On an orbit algebra induced by the Auslander-Reiten translation for the enveloping algebra of a self-injective Nakayama algebra

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Abstract. Let A be a basic self-injective Nakayama algebra over an algebraically closed field. In this paper, we investigate the ring structure of the orbit algebra $\mathbb{A}(\tau_{A^e}; A) = \bigoplus_{i \geq 0} \underline{\mathrm{Hom}}_{A^e} \left(\tau_{A^e}^i(A), A \right)$, where A^e is the enveloping algebra of A and τ_{A^e} is the Auslander-Reiten translation for A^e.

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§1. Introduction

Let K be an algebraically closed field, and let R be a finite dimensional self-injective algebra over K. We denote by R^{op} the opposite algebra of R, and by $\mathrm{mod}(R)$ the category of finitely generated left R-modules. Recall from [ASS] that the *projectively stable category* $\underline{\mathrm{mod}}(R)$ of $\mathrm{mod}(R)$ is defined to be the category whose objects are the same as those of $\mathrm{mod}(R)$ and the morphism set $\underline{\mathrm{Hom}}_R(M, N)$ for M, N in $\underline{\mathrm{mod}}(R)$ is the factor space $\mathrm{Hom}_R(M, N)/\mathscr{P}(M, N)$, where $\mathscr{P}(M, N)$ is a subspace of $\mathrm{Hom}_R(M, N)$ consisting of all morphisms which factor through a projective module in $\mathrm{mod}(R)$. Dually, the *injectively stable category* $\overline{\mathrm{mod}}(R)$ of $\mathrm{mod}(R)$ is also defined. However, since R is self-injective, we obtain $\underline{\mathrm{mod}}(R) = \overline{\mathrm{mod}}(R)$.

Let M be a module in $\mathrm{mod}(R)$, and let $P_1 \xrightarrow{\rho_1} P_0 \xrightarrow{\rho_0} M \longrightarrow 0$ be a minimal projective presentation of M. Applying the functor $(-)^t := \mathrm{Hom}_R(-, R)$, we have the exact sequence of right R-modules:

$$0 \longrightarrow M^t \xrightarrow{\rho_0^t} P_0^t \xrightarrow{\rho_1^t} P_1^t \longrightarrow \mathrm{Coker}\, \rho_1^t \longrightarrow 0.$$

Then, by setting $\mathrm{Tr}_R(M) := \mathrm{Coker}\,\rho_1^t$, we obtain the duality $\mathrm{Tr}_R : \underline{\mathrm{mod}}(R) \longrightarrow \underline{\mathrm{mod}}(R^{\mathrm{op}})$ called the *transpose duality*. Moreover, we have the self-duality $\tau_R := D\mathrm{Tr}_R : \underline{\mathrm{mod}}(R) \longrightarrow \underline{\mathrm{mod}}(R)$ called the *Auslander-Reiten translation* (see [ARS], [ASS]), where D denotes the usual duality $\mathrm{Hom}_K(-, K)$. In this paper, we study a graded algebra over K induced by τ_R in the case where R is the enveloping algebra of a self-injective Nakayama algebra.

Let s be a positive integer and K an algebraically closed field, and let Γ be the cyclic quiver with s vertices $e_0, e_1, \ldots, e_{s-1}$ and s arrows $a_0, a_1, \ldots, a_{s-1}$, where each a_t $(0 \le t \le s-1)$ starts at e_t and ends at e_{t+1}. Here, we regard the index t of e_t modulo s. We denote by $K\Gamma$ the path algebra of Γ over K, and by X the sum of all arrows in $K\Gamma$: $X = a_0 + \cdots + a_{s-1}$. Moreover, we denote the K-algebra $K\Gamma/(X^k)$ $(k \ge 2)$ by A. It is known that A is a basic self-injective Nakayama algebra (see [ASS]). Note that the enveloping algebra $A^e := A \otimes_K A^{\mathrm{op}}$ is also a self-injective algebra. Recall that the τ_{A^e}-*orbit algebra* of A, denoted by $\mathbb{A}(\tau_{A^e}; A)$ as in [P], is a graded K-algebra defined as follows: $\mathbb{A}(\tau_{A^e}; A)$ is the direct sum of the K-vector spaces

$$\mathbb{A}(\tau_{A^e}; A) = \bigoplus_{i \ge 0} \underline{\mathrm{Hom}}_{A^e}(\tau_{A^e}^i(A), A).$$

The multiplication $\underline{f} \cdot \underline{g}$ of homogeneous elements $\underline{f} \in \underline{\mathrm{Hom}}_{A^e}(\tau_{A^e}^m(A), A)$ and $\underline{g} \in \underline{\mathrm{Hom}}_{A^e}(\tau_{A^e}^n(A), A)$ is the composition $\underline{f} \circ \tau_{A^e}^m(\underline{g}) \in \underline{\mathrm{Hom}}_{A^e}(\tau_{A^e}^{m+n}(A), A)$.

In [P], Pogorzały describes the ring structure of $\mathbb{A}(\tau_{A^e}; A)$ by using a Galois covering of A^e in the case where the τ_{A^e}-period of A equals one, that is, $k \equiv 2 \pmod{s}$. See Remark 2.6 for $k \equiv 1 \pmod{s}$. In this paper, under the condition that $s \ge 2$ and $k \equiv 0 \pmod{s}$, we find a basis of the K-space $\underline{\mathrm{Hom}}_{A^e}(\tau_{A^e}^i(A), A)$ $(i \ge 0)$ by using an injective hull of $\tau_{A^e}^i(A)$ and determine the ring structure of $\mathbb{A}(\tau_{A^e}; A)$.

This paper is organized as follows: In Section 2, we will define an automorphism of categories $(-)_{\alpha^n} : \underline{\mathrm{mod}}(A^e) \longrightarrow \underline{\mathrm{mod}}(A^e)$ for any integer n and an automorphism α of A, and prove that $\mathbb{A}(\tau_{A^e}; A)$ is isomorphic to the orbit algebra $\bigoplus_{i \ge 0} \underline{\mathrm{Hom}}_{A^e}(A_{\alpha^{i(k-2)}}, A)$ induced by $(-)_{\alpha^{k-2}}$ (Lemma 2.1). Next, we explicitly give a K-basis of $\mathrm{Hom}_{A^e}(A_{\alpha^{i(k-2)}}, A)$ (Proposition 2.3). Moreover, in the case $s \ge 2$ and $k \equiv 0 \pmod{s}$, we find a K-basis of $\mathscr{P}(A_{\alpha^{-2i}}, A)$ $(i \ge 0)$ by means of the injective hull of A^e-module $A_{\alpha^{-2i}}$ given in [F], and we give a K-basis of $\underline{\mathrm{Hom}}_{A^e}(\tau_{A^e}^i(A), A)$ $(i \ge 0)$ (Theorem 2.5). In Section 3, we give a presentation of $\mathbb{A}(\tau_{A^e}; A)$ by the generators and the relations in the case $s \ge 2$ and $k \equiv 0 \pmod{s}$ (Theorems 3.2, 3.3).

§2.   The stable homomorphisms

Let s be a positive integer, and let Γ be the cyclic quiver with s vertices $e_0, e_1, \ldots, e_{s-1}$ and s arrows $a_0, a_1, \ldots, a_{s-1}$, where each a_i starts at e_i and ends at e_{i+1}. Here, we regard the index i of e_i modulo s. Denote by X the sum of all arrows in the path algebra $K\Gamma$, and by A the algebra $K\Gamma/(X^k)$ ($k \geq 2$) as in Section 1. Furthermore, for simplicity, we denote a coset in A by one of its representative elements in $K\Gamma$. Then clearly the set $\{X^j e_\ell \mid 0 \leq \ell \leq s-1,\ 0 \leq j \leq k-1\}$ is a K-basis of A, and so $\dim_K A = ks$.

