin{cases} 0, & (\Gamma(-i0) \geqq 0), \\ -\langle Pf, g\rangle, & (\Gamma(-i0) < 0) \end{cases} \tag{4.6}$$

and

$$\int_{-\infty}^{\infty} \frac{1}{2\pi} \frac{\langle f, v(\lambda - i0)\rangle \langle v(\lambda + i0), g\rangle}{\Gamma(\lambda + i0)}\, d\lambda = 0 \tag{4.7}$$

for any $f \in \mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R})$ and

$$g \in \begin{cases} \mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R}), & (\Gamma(-i0) \neq 0), \\ \mathcal{E}, & (\Gamma(-i0) = 0). \end{cases}$$

Indeed , for $\lambda \in \mathbb{R}$, integrating the both side of (4.5) and noting

$$\int_{-\infty}^{\infty} \langle (\mathfrak{F}_0 f)(\lambda), (\mathfrak{F}_0 g)(\lambda))\rangle_{\mathbb{C}^2}\, d\lambda = \langle f, g\rangle,$$

by (4.6) and (4.7) we have

$$\int_{-\infty}^{\infty} \langle (\mathfrak{F}f)(\lambda), (\mathfrak{G}g)(\lambda) \rangle_{\mathbb{C}^2} \, d\lambda = \begin{cases} \langle f, g \rangle \,, & (\mathit{\Gamma}(-i0) \geqq 0), \\ \langle f, g \rangle - \langle Pf, g \rangle \,, & (\mathit{\Gamma}(-i0) < 0). \end{cases}$$

Hence the density argument implies Theorem 2.5.

In the following, we prove (4.6) and (4.7).

Let X_1, X_2 and Y be some sufficiently large positive numbers and let ε be a sufficiently small positive number. We give the proof of (4.6). Put

$$F(z) = -\frac{1}{2\pi} \frac{\langle f, v(\overline{z}) \rangle \langle v(z), g \rangle}{\mathit{\Gamma}(z)}$$

for $z \in \mathbb{C}_-$.

Then by Proposition 2.1 (together with (1.5) for $\mathit{\Gamma}(-i0) < 0$) we have

$$\sum_{j=1}^{4} \int_{\mathit{\Gamma}_j} F(z) dz = \begin{cases} 0, & (\mathit{\Gamma}(-i0) \geqq 0), \\ -\langle Pf, g \rangle \,, & (\mathit{\Gamma}(-i0) < 0), \end{cases}$$

where

$$\begin{aligned} &\mathit{\Gamma}_1 : z = \lambda - i\varepsilon, && (\lambda = X_2 \to -X_1), \\ &\mathit{\Gamma}_2 : z = -X_1 + i\kappa, && (\kappa = -\varepsilon \to -Y), \\ &\mathit{\Gamma}_3 : z = \lambda - iY, && (\lambda = -X_1 \to X_2) \end{aligned}$$

and

$$\mathit{\Gamma}_4 : z = X_2 + i\kappa, \qquad (\kappa = -Y \to -\varepsilon).$$

We estimate

$$I_j = \int_{\mathit{\Gamma}_j} F(z) \, dz, \quad (j = 2, 3, 4).$$

In the case $\mathit{\Gamma}(-i0) = 0$, note that L'Hospital's theorem together with $\mathit{\Gamma}'(-i0) \neq 0$ (cf. Proposition 2.1) implies that for any $g \in \mathcal{E}$,

$$\lim_{\lambda \to 0} \frac{\langle v(\lambda - i0), g \rangle}{\mathit{\Gamma}(\lambda - i0)} \quad \text{exists.} \tag{4.8}$$

