

# On the topology of the complements of quartic and line configurations

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**Abstract.** For a reduced plane curve  $C$  and a line  $L$  in  $\mathbb{P}^2$ , we put  $\mathbb{C}_L^2 := \mathbb{P}^2 - L$ , and  $C_L := C - (C \cap L)$ . If  $C$  and  $L$  intersect transversally and  $\pi_1(\mathbb{P}^2 - C, b_0)$  is abelian, it is known that  $\pi_1(\mathbb{C}_L^2 - C_L)$  is also abelian. In this article, we study  $\pi_1(\mathbb{C}_L^2 - C_L)$  and the Alexander polynomial for the case when a quartic curve  $C$  and a line  $L$  do not intersect transversally.

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## §1. Introduction

Let  $C$  be a reduced plane curve in  $\mathbb{P}^2$ . We choose a line  $L \subset \mathbb{P}^2$  and we put  $\mathbb{C}_L^2 := \mathbb{P}^2 - L$ , and  $C_L := C - (C \cap L)$ . The line  $L$  is said to be *generic* with respect to  $C$  if  $L$  intersects  $C$  transversally.

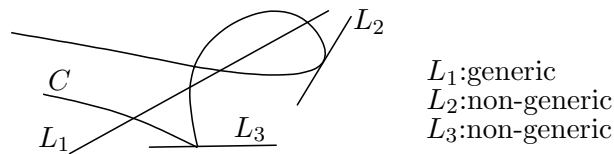


Figure 1.1

An element  $\omega \in \pi_1(\mathbb{P}^2 - L, b_0)$  is called a *lasso* for  $L$  if it is represented by a loop  $l \circ \tau \circ l^{-1}$  where  $\tau$  is a counter-clockwise oriented boundary of a small disc  $D(p)$  of  $L$  at a point  $p \in L$  and  $l$  is a path connecting  $b_0$  and  $\tau$ . In [O1], Oka proved the following:

**Proposition 1.1 ([O1])** *Let  $\omega