Our purpose in this section is to give a K-basis of $\underline{\mathrm{Hom}}_{A^e}(\tau_{A^e}^i(A), A)$ for $i \geq 0$ in the case $k \equiv 0 \pmod{s}$ (Theorem 2.5). However, the discussion in the subsections 2.1 and 2.2 are valid for arbitrary $k \geq 2$.

2.1.   The algebra $\mathbb{A}(\tau_{A^e}; A)$ and an automorphism α of A

Let $\alpha : A \longrightarrow A$ be an algebra automorphism defined by $\alpha(e_t) = e_{t-1}$, $\alpha(a_t) = a_{t-1}$ for $0 \leq t \leq s-1$. Then clearly $\alpha^s = \mathrm{id}_A$ holds. For any integer n and M in $\mathrm{mod}(A^e)$, we denote by M_{α^n} the left A^e-module, equivalently, the A-bimodule defined as follows: M_{α^n} has the underlying K-space M, and the operation $\cdot$ of A from the right is given by $m \cdot a = m\alpha^n(a)$ for $a \in A$, $m \in M_{\alpha^n}$, and the operation of A from the left is the usual one. Moreover, for any left A^e-homomorphism $f : M \longrightarrow N$, we define the A^e-homomorphism $f_{\alpha^n} : M_{\alpha^n} \longrightarrow N_{\alpha^n}$ by $f_{\alpha^n}(m) = f(m)$ for $m \in M_{\alpha^n}$. Then we have the automorphism of categories $(-)_{\alpha^n} : \mathrm{mod}(A^e) \longrightarrow \mathrm{mod}(A^e)$ with the inverse $(-)_{\alpha^{-n}} : \mathrm{mod}(A^e) \longrightarrow \mathrm{mod}(A^e)$ (see [H]). It is easy to see that φ is in $\mathscr{P}(M, N)$ if and only if φ_{α^n} is in $\mathscr{P}(M_{\alpha^n}, N_{\alpha^n})$. Hence the functor $(-)_{\alpha^n}$ induces the automorphism of $\underline{\mathrm{mod}}(A^e)$. We also denote this functor by $(-)_{\alpha^n}$.

It is shown in [F, Theorem] that $\tau_{A^e}^i(A) \simeq A_{\alpha^{i(k-2)}}$ as left A^e-modules for each $i \geq 0$. So, we immediately have an isomorphism $\underline{\mathrm{Hom}}_{A^e}(\tau_{A^e}^i(A), A) \xrightarrow{\sim} \underline{\mathrm{Hom}}_{A^e}(A_{\alpha^{i(k-2)}}, A)$ of K-spaces. In the following, we show that, in fact, there is an isomorphism $\underline{\mathrm{Hom}}_{A^e}(\tau_{A^e}^i(A), A) \xrightarrow{\sim} \underline{\mathrm{Hom}}_{A^e}(A_{\alpha^{i(k-2)}}, A)$ for each $i \geq 0$ which provides an isomorphism of algebras between $\mathbb{A}(\tau_{A^e}; A)$ and the orbit algebra $\bigoplus_{i \geq 0} \underline{\mathrm{Hom}}_{A^e}(A_{\alpha^{i(k-2)}}, A)$ induced by $(-)_{\alpha^{k-2}}$.

Lemma 2.1. *There exists an isomorphism of K-spaces*

$$\theta_i : \underline{\mathrm{Hom}}_{A^e}(\tau_{A^e}^i(A), A) \xrightarrow{\sim} \underline{\mathrm{Hom}}_{A^e}(A_{\alpha^{i(k-2)}}, A)$$

for each $i \geq 0$ such that

$$\bigoplus_{i \geq 0} \theta_i : \mathbb{A}(\tau_{A^e}; A) \xrightarrow{\sim} \bigoplus_{i \geq 0} \underline{\mathrm{Hom}}_{A^e}(A_{\alpha^{i(k-2)}}, A)$$

is an isomorphism of graded K-algebras.

Proof. First note that $\tau_{A^e} \simeq \mathscr{N}\Omega^2_{A^e}$ as functors, where $\Omega_{A^e} : \underline{\mathrm{mod}}(A^e) \longrightarrow \underline{\mathrm{mod}}(A^e)$ is the *syzygy functor* and $\mathscr{N} : \underline{\mathrm{mod}}(A^e) \longrightarrow \underline{\mathrm{mod}}(A^e)$ is the *Nakayama functor* $D\mathrm{Hom}_{A^e}(-, A^e)$ (see [ARS]). Moreover Ω_{A^e} and $\mathscr{N}$ are commutative as functors, and so $\tau^i_{A^e} \simeq \mathscr{N}^i\Omega^{2i}_{A^e}$ for all $i \geq 0$ as functors.

We show the following statement from which the lemma easily follows: For each integers $i, j \geq 0$, there exists an isomorphism $\eta_{i,j} : \mathscr{N}^i\Omega^{2i}_{A^e}(A_{\alpha^j}) \longrightarrow A_{\alpha^{i(k-2)+j}}$ in $\underline{\mathrm{mod}}(A^e)$ such that, for any integers $\ell, p, q \geq 0$ and a morphism $\underline{f} : A_{\alpha^p} \longrightarrow A_{\alpha^q}$ in $\underline{\mathrm{Hom}}_{A^e}(A_{\alpha^p}, A_{\alpha^q})$, the square

$$
\begin{array}{ccc}
\mathscr{N}^\ell\Omega^{2\ell}_{A^e}(A_{\alpha^p}) & \xrightarrow{\mathscr{N}^\ell\Omega^{2\ell}_{A^e}(\underline{f})} & \mathscr{N}^\ell\Omega^{2\ell}_{A^e}(A_{\alpha^q}) \\
\wr \downarrow {\scriptstyle \eta_{\ell,p}} & & \wr \downarrow {\scriptstyle \eta_{\ell,q}} \\
A_{\alpha^{\ell(k-2)+p}} & \xrightarrow{(\underline{f})_{\alpha^{\ell(k-2)}}} & A_{\alpha^{\ell(k-2)+q}}
\end{array}
$$

in $\underline{\mathrm{mod}}(A^e)$ commutes.

It is shown in [EH, Section 4] that $\Omega^{2\ell}_{A^e}(A) \simeq A_{\alpha^{-\ell k}}$ for $\ell \geq 0$ as left A^e-modules, and then we easily have an isomorphism of A^e-modules $\mu_{t,r} : \Omega^{2t}_{A^e}(A_{\alpha^r}) \longrightarrow A_{\alpha^{-tk+r}}$ for $t, r \geq 0$ such that the following square in $\underline{\mathrm{mod}}(A^e)$ commutes for any $\ell, p, q \geq 0$ and $\underline{f} \in \underline{\mathrm{Hom}}_{A^e}(A_{\alpha^p}, A_{\alpha^q})$:

$$
\begin{array}{ccc}
\Omega^{2\ell}_{A^e}(A_{\alpha^p}) & \xrightarrow{\Omega^{2\ell}_{A^e}(\underline{f})} & \Omega^{2\ell}_{A^e}(A_{\alpha^q}) \\
\wr \downarrow {\scriptstyle \mu_{\ell,p}} & & \wr \downarrow {\scriptstyle \mu_{\ell,q}} \\
A_{\alpha^{-\ell k+p}} & \xrightarrow{(\underline{f})_{\alpha^{-\ell k}}} & A_{\alpha^{-\ell k+q}}.
\end{array}
$$

Since $\nu := \alpha^{1-k} \otimes \alpha^{k-1} : A^e \longrightarrow A^e$ is a Nakayama automorphism of A^e (see [F, Appendix]), we have $\mathscr{N} \simeq F_\nu$ as functors, where $F_\nu : \underline{\mathrm{mod}}(A^e) \longrightarrow \underline{\mathrm{mod}}(A^e)$ is the functor defined as follows: For M in $\underline{\mathrm{mod}}(A^e)$, $F_\nu(M)$ has the underlying K-space M, and the operation $\cdot$ of A^e is given by $(a \otimes b^{\mathrm{op}}) \cdot m = \nu(a \otimes b^{\mathrm{op}})m = \alpha^{1-k}(a)m\alpha^{k-1}(b)$ for $a \otimes b^{\mathrm{op}} \in A^e$ and $m \in F_\nu(M)$. Also, for $\underline{f} \in \underline{\mathrm{Hom}}_{A^e}(M, N)$, $F_\nu(\underline{f})$ is the coset $\underline{{}_\nu f} \in \underline{\mathrm{Hom}}_{A^e}(F_\nu(M), F_\nu(N))$, where $_\nu f \in \mathrm{Hom}_{A^e}(F_\nu(M), F_\nu(N))$ is given by $_\nu f(m) := f(m)$ for $m \in F_\nu(M)$.