Note that

$$\begin{aligned} I_2 &= \int_{\varepsilon}^{Y} F(-X_1 + i\kappa)(-i) \, d\kappa, \\ I_3 &= \int_{-X_1}^{X_2} F(\lambda - iY) \, d\lambda \end{aligned}$$

and

$$I_4 = \int_Y^{\varepsilon} F(X_2 + i\kappa)(-i)\, d\kappa.$$

Using Proposition 2.1, by Lemma 3.1 and (3.2) of Lemma 3.2 we have

$$|I_2| \leqq C \int_{\varepsilon}^{Y} \frac{1}{X_1^2 + \kappa^2}\, d\kappa = \frac{C}{X_1}(\tan^{-1}\frac{Y}{X_1} - \tan^{-1}\frac{\varepsilon}{X_1}) \leqq \frac{C}{X_1}\frac{\pi}{2},$$

$$|I_3| \leqq C \int_{-X_1}^{X_2} \frac{1}{\lambda^2 + Y^2}\, d\lambda = \frac{C}{Y}(\tan^{-1}\frac{X_2}{Y} + \tan^{-1}\frac{X_1}{Y}) \leqq \frac{C}{Y}\pi$$

and

$$|I_4| \leqq C \int_{\varepsilon}^{Y} \frac{1}{X_2^2 + \kappa^2} d\kappa = \frac{C}{X_2}(\tan^{-1}\frac{Y}{X_2} - \tan^{-1}\frac{\varepsilon}{X_2}) \leqq \frac{C}{X_2}\frac{\pi}{2},$$

where $C = C(f, g, \varphi) > 0$.

Noting that

$$I_1 = \int_{X_2}^{-X_1} F(\lambda - i\varepsilon)\, d\lambda,$$

we have

$$C(\frac{1}{X_1} + \frac{1}{X_2} + \frac{1}{Y}) \geqq \begin{cases} \left| - \int_{-X_1}^{X_2} F(\lambda - i\varepsilon)\, d\lambda \right|, & (\mathit{\Gamma}(-i0) \geqq 0) \\ \left| - \int_{-X_1}^{X_2} F(\lambda - i\varepsilon)\, d\lambda - \langle Pf, g\rangle \right|, & (\mathit{\Gamma}(-i0) < 0). \end{cases}$$

Thus taking X_1, X_2, $Y \to \infty$, we have

$$- \int_{-\infty}^{\infty} F(\lambda - i\varepsilon)\, d\lambda = \begin{cases} 0, & (\mathit{\Gamma}(-i0) \geqq 0), \\ - \langle Pf, g\rangle\, , & (\mathit{\Gamma}(-i0) < 0). \end{cases} \tag{4.9}$$

For each λ, note that

$$\lim_{\varepsilon \to 0} F(\lambda - i\varepsilon) = -\frac{1}{2\pi} \frac{\langle f, v(\lambda + i0)\rangle \langle v(\lambda - i0), g\rangle}{\mathit{\Gamma}(\lambda - i0)}. \tag{4.10}$$

Noting (2.7) of Proposition 2.1, by Lemma 3.1 and (3.2) of Lemma 3.2 we have that there exist positive numbers C_1 and $C_2 > 0$ which are independent of $\varepsilon > 0$ such that

$$|F(\lambda - i\varepsilon)| \leqq \begin{cases} \frac{C_1}{\lambda^2}, & (|\lambda| > 1), \\ C_2, & (|\lambda| \leqq 1). \end{cases}$$

Thus using (4.9) and (4.10), we have (4.6) by Lebesgue's theorem.

Next we give the sketch of the proof of (4.7). Put

$$G(z) = \frac{1}{2\pi} \frac{\langle f, v(\overline{z}) \rangle \langle v(z), g \rangle}{\Gamma(z)}$$

for $z \in \mathbb{C}_+$.

Then by Lemma 3.1 and (3.1) of Lemma 3.2, we have

$$\sum_{j=1}^{4} \int_{\tilde{\gamma}_j} G(z) dz = 0,$$

where

$$\begin{aligned} &\tilde{\gamma}_1 : z = \lambda + i\varepsilon, && (\lambda = -X_1 \to X_2), \\ &\tilde{\gamma}_2 : z = X_2 + i\kappa, && (\kappa = \varepsilon \to Y), \\ &\tilde{\gamma}_3 : z = \lambda + iY, && (\lambda = X_2 \to -X_1) \end{aligned}$$

and

$$\tilde{\gamma}_4 : z = -X_1 + i\kappa, \quad (\kappa = Y \to \varepsilon).$$

Therefore (4.7) follows from a similar way as in the proof of (4.6). □

§5. The proof of Lemma 2.4

Proof of Lemma 2.4. First of all, we show (1). Noting (4.8), by Lemma 3.1 and (3.2) of Lemma 3.2 we have

$$\left| \frac{\langle v(\lambda - i0), g \rangle}{\Gamma(\lambda - i0)} \right| \leqq C$$

for $g \in \mathcal{E}$, where $C = C(\varphi, g) > 0$ is independent of λ.