Applying $\mathscr{N}^\ell$ to the square above yields the following commutative square in $\underline{\mathrm{mod}}(A^e)$:

$$
\begin{array}{ccc}
\mathscr{N}^\ell\Omega^{2\ell}_{A^e}(A_{\alpha^p}) & \xrightarrow{\mathscr{N}^\ell\Omega^{2\ell}_{A^e}(\underline{f})} & \mathscr{N}^\ell\Omega^{2\ell}_{A^e}(A_{\alpha^q}) \\
\wr \downarrow {\scriptstyle \xi_{\ell,p}} & & \wr \downarrow {\scriptstyle \xi_{\ell,q}} \\
F^\ell_\nu(A_{\alpha^{-\ell k+p}}) & \xrightarrow{\nu\left(f_{\alpha^{-\ell k}}\right)} & F^\ell_\nu(A_{\alpha^{-\ell k+q}}).
\end{array}
$$

Moreover there exists the following commutative square in $\underline{\mathrm{mod}}(A^e)$:

$$
\begin{array}{ccc}
F_\nu^\ell(A_{\alpha^{-\ell k+p}}) & \xrightarrow{\ \nu(f_{\alpha^{-\ell k}})\ } & F_\nu^\ell(A_{\alpha^{-\ell k+q}}) \\
\wr \downarrow \underline{\alpha^{\ell(k-1)}} & & \wr \downarrow \underline{\alpha^{\ell(k-1)}} \\
A_{\alpha^{\ell(k-2)+p}} & \xrightarrow{\ f_{\alpha^{\ell(k-2)}}\ } & A_{\alpha^{\ell(k-2)+q}}.
\end{array}
$$

In the above square, the left vertical map $\underline{\alpha^{\ell(k-1)}}$ is defined by

$$
\underline{\alpha^{\ell(k-1)}}(x) = \alpha^{\ell(k-1)}(x) \quad \text{for } x \in F_\nu^\ell(A_{\alpha^{-\ell k+p}}),
$$

and it is verified that $\underline{\alpha^{\ell(k-1)}}$ is an A^e-homomorphism between the A^e-modules $F_\nu^\ell(A_{\alpha^{-\ell k+p}})$ and $A_{\alpha^{\ell(k-2)+p}}$. Similarly the right vertical map $\underline{\alpha^{\ell(k-1)}}$ is defined and it is also an A^e-homomorphism.

We will show the commutativity of this square. Let $q - p \equiv z \pmod s$ $(0 \leq z \leq s-1)$. Let $\underline{f}(e_t) = \sum_{u=0}^{k-1} \sum_{v=1}^{s} k_{u,v}^{(t)} X^u e_v$ for each t $(1 \leq t \leq s)$, where $k_{u,v}^{(t)} \in K$. Then we have

$$
\underline{f}(e_t) = \sum_{j_t=0}^{n_t} k_{z+j_t s, w_t}^{(t)} X^{z+j_t s} e_{w_t},
$$

where $t+p-q \equiv w_t \pmod s$ $(1 \leq w_t \leq s)$, because $\underline{f}(e_t) = \underline{f}((e_t \otimes e_{t+p}^{\mathrm{op}}) \cdot e_t) = (e_t \otimes e_{t+p}^{\mathrm{op}}) \cdot \underline{f}(e_t) = e_t \underline{f}(e_t) e_{t+p-q}$. Furthermore, $\alpha(X) = X$ implies $X \underline{f}(e_t) = \underline{f}(e_{t+1}) X$. Hence it follows that $k_{z+rs,w_1}^{(1)} = k_{z+rs,w_2}^{(2)} = \cdots = k_{z+rs,w_s}^{(s)}$ for each r $(0 \leq r \leq n)$ and $k_{z+r's,w_t}^{(t)} = 0$ for $r' > n$, where $n = \min\{n_1, \ldots, n_s\}$. Therefore we have

$$
\underline{f}(1) = \sum_{t=1}^{s} \underline{f}(e_t) = \sum_{i=0}^{n} k_{z+is,w_1}^{(1)} X^{z+is}.
$$

Hence we have $\underline{\alpha^j}(\underline{f}(1)) = \underline{f}(1)$ for any j. Finally, for $x \in F_\nu^\ell(A_{\alpha^{-\ell k+p}})$, we get

$$
\begin{aligned}
(\underline{f_{\alpha^{\ell(k-2)}}} \circ \underline{\alpha^{\ell(k-1)}})(x) &= \underline{f_{\alpha^{\ell(k-2)}}}(\underline{\alpha^{\ell(k-1)}}(x)) \\
&= \underline{f_{\alpha^{\ell(k-2)}}}(\underline{\alpha^{\ell(k-1)}}((\alpha^{\ell(k-1)}(x) \otimes 1^{\mathrm{op}}) \cdot 1)) \\
&= \underline{f_{\alpha^{\ell(k-2)}}}((\alpha^{\ell(k-1)}(x) \otimes 1^{\mathrm{op}}) \cdot 1) \\
&= (\alpha^{\ell(k-1)}(x) \otimes 1^{\mathrm{op}}) \cdot \underline{f_{\alpha^{\ell(k-2)}}}(1) \\
&= \alpha^{\ell(k-1)}(x) \underline{f}(1)
\end{aligned}
$$

and

$$
(\underline{\alpha^{\ell(k-1)}} \circ \underline{\nu(f_{\alpha^{-\ell k}})})(x) = \underline{\alpha^{\ell(k-1)}}(\underline{\nu(f_{\alpha^{-\ell k}})}(x))
$$

$$
\begin{aligned}
&= \underline{\alpha^{\ell(k-1)}}(\underline{\nu\ (f_{\alpha^{-\ell k}})}((\alpha^{\ell(k-1)}(x) \otimes 1^{\mathrm{op}}) \cdot 1)) \\
&= \underline{\alpha^{\ell(k-1)}}((\alpha^{\ell(k-1)}(x) \otimes 1^{\mathrm{op}}) \cdot \underline{\nu\ (f_{\alpha^{-\ell k}})}(1)) \\
&= (\alpha^{\ell(k-1)}(x) \otimes 1^{\mathrm{op}}) \cdot \underline{\alpha^{\ell(k-1)}}(\underline{\nu\ (f_{\alpha^{-\ell k}})}(1)) \\
&= \alpha^{\ell(k-1)}(x)\underline{f}(1).
\end{aligned}
$$

So the square is commutative.

Combining the last two squares, we have the desired isomorphism $\eta_{i,j}$. $\square$

2.2.   The spaces of homomorphisms

Next we will give a K-basis of $\mathrm{Hom}_{A^e}(\tau^i_{A^e}(A), A)$ for $i \geq 0$. We will use the following lemma, which is an analogue of [EH, Lemma 2.1]. The proof is straightforward.

Lemma 2.2. *Let n be any integer. Then the map*

$$
\mathrm{Hom}_{A^e}(A_{\alpha^n}, A) \longrightarrow {}_{\alpha^n}Z := \{x \in A \mid xy = \alpha^n(y)x \text{ for any } y \in A\}
$$

given by $f \longmapsto f(1)$ is an isomorphism of K-spaces.

If $s = 1$, then we easily see that the τ_{A^e}-period of A equals one by [F, Corollary 3.7], and so the ring structure of $\mathbb{A}(\tau_{A^e}; A)$ is described in [P]. Therefore, in the rest of this paper, we assume $s \geq 2$. Also, for any integer z, denote by $\overline{z}$ the unique integer r $(0 \leq r \leq s-1)$ such that $z \equiv r \pmod{s}$, and let m be the unique integer such that $k = ms + \overline{k}$.

First we consider the K-space $\mathrm{Hom}_{A^e}(A_{\alpha^{i(k-2)}}, A)$ for each $i \geq 0$. We identify $\mathrm{Hom}_{A^e}(A_{\alpha^{i(k-2)}}, A)$ with ${}_{\alpha^{i(k-2)}}Z$ via the isomorphism in Lemma 2.2. Then we have the following proposition.

Proposition 2.3. *Let i be any non-negative integer, and set $d = -i(k-2)$. Then we have the isomorphism of K-spaces*

$\operatorname{Hom}_{A^e}(A_{\alpha^{-d}}, A) = {}_{\alpha^{-d}}Z$

$$
= \begin{cases}
\displaystyle\bigoplus_{j=0}^{m} KX^{js+\overline{d}} & \text{if } \overline{k}-1 \neq \overline{d} < \overline{k}, \\
\displaystyle\Big(\bigoplus_{j=0}^{m-1} KX^{js+\overline{k}-1}\Big) \oplus \Big(\bigoplus_{\ell=0}^{s-1} KX^{ms+\overline{k}-1}e_\ell\Big) & \text{if } \overline{k}-1 = \overline{d}, \\
\displaystyle\bigoplus_{j=0}^{m-1} KX^{js+\overline{d}} & \text{if } \overline{k} \leq \overline{d} \neq s-1, \\
\displaystyle\bigoplus_{j=0}^{m-1} KX^{js+s-1} & \text{if } \overline{d} = s-1 \text{ and } \overline{k} \neq 0, \\
\displaystyle\Big(\bigoplus_{j=0}^{m-2} KX^{js+s-1}\Big) \oplus \Big(\bigoplus_{\ell=0}^{s-1} KX^{ms-1}e_\ell\Big) & \text{if } \overline{d} = s-1 \text{ and } \overline{k} = 0.
\end{cases}
$$

Proof. Take any $x \in {}_{\alpha^{-d}}Z$ and let $x = \sum_{j=0}^{k-1}\sum_{\ell=0}^{s-1} k_{j,\ell}X^j e_\ell$, where $k_{j,\ell} \in K$. Then we have $xe_t = xe_t e_t = \alpha^{-d}(e_t)xe_t = e_{t+d}xe_t$ for each t $(0 \leq t \leq s-1)$. Furthermore, if j $(0 \leq j \leq k-1)$ satisfies $j \not\equiv d \pmod{s}$, then since $e_{t+d-j}e_t = 0$ we get $e_{t+d}X^j e_t = X^j e_{t+d-j}e_t = 0$. Thus we have

$$
\sum_{j=0}^{k-1} k_{j,t}X^j e_t = \sum_{\substack{0 \leq j \leq k-1, \\ j \equiv d\,(\mathrm{mod}\,s)}} k_{j,t}X^j e_t \quad \text{for each } t\ (0 \leq t \leq s-1),
$$

and hence $k_{j,t} = 0$ for every t $(0 \leq t \leq s-1)$ and j $(0 \leq j \leq k-1)$ such that $j \not\equiv d \pmod{s}$. Then we have

$$
x = \begin{cases}
\displaystyle\sum_{j=0}^{m}\sum_{\ell=0}^{s-1} k_{js+\overline{d},\ell}X^{js+\overline{d}}e_\ell & \text{if } \overline{d} < \overline{k}, \\
\displaystyle\sum_{j=0}^{m-1}\sum_{\ell=0}^{s-1} k_{js+\overline{d},\ell}X^{js+\overline{d}}e_\ell & \text{if } \overline{k} \leq \overline{d}.
\end{cases}
$$

Next, note that $xX = \alpha^{-d}(X)x = Xx$ holds. We consider the case $\overline{d} < \overline{k}$. If $\overline{d} \neq \overline{k}-1$, then since $xX = Xx$ we have

$$
\sum_{j=0}^{m}\sum_{\ell=0}^{s-1} k_{js+\overline{d},\ell}X^{js+\overline{d}+1}e_{\ell-1} = \sum_{j=0}^{m}\sum_{\ell=0}^{s-1} k_{js+\overline{d},\ell}X^{js+\overline{d}+1}e_\ell.
$$

So, for every $0 \leq j \leq m$ and $0 \leq \ell \leq s-1$, we obtain $k_{js+\overline{d},\ell+1} = k_{js+\overline{d},\ell}$, where we put $k_{js+\overline{d},s} := k_{js+\overline{d},0}$. Hence $k_{js+\overline{d},0} = k_{js+\overline{d},\ell}$ for $0 \leq j \leq m$ and

$0 \le \ell \le s - 1$. This yields

$$x = \sum_{j=0}^{m} \sum_{\ell=0}^{s-1} k_{js+\overline{d},0} X^{js+\overline{d}} e_\ell = \sum_{j=0}^{m} k_{js+\overline{d},0} X^{js+\overline{d}} \in \bigoplus_{j=0}^{m} K X^{js+\overline{d}}.$$

Therefore $_{\alpha^{-d}}Z \subseteq \bigoplus_{j=0}^{m} K X^{js+\overline{d}}$. Conversely, $X^{js+\overline{d}}$ belongs to $_{\alpha^{-d}}Z$, because $X^{js+\overline{d}} e_u = e_{u+d} X^{js+\overline{d}} = \alpha^{-d}(e_u) X^{js+\overline{d}}$ and $X^{js+\overline{d}}X = X^{js+\overline{d}+1} = XX^{js+\overline{d}} = \alpha^{-d}(X) X^{js+\overline{d}}$ for any $0 \le j \le m$ and $0 \le u \le s - 1$. This shows $\bigoplus_{j=0}^{m} K X^{js+\overline{d}} \subseteq {}_{\alpha^{-d}}Z$. Therefore $_{\alpha^{-d}}Z = \bigoplus_{j=0}^{m} K X^{js+\overline{d}}$. On the other hand, if $\overline{d} = \overline{k} - 1$, then since $xX = Xx$ we have

$$\sum_{j=0}^{m-1} \sum_{\ell=0}^{s-1} k_{js+\overline{k}-1,\ell} X^{js+\overline{k}} e_{\ell-1} = \sum_{j=0}^{m-1} \sum_{\ell=0}^{s-1} k_{js+\overline{k}-1,\ell} X^{js+\overline{k}} e_\ell.$$

So, for every $0 \le j \le m - 1$ and $0 \le \ell \le s - 1$, we obtain $k_{js+\overline{k}-1,\ell+1} = k_{js+\overline{k}-1,\ell}$, where we put $k_{js+\overline{k}-1,s} := k_{js+\overline{k}-1,0}$. Hence $k_{js+\overline{k}-1,0} = k_{js+\overline{k}-1,\ell}$ for $0 \le j \le m - 1$ and $0 \le \ell \le s - 1$. Then it follows that

$$x = \sum_{j=0}^{m-1} \sum_{\ell=0}^{s-1} k_{js+\overline{k}-1,0} X^{js+\overline{k}-1} e_\ell + \sum_{\ell=0}^{s-1} k_{ms+\overline{k}-1,\ell} X^{ms+\overline{k}-1} e_\ell$$

$$= \sum_{j=0}^{m-1} k_{js+\overline{k}-1,0} X^{js+\overline{k}-1} + \sum_{\ell=0}^{s-1} k_{ms+\overline{k}-1,\ell} X^{ms+\overline{k}-1} e_\ell$$

$$\in \Big(\bigoplus_{j=0}^{m-1} K X^{js+\overline{k}-1} \Big) \oplus \Big(\bigoplus_{\ell=0}^{s-1} K X^{ms+\overline{k}-1} e_\ell \Big).$$