Thus since $\mathfrak{F}_0$ is a unitary operator, (1) is true.

Next, we give the proof of (2).

Let $f = {}^t(f_1, f_2) \in \mathcal{H}$. Then for any $\varepsilon > 0$, there exists $g_1 \in \mathcal{S}(\mathbb{R})$ such that

$$\|f_1 - g_1\|_{\dot{H}^1(\mathbb{R})} < \varepsilon,$$

where $\| \cdot \|_{\dot{H}^1(\mathbb{R})}$ is the norm of $\dot{H}^1(\mathbb{R})$.

Note that

$$\langle v(-i0), g \rangle = 0 \Longleftrightarrow \langle \varphi, g_1 \rangle_0 - \frac{1}{2} \langle \varphi, 1 \rangle_0 \langle 1, g_2 \rangle_0 = 0 \Longleftrightarrow \hat{g}_2(0) = \frac{2 \langle g_1, \varphi \rangle_0}{\displaystyle\int_{\mathbb{R}} \overline{\varphi(y)}\, dy}.$$

Then we put

$$\alpha = \frac{2\langle g_1, \varphi\rangle_0}{\displaystyle\int_{\mathbb{R}} \overline{\varphi(y)}\, dy}.$$

We can find $h \in \mathcal{S}(\mathbb{R})$ which satisfies

$$\hat{h}(k) \in C_0^\infty(\mathbb{R} \setminus 0), \quad \|h - f_2\|_0 < \varepsilon/4.$$

Taking $\delta > 0$ such that

$$|\alpha|^2 2\delta < \varepsilon^2/16,$$

we put

$$H_\delta(k) = \begin{cases} \hat{h}(k) + \alpha, & (|k| < \delta), \\ \hat{h}(k), & (|k| \geqq \delta). \end{cases}$$

Then we have $H_\delta(0) = \alpha$ and

$$\|\hat{f}_2 - H_\delta\|_0 = \|(\hat{f}_2 - \hat{h}) + \alpha\chi_\delta\|_0 < \varepsilon/4 + |\alpha|(2\delta)^{1/2} < \varepsilon/2,$$

where χ_δ is the characteristic function of the open interval $|k| < \delta$.

We approximate H_δ by $C_0^\infty(\mathbb{R})$-function. Let $\tilde{\chi}_\delta \in C_0^\infty(\mathbb{R})$ a function satisfying

(1) $0 \leqq \tilde{\chi}_\delta \leqq 1$

(2) $\operatorname{supp} \tilde{\chi}_\delta = \{k \in \mathbb{R}; |k| \leqq 3\delta/4\}$

(3) $\tilde{\chi}_\delta(k) = 1 \Longleftrightarrow |k| \leqq 2\delta/3$

Let be $R > 0$ a sufficiently large number and let be ρ_τ a modifier function for a suffiently small number $\tau > 0$. We define a function $\tilde{H}_{\delta,R,\tau}(k)$ as

$$\tilde{H}_{\delta,R,\tau}(k) = \tilde{\chi}_\delta(k)H_\delta(k) + \{\rho_\tau * \chi_R(1 - \tilde{\chi}_\delta)H_\delta\}(k).$$

Then $\tilde{H}_{\delta,R,\tau}$ belongs to $C_0^\infty(\mathbb{R})$. Taking τ as $2\delta/3 > \tau$, we have

$$0 \notin \operatorname{supp} \rho_\tau * \chi_R(1 - \tilde{\chi}_\delta)H_\delta \subset \left\{k \in \mathbb{R}; \frac{2\delta}{3} - \tau \leqq |k| \leqq \tau + R\right\}.$$