Thus $_{\alpha^{-d}}Z = {}_{\alpha^{-\overline{k}+1}}Z \subseteq (\bigoplus_{j=0}^{m-1} K X^{js+\overline{k}-1}) \oplus (\bigoplus_{\ell=0}^{s-1} K X^{ms+\overline{k}-1} e_\ell)$. Conversely, it is easy to check that the equations $X^{js+\overline{k}-1} e_u = \alpha^{-\overline{k}+1}(e_u) X^{js+\overline{k}-1}$ and $X^{js+\overline{k}-1}X = \alpha^{-\overline{k}+1}(X) X^{js+\overline{k}-1}$ hold for every $0 \le j \le m - 1$ and $0 \le u \le s - 1$. Hence $X^{js+\overline{k}-1}$ is in $_{\alpha^{-\overline{k}+1}}Z$ for $0 \le j \le m - 1$. Moreover, it follows that $X^{ms+\overline{k}-1} e_\ell$ is in $_{\alpha^{-\overline{k}+1}}Z$ for $0 \le \ell \le s - 1$. Actually, for $0 \le \ell \le s - 1$ and $0 \le u \le s - 1$, we easily obtain the equations

$$(X^{ms+\overline{k}-1} e_\ell) e_u = \left\{ \begin{array}{ll} 0 & \text{if } u \ne \ell \\ X^{ms+\overline{k}-1} e_\ell & \text{if } u = \ell \end{array} \right\} = \alpha^{-\overline{k}+1}(e_u)(X^{ms+\overline{k}-1} e_\ell)$$

and

$$(X^{ms+\overline{k}-1} e_\ell) X = 0 = \alpha^{-\overline{k}+1}(X)(X^{ms+\overline{k}-1} e_\ell),$$

which mean that $X^{ms+\overline{k}-1} e_\ell$ is in $_{\alpha^{-\overline{k}+1}}Z$ for each $0 \le \ell \le s-1$. Accordingly, it follows that $(\bigoplus_{j=0}^{m-1} K X^{js+\overline{k}-1}) \oplus (\bigoplus_{\ell=0}^{s-1} K X^{ms+\overline{k}-1} e_\ell) \subseteq {}_{\alpha^{-\overline{k}+1}}Z$. Therefore, we get the desired equation in this case.

The desired equations in the case $\overline{k} \leq \overline{d}$ are shown in the similar way above. $\square$

2.3.   Factor through projectives

Next we will give a basis of the K-space $\mathscr{P}(A_{\alpha^{-2i}}, A)$ for $i \geq 0$ in the case $\overline{k} = 0$. Until the end of this paper, we assume $\overline{k} = 0$, i.e., $k = ms$.

Let i be an integer. Then, from [F, Lemma 4.5], we can describe an injective hull of the left A^e-module $A_{\alpha^{i(k-2)}} = A_{\alpha^{-2i}}$ as follows:

$$0 \longrightarrow A_{\alpha^{-2i}} \overset{\iota}{\longrightarrow} \bigoplus_{\ell=0}^{s-1} Ae_{\ell+1} \otimes e_{\ell-2i}A,$$

where ι is given by

$$\iota(e_u) = e_u\Big(\sum_{j=0}^{ms-1} X^j \otimes X^{ms-j-1} \Big)e_{u-2i} \quad \text{for } 0 \leq u \leq s-1.$$

In the following lemma, we regard $\mathscr{P}(A_{\alpha^{-2i}}, A)$ as a subspace of $_{\alpha^{-2i}}Z$ by means of the isomorphism in Lemma 2.2.

Lemma 2.4. *Let i be any non-negative integer.*

(1) *If $-2i \not\equiv 1 \pmod{s}$, then we have $\mathscr{P}(A_{\alpha^{-2i}}, A) = 0$.*

(2) *If $-2i \equiv 1 \pmod{s}$, then we have*
   (a) *if $\operatorname{char} K \mid m$, then $\mathscr{P}(A_{\alpha^{-2i}}, A) = 0$; and*
   (b) *if $\operatorname{char} K \nmid m$, then the set $\{X^{ms-1}\}$ is a basis of $\mathscr{P}(A_{\alpha^{-2i}}, A)$.*

Proof. Let φ be in $\mathscr{P}(A_{\alpha^{-2i}}, A)$. Then, we easily obtain an A^e-homomorphism $h : \bigoplus_{\ell=0}^{s-1} Ae_{\ell+1} \otimes e_{\ell-2i}A \longrightarrow A$ such that $\varphi = h\iota$. Hence, for each u ($0 \leq u \leq s-1$), we have

$$\varphi(e_u) = h\iota(e_u) = \sum_{j=0}^{ms-1} X^j h(e_{u-j} \otimes e_{u-2i-j-1})X^{ms-j-1}.$$

Case $-2i \not\equiv 1 \pmod{s}$: Since $u - j \not\equiv u - 2i - j - 1 \pmod{s}$ for j ($0 \leq j \leq ms-1$), we obtain $e_{u-j} \neq e_{u-2i-j-1}$ for j ($0 \leq j \leq ms-1$). Then it is easy to see that $h(e_{u-j} \otimes e_{u-2i-j-1})$ is in the radical $(X)/(X^{ms})$ of A, and so $\varphi(e_u) = 0$ for each $0 \leq u \leq s-1$. This means $\mathscr{P}(A_{\alpha^{-2i}}, A) = 0$.

Case $-2i \equiv 1 \pmod{s}$: Since $u - j \equiv u - 2i - j - 1 \pmod{s}$ for j ($0 \leq j \leq ms-1$), we have $e_{u-j} = e_{u-2i-j-1}$ for j ($0 \leq j \leq ms-1$). Thus $h(e_{u-j} \otimes e_{u-2i-j-1}) = h(e_{u-j} \otimes e_{u-j})$ holds. We write $h(e_w \otimes e_w) = b_w e_w +$

$\sum_{i=1}^{m-1} b_{w,i} X^{is} e_w$ with $b_w, b_{w,i} \in K$ $(1 \le i \le m-1)$ for each $0 \le w \le s-1$. Then it follows that

$$\varphi(e_u) = \sum_{j=0}^{ms-1} X^j \Big(b_{\overline{u-j}} e_{u-j} + \sum_{i=1}^{m-1} b_{\overline{u-j},i} X^{is} e_{u-j} \Big) X^{ms-j-1}$$

$$= \sum_{j=0}^{ms-1} X^j b_{\overline{u-j}} e_{u-j} X^{ms-j-1}$$

$$= \Big(\sum_{j=0}^{ms-1} b_{\overline{u-j}} \Big) e_u X^{ms-1} = m \Big(\sum_{j=0}^{s-1} b_j \Big) e_u X^{ms-1}.$$

So we get

$$\varphi(1) = \sum_{u=0}^{s-1} \varphi(e_u) = m \Big(\sum_{j=0}^{s-1} b_j \Big) \Big(\sum_{u=0}^{s-1} e_u \Big) X^{ms-1} = m \Big(\sum_{j=0}^{s-1} b_j \Big) X^{ms-1}.$$

Conversely, take any $c \in K$, and let $\varphi : A_{\alpha^{-2i}} \longrightarrow A$ be the A^e-homomorphism given by $\varphi(1) = mcX^{ms-1}$. Then φ factors through ι. In fact, let $\eta : \bigoplus_{\ell=0}^{s-1} Ae_\ell \otimes e_\ell A \longrightarrow A$ be the A^e-homomorphism given by

$$\eta(e_\ell \otimes e_\ell) = \begin{cases} ce_0 & \text{if } \ell = 0, \\ 0 & \text{if } 1 \le \ell \le s-1. \end{cases}$$

Then, for every u $(0 \le u \le s-1)$, we obtain

$$\eta\iota(e_u) = \eta \Bigg(e_u \Big(\sum_{j=0}^{ms-1} X^j \otimes X^{ms-j-1} \Big) e_{u+1} \Bigg)$$

$$= \sum_{j=0}^{ms-1} X^j \eta \left(e_{u-j} \otimes e_{u-j} \right) X^{ms-j-1}$$

$$= \sum_{\ell=0}^{m-1} X^{u+\ell s} ce_0 X^{ms-u-\ell s-1}$$

$$= mce_u X^{ms-1}.$$

So one have $\eta\iota(1) = \sum_{u=0}^{s-1} \eta\iota(e_u) = \sum_{u=0}^{s-1} mce_u X^{ms-1} = mcX^{ms-1} = \varphi(1)$, which shows $\varphi = \eta\iota$. Consequently, we obtain

$$\mathscr{P}(A_{\alpha^{-2i}}, A) = \{ \varphi \in \operatorname{Hom}_{A^e}(A_{\alpha^{-2i}}, A) \mid \varphi(1) = mcX^{ms-1} \text{ for } c \in K \}.$$

Thus, if char $K \mid m$, then $\mathscr{P}(A_{\alpha^{-2i}}, A) = 0$; and if char $K \nmid m$, then by identifying $\mathscr{P}(A_{\alpha^{-2i}}, A)$ with a subspace of $_{\alpha^{-2i}}Z$ via the isomorphism in Lemma 2.2 we obtain a K-basis $\{X^{ms-1}\}$ of $\mathscr{P}(A_{\alpha^{-2i}}, A)$. This completes the proof. $\square$

2.4.   The spaces of stable homomorphisms

Finally, we will find a K-basis of $\underline{\mathrm{Hom}}_{A^e}(\tau_{A^e}^i(A), A)$ $(i \geq 0)$. If, for each $i \geq 0$, we denote by $_{\alpha^{-2i}}Z_{\mathrm{pr}}$ the image of $\mathscr{P}(A_{\alpha^{-2i}}, A)$ under the isomorphism in Lemma 2.2, then we have the isomorphism of K-spaces

$$(2.1) \qquad \underline{\mathrm{Hom}}_{A^e}(A_{\alpha^{-2i}}, A) \xrightarrow{\sim} {}_{\alpha^{-2i}}Z/_{\alpha^{-2i}}Z_{\mathrm{pr}}; \ \underline{f} \longmapsto f(1) + {}_{\alpha^{-2i}}Z_{\mathrm{pr}}.$$

In the following theorem, we regard $\underline{\mathrm{Hom}}_{A^e}(A_{\alpha^{-2i}}, A)$ $\left(\simeq \underline{\mathrm{Hom}}_{A^e}(\tau_{A^e}^i(A), A) \right)$ as $_{\alpha^{-2i}}Z/_{\alpha^{-2i}}Z_{\mathrm{pr}}$ for $i \geq 0$ by using the isomorphism above.

Theorem 2.5. *Let $k = ms$ for $m \geq 1$ and $s \geq 2$. Then, for any non-negative integer i, we have the following:*

(1) *If $-2i \not\equiv 1 \pmod{s}$, then the set*

$$\{X^{\overline{2i}+js} \mid 0 \leq j \leq m - 1\}$$

is a K-basis of $\underline{\mathrm{Hom}}_{A^e}(\tau_{A^e}^i(A), A)$.

(2) *If $-2i \equiv 1 \pmod{s}$, then we have*

(a) *if $\mathrm{char}\, K \mid m$, then the set*

$$\{X^{js+s-1}, X^{ms-1}e_\ell \mid 0 \leq j \leq m - 2, \ 0 \leq \ell \leq s - 1\}$$

is a K-basis of $\underline{\mathrm{Hom}}_{A^e}(\tau_{A^e}^i(A), A)$; and

(b) *if $\mathrm{char}\, K \nmid m$, then the set*

$$\{X^{js+s-1}, Y_\ell \mid 0 \leq j \leq m - 2, \ 0 \leq \ell \leq s - 2\}$$

is a K-basis of $\underline{\mathrm{Hom}}_{A^e}(\tau_{A^e}^i(A), A)$ where $Y_\ell := \sum_{j=0}^{\ell} X^{ms-1}e_j$ for $0 \leq \ell \leq s - 2$.

Proof. Since $\overline{k} = 0$ and $\overline{-i(k-2)} = \overline{2i}$, it follows from Proposition 2.3 that, if $\overline{2i} \neq s - 1$, that is, $-2i \not\equiv 1 \pmod{s}$, then the set

$$(2.2) \qquad\qquad \left\{X^{\overline{2i}+js} \mid 0 \leq j \leq m - 1\right\}$$

is a K-basis of $\mathrm{Hom}_{A^e}(A_{\alpha^{-2i}}, A)$; and if $\overline{2i} = s - 1$, that is, $-2i \equiv 1 \pmod{s}$, then the set

$$(2.3) \qquad \left\{X^{js+s-1}, X^{ms-1}e_\ell \mid 0 \leq j \leq m - 2, \ 0 \leq \ell \leq s - 1\right\}$$

is a K-basis of $\mathrm{Hom}_{A^e}(A_{\alpha^{-2i}}, A)$. So, if $-2i \not\equiv 1 \pmod{s}$, then by Lemma 2.4 (1) we have a K-basis

$$\left\{X^{\overline{2i}+js} \mid 0 \leq j \leq m - 1\right\}$$

of $\underline{\mathrm{Hom}}_{A^e}(A_{\alpha^{-2i}}, A)$; and if $-2i \equiv 1 \pmod{s}$ and $\mathrm{char}\, K \mid m$, then by Lemma 2.4 (2)(a) we obtain a K-basis

$$\left\{ X^{js+s-1}, X^{ms-1}e_\ell \mid 0 \le j \le m-2,\ 0 \le \ell \le s-1 \right\}$$

of $\underline{\mathrm{Hom}}_{A^e}(A_{\alpha^{-2i}}, A)$. On the other hand, if $-2i \equiv 1 \pmod{s}$ and $\mathrm{char}\, K \nmid m$, then by Lemma 2.4 (2)(b) we have a K-basis

$$\left\{ X^{js+s-1}, Y_\ell \mid 0 \le j \le m-2,\ 0 \le \ell \le s-2 \right\}$$

of $\underline{\mathrm{Hom}}_{A^e}(A_{\alpha^{-2i}}, A)$, where we put $Y_\ell := \sum_{j=0}^{\ell} X^{ms-1}e_j \in \underline{\mathrm{Hom}}_{A^e}(A_{\alpha^{-2i}}, A)$ for $0 \le \ell \le s-2$. $\square$

Remark 2.6. We consider the case $k \equiv 1 \pmod{s}$. Then, A is exactly a symmetric algebra (see [T, Lemma 3.1]), and hence A^e is also a symmetric algebra (see [EN, Proposition 2]). So $\tau_{A^e}^i(A) \simeq \Omega_{A^e}^{2i}(A)$ for $i \ge 0$ as A^e-modules, which yields $\underline{\mathrm{Hom}}_{A^e}(\tau_{A^e}^i(A), A) \simeq \underline{\mathrm{Hom}}_{A^e}(\Omega_{A^e}^{2i}(A), A)$ as K-spaces for each $i \ge 0$. Moreover, since A is self-injective, we have $\underline{\mathrm{Hom}}_{A^e}(\Omega_{A^e}^{2i}(A), A) \simeq \mathrm{Ext}_{A^e}^{2i}(A, A)$ for each $i \ge 1$. Therefore $\underline{\mathrm{Hom}}_{A^e}(\tau_{A^e}^i(A), A)$ is isomorphic to the $2i$th Hochschild cohomology group $\mathrm{HH}^{2i}(A) := \mathrm{Ext}_{A^e}^{2i}(A, A)$ for each $i \ge 1$. In [H], Holm computes the dimension of $\mathrm{HH}^{2i}(A)$ $(i \ge 0)$ and describes the even Hochschild cohomology ring $\mathrm{HH}^{ev}(A) = \bigoplus_{i \ge 0} \mathrm{HH}^{2i}(A)$ (see also [EH]).