This means

$$\tilde{H}_{\delta,R,\tau}(0) = \alpha.$$

Since

$$\lim_{\tau \to 0, R \to \infty} \|H_\delta - \tilde{H}_{\delta,R,\tau}\|_0 = 0$$

for the above ε and δ, there exists positive numbers τ_0 and $R_0 > 0$ such that

$$\|H_\delta - \tilde{H}_{\delta,R_0,\tau_0}\|_0 < \varepsilon/2.$$

We choose $g_2 \in \mathcal{S}(\mathbb{R})$ satisfying $\hat{g}_2 = \tilde{H}_{\delta,R_0,\tau_0}$. Then we have

$$\|g_2 - f_2\|_0 = \|\hat{g}_2 - \hat{f}_2\|_0 \leqq \|\hat{g}_2 - H_\delta\|_0 + \|H_\delta - \hat{f}_2\|_0 < \varepsilon.$$

Put $g = {}^t(g_1, g_2)$. Then g belongs to $\mathcal{E}$ and satisfies $\|f - g\| < 2\varepsilon$. The proof is completed. □

References

[1] S. Albeverio and P. Kurasov, *Singular perturbations of differential operators*, London Math. Soc. Lect. Note Ser. No. 271, Cambridge Univ. Press, 2000.

[2] D. S. Gilliam and J. R. Schulenberger *Electromagnetic waves in a three-dimensional half space with a dissipative boundary*, J. Math. Anal. Appl., **89**, 1982, 129–185.

[3] D. S. Gilliam and J. R. Schulenberger, *Spectral analysis of a dissipative problem in electrodynamics: The Sommerfeld problem*, Acta Appl. Math., **6**, 1986, 63–94.

[4] M. Kadowaki *Resolvent estimates and scattering states for dissipative systems*, Publ. RIMS Kyoto Univ., **38**, 2002, 191–209.

[5] M. Kadowaki, H. Nakazawa and K. Watanabe, *On the asymptotics of solutions for some Schrödinger equations with dissipative perturbations of rank one*, Hiroshima Math.J., **34**, 2004, 345–369.

[6] T. Kato, *Wave operators and similarity for some non-self adjoint operators*, Math. Ann. **162**, 1966, 258–279.

[7] S. T. Kuroda, *Spectral theory II*, Iwanami, Tokyo 1978 (in Japanese).

[8] S. T. Kuroda, *An Introduction to Scattering Theory*, Lecture Note Series N^o 51, Matematisk Institut, Aarhus University, 1980.

[9] K. Mochizuki, *Scattering theory for wave equations with dissipative terms*, Publ. RIMS Kyoto Univ., **12**, 1976, 383–390.

[10] K. Mochizuki, *Inverse scattering for a small nonselfadjoint perturbation of the wave equation*, Anaiysis and application ISAAC 2001 (Berlin), 303-316.

[11] M. A. Naimark , *Investigation of the spectrum and the expansion in eigenfunction of a non-selfadjoint operator of second order on a semi axis*, Amer. Math. Soc. Transl. Ser, **216**, 1960, 103–193.

[12] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, II. Fourier Analysis, Self-Adjointness*, Academic Press, 1975, New York, San Francisco, London.

[13] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, IV, Analysis of Operators*, Academic Press, 1978, New York, San Francisco, London.

[14] J. T. Schwartz, *Some non selfadjoint operators* , Comm. Pure Appl. Math., **13**, 1960, 609–639.

[15] B. Simon *Phase space analysis of simple scattering systems: extensions of some work of Enss*, Duke Math. J, **46**, 1979, 119–168.

[16] S. A. Stepin, *Scattering and spectral singularities for some dissipative operators of mathematical physics*, J. of Nonliear Math. Phys., **11**, 2004, pp. 194–203.

Mitsuteru Kadowaki
Department of Mechanical Engineering, Ehime University
Matsuyama, Ehime 790-8577, Japan
E-mail: mkadowaki@eng.ehime-u.ac.jp

Hideo Nakazawa
Department of Mathematics, Chiba Institute of Technology
Narashino, Chiba 275-0023, Japan
E-mail: nakazawa.hideo@it-chiba.ac.jp

Kazuo Watanabe
Department of Mathematics, Gakushuin University
Toshima-ku, Tokyo 171-8588, Japan
E-mail: kazuo.watanabe@gakushuin.ac.jp