§3.   The ring structure of $\mathbb{A}(\tau_{A^e}; A)$

Throughout this section, we keep the notation from Section 2, and assume that $\overline{k} = 0$, i.e., $k = ms$ ($m \ge 1$, $s \ge 2$). The purpose in this section is to give the generators and the relations of $\mathbb{A}(\tau_{A^e}; A) = \bigoplus_{i \ge 0} \underline{\mathrm{Hom}}_{A^e}(\tau_{A^e}^i(A), A)$ as K-algebra, explicitly, in the similar way in [EH] and [H].

Since, by Lemma 2.1, the algebra $\mathbb{A}(\tau_{A^e}; A)$ is isomorphic to the orbit algebra $\bigoplus_{i \ge 0} \underline{\mathrm{Hom}}_{A^e}(A_{\alpha^{-2i}}, A)$ induced by the functor $(-)_{\alpha^{-2}}$, it suffices to consider the algebra $\bigoplus_{i \ge 0} \underline{\mathrm{Hom}}_{A^e}(A_{\alpha^{-2i}}, A)$. As in Theorem 2.5, for each $i \ge 0$, we identify $\underline{\mathrm{Hom}}_{A^e}(A_{\alpha^{-2i}}, A)$ with ${}_{\alpha^{-2i}}Z/{}_{\alpha^{-2i}}Z_{\mathrm{pr}}$ via the isomorphism (2.1).

The following lemma says that the multiplication $\cdot$ in $\bigoplus_{i \ge 0} {}_{\alpha^{-2i}}Z/{}_{\alpha^{-2i}}Z_{\mathrm{pr}} = \bigoplus_{i \ge 0} \underline{\mathrm{Hom}}_{A^e}(A_{\alpha^{-2i}}, A)$ is induced by that of A. Here, for simplicity, we set $\mathbb{A}_i := {}_{\alpha^{-2i}}Z/{}_{\alpha^{-2i}}Z_{\mathrm{pr}}$ $(i \ge 0)$ and denote a coset $x + {}_{\alpha^{-2i}}Z_{\mathrm{pr}}$ in $\mathbb{A}_i$ $(i \ge 0)$ by $[x]$.

Lemma 3.1. *Let i and j be any non-negative integers. Then $xy = yx$ in A for $x \in {}_{\alpha^{-2i}}Z$ and $y \in {}_{\alpha^{-2j}}Z$. Furthermore, for $[x] = x + {}_{\alpha^{-2i}}Z_{\mathrm{pr}} \in \mathbb{A}_i$ and $[y] = y + {}_{\alpha^{-2j}}Z_{\mathrm{pr}} \in \mathbb{A}_j$, the multiplication $[x] \cdot [y]$ in $\bigoplus_{i \ge 0} \mathbb{A}_i$ is given by $[x] \cdot [y] = [xy] \in \mathbb{A}_{i+j}$. Consequently, $[x]$ and $[y]$ are commutative.*

Proof. For each $\ell \geq 0$, $_{\alpha^{-2\ell}}Z$ has K-basis (2.2) if $-2\ell \not\equiv 1 \pmod{s}$, and has K-basis (2.3) if $-2\ell \equiv 1 \pmod{s}$. Therefore, we easily see that $x \in {}_{\alpha^{-2i}}Z$ and $y \in {}_{\alpha^{-2j}}Z$ are commutative.

Now, by Lemma 2.2, there exist A^e-homomorphisms $f \in \mathrm{Hom}_{A^e}(A_{\alpha^{-2i}}, A)$ and $g \in \mathrm{Hom}_{A^e}(A_{\alpha^{-2j}}, A)$ satisfying $f(1) = x$ and $g(1) = y$. Moreover the multiplication $\underline{f} \cdot \underline{g}$ in the orbit algebra $\bigoplus_{i \geq 0} \underline{\mathrm{Hom}}_{A^e}(A_{\alpha^{-2i}}, A)$ is given by

$$\underline{f} \cdot \underline{g} = \underline{f} \circ (\underline{g})_{\alpha^{-2i}} = \underline{f} \circ \underline{g_{\alpha^{-2i}}} = \underline{f \circ g_{\alpha^{-2i}}} \in \underline{\mathrm{Hom}}_{A^e}(A_{\alpha^{-2(i+j)}}, A).$$

Then, since

$$[f \circ g_{\alpha^{-2i}}(1)] = [f \circ g(1)] = [f(g(1))] = [f(g(1)1)] = [g(1)f(1)] = [yx] = [xy],$$

it follows that $[x] \cdot [y] = [xy]$. $\qquad\qquad\square$

Now we give the generators and the relations of the algebra $\bigoplus_{i \geq 0} \mathbb{A}_i$ ($\simeq \mathbb{A}(\tau_{A^e}; A)$). Note that $\bigoplus_{i \geq 0} \mathbb{A}_i$ is a commutative graded K-algebra by Lemma 3.1.

First we consider the case when s is even. We put $s = 2t$ for an integer $t \geq 1$. Then, for each $i \geq 0$, since $-2i \not\equiv 1 \pmod{2t}$, by Theorem 2.5 (1) we obtain the K-basis

$$\{X^{\overline{2i+2tj}} \mid 0 \leq j \leq m-1\}$$

of $\mathbb{A}_i$. It is easy to see that, if we set $i = qt + r$ ($0 \leq r \leq t-1$), then this basis can be written as

$$\{X^{2r+2tj} \mid 0 \leq j \leq m-1\}.$$

Here note that $\mathbb{A}_{i+t} = \mathbb{A}_i$ holds for each $i \geq 0$. We set $y_0 := X^{2t} \in \mathbb{A}_0$. Then, by Lemma 3.1, we have

$$y_0^j = X^{2tj} \quad \text{for } 0 \leq j \leq m-1$$

in $\mathbb{A}_0$, and we have the following relation:

(1) $y_0^m = 0$.

Next, we put $y_1 := X^2 \in \mathbb{A}_1$. Then, for $1 \leq i \leq t-1$, we obtain

$$y_0^j \cdot y_1^i = X^{2jt+2i} \quad \text{for } 0 \leq j \leq m-1$$

in $\mathbb{A}_i$. Furthermore, we set $y_t := 1 \in \mathbb{A}_t(= \mathbb{A}_0)$. Then, for any $\ell \geq t$, by letting $\ell = qt + r$ ($0 \leq r \leq t-1$), we have

$$y_0^j \cdot y_1^r \cdot y_t^q = X^{2jt+2r} \quad \text{for } 0 \leq j \leq m-1$$

in $\mathbb{A}_\ell$, and we have the following relation:

(2) $y_0 \cdot y_t = y_1^t$.

Summarizing these results, we have the following theorem.

Theorem 3.2. *Let $k = ms$ and $s = 2t$ $(t \geq 1)$. Then $\mathbb{A}(\tau_{A^e}; A)$ is isomorphic to the commutative graded K-algebra $K[y_0, y_1, y_t]/(y_0^m, y_0 \cdot y_t - y_1^t)$, where $\deg y_i = i$ $(i = 0, 1, t)$.*

Next we consider the case when s is odd. We put $s = 2t + 1$ for an integer $t \geq 1$. For each $i \geq 0$ with $i \not\equiv t \pmod{2t+1}$, since $-2i \not\equiv 1 \pmod{2t+1}$, by Theorem 2.5 (1) we obtain the K-basis

$$\{X^{\overline{2i} + (2t+1)j} \mid 0 \leq j \leq m - 1\}$$

of $\mathbb{A}_i$. It is easy to see that, if we set $i = q(2t+1) + r$ $(0 \leq r \leq 2t, r \neq t)$, then this basis can be written as follows:

$$\{X^{2r + (2t+1)j} \mid 0 \leq j \leq m - 1\} \quad \text{if } 0 \leq r \leq t - 1,$$

and

$$\{X^{2r - (2t+1) + (2t+1)j} \mid 0 \leq j \leq m - 1\} \quad \text{if } t + 1 \leq r \leq 2t.$$

On the other hand, for each $i \geq 0$ with $i \equiv t \pmod{2t+1}$, by Theorem 2.5 (2) we have the following K-basis of $\mathbb{A}_i$:

$$\{X^{2t + (2t+1)j}, X^{(2t+1)m-1}e_\ell \mid 0 \leq j \leq m - 2, \ 0 \leq \ell \leq 2t\} \quad \text{if char } K \mid m,$$

and

$$\{X^{2t + j(2t+1)}, Y_\ell \mid 0 \leq j \leq m - 2, \ 0 \leq \ell \leq 2t - 1\} \quad \text{if char } K \nmid m,$$

where $Y_\ell := \sum_{j=0}^{\ell} X^{(2t+1)m-1}e_j \in \mathbb{A}_i$ for $0 \leq \ell \leq 2t - 1$.

First assume char $K \mid m$. We put

$$z_0 := X^{2t+1} \in \mathbb{A}_0.$$

Then by Lemma 3.1 we have

$$X^{(2t+1)j} = z_0^j \quad \text{for } 1 \leq j \leq m - 1$$

in $\mathbb{A}_0$, and we obtain the following relation:

(1) $z_0^m = 0$.

We set

$$z_1 := X^2 \in \mathbb{A}_1 \quad \text{and} \quad z_{t,\ell} := X^{m(2t+1)-1}e_\ell \in \mathbb{A}_t \quad \text{for } 0 \leq \ell \leq 2t.$$

Then for each $1 \leq i \leq t$ we have

$$X^{2i + (2t+1)j} = z_0^j \cdot z_1^i \quad \text{for } 0 \leq j \leq m - 1$$

in $\mathbb{A}_i$, and we obtain the following relations:

(2) $z_0^{m-1} \cdot z_1^t = \sum_{\ell=0}^{2t} z_{t,\ell}$,

(3) $z_0 \cdot z_{t,\ell} = 0$ for $0 \le \ell \le 2t$,

(4) $z_1 \cdot z_{t,\ell} = 0$ for $0 \le \ell \le 2t$,

(5) $z_{t,u} \cdot z_{t,v} = 0$ for $0 \le u, v \le 2t$.

Next we set $z_{t+1} := X \in \mathbb{A}_{t+1}$. Then for each $t+1 \le i \le 2t$ we have

$$X^{2i-(2t+1)+(2t+1)j} = z_0^j \cdot z_1^{i-(t+1)} \cdot z_{t+1} \quad \text{for } 0 \le j \le m-1$$

in $\mathbb{A}_i$, and we obtain the following relations:

(6) $z_1^{t+1} = z_0 \cdot z_{t+1}$,

(7) $z_{t+1} \cdot z_{t,\ell} = 0$ for $0 \le \ell \le 2t$.

Furthermore, we set $z_{2t+1} := 1 \in \mathbb{A}_{2t+1}(= \mathbb{A}_0)$. Then, for any $\ell \ge 2t+1$, let $\ell = q(2t+1) + r$ $(0 \le r \le 2t)$. If $0 \le r \le t-1$, then

$$X^{2r+(2t+1)j} = z_0^j \cdot z_1^r \cdot z_{2t+1}^q \quad \text{for } 0 \le j \le m-1.$$

If $r = t$, then

$$X^{2t+(2t+1)j} = z_0^j \cdot z_1^t \cdot z_{2t+1}^q, \quad X^{m(2t+1)-1}e_\ell = z_{t,\ell} \cdot z_{2t+1}^q \quad \text{for } 0 \le j \le m-2.$$

If $t+1 \le r \le 2t$, then

$$X^{2r-(2t+1)+(2t+1)j} = z_0^j \cdot z_1^{r-(t+1)} \cdot z_{t+1} \cdot z_{2t+1}^q \quad \text{for } 0 \le j \le m-1.$$

So we obtain the following relations:

(8) $z_0 \cdot z_{2t+1} = z_1^t \cdot z_{t+1}$,

(9) $z_{t+1}^2 = z_1 \cdot z_{2t+1}$.

Next we assume char $K \nmid m$. As in the above we put $z_0 := X^{2t+1} \in \mathbb{A}_0$, $z_1 := X^2 \in \mathbb{A}_1$, $z_{t+1} := X \in \mathbb{A}_{t+1}$, and $z_{2t+1} := 1 \in \mathbb{A}_{2t+1}$. Moreover, we set $z'_{t,\ell} := Y_\ell$ for $0 \le \ell \le 2t-1$. Then these elements are generators of $\bigoplus_{i \ge 0} \mathbb{A}_i$. Thus we obtain the relations (1), (6), (8) and (9) above and the following relations:

(2′) $z_0^{m-1} \cdot z_1^t = 0$,

(3′) $z_0 \cdot z'_{t,\ell} = 0$ for $0 \le \ell \le 2t-1$,

(4′) $z_1 \cdot z'_{t,\ell} = 0$ for $0 \le \ell \le 2t-1$,

(5′) $z_{t,u} \cdot z_{t,v}$ for $0 \le u, v \le 2t - 1$,

(7′) $z_{t+1} \cdot z'_{t,\ell} = 0$ for $0 \le \ell \le 2t - 1$.

Summarizing these results, we have the following theorem.

Theorem 3.3. *Let $k = ms$ and $s = 2t + 1$ $(t \ge 1)$. If $\operatorname{char} K \mid m$, then $\mathbb{A}(\tau_{A^e}; A)$ is a commutative graded algebra with generators $z_0, z_1, z_{t,\ell}$ $(0 \le \ell \le 2t), z_{t+1}, z_{2t+1}$ where $\deg z_i = i$ $(i = 0, 1, t + 1, 2t + 1)$ and $\deg z_{t,\ell} = t$, and relations*

$$z_0^m = 0, \quad z_{t+1}^2 = z_1 \cdot z_{2t+1}, \quad z_1^{t+1} = z_0 \cdot z_{t+1}, \quad z_0 \cdot z_{2t+1} = z_1^t \cdot z_{t+1},$$

$$z_0^{m-1} \cdot z_1^t = \sum_{\ell=0}^{2t} z_{t,\ell}, \quad z_{t,u} \cdot z_{t,v} = 0 \ for \ 0 \le u, v \le 2t,$$

$$z_j \cdot z_{t,\ell} = 0 \ for \ j = 0, 1, t + 1 \ and \ 0 \le \ell \le 2t.$$

And if $\operatorname{char} K \nmid m$, then $\mathbb{A}(\tau_{A^e}; A)$ is a commutative graded algebra with generators $z_0, z_1, z'_{t,\ell}$ $(0 \le \ell \le 2t - 1), z_{t+1}, z_{2t+1}$ where $\deg z_i = i$ $(i = 0, 1, t + 1, 2t + 1)$ and $\deg z'_{t,\ell} = t$, and relations

$$z_0^m = 0, \quad z_{t+1}^2 = z_1 \cdot z_{2t+1}, \quad z_1^{t+1} = z_0 \cdot z_{t+1}, \quad z_0 \cdot z_{2t+1} = z_1^t \cdot z_{t+1},$$

$$z_0^{m-1} \cdot z_1^t = 0, \quad z'_{t,u} \cdot z'_{t,v} = 0 \ for \ 0 \le u, v \le 2t - 1,$$

$$z_j \cdot z'_{t,\ell} = 0 \ for \ j = 0, 1, t + 1 \ and \ 0 \le \ell \le 2t - 1.$$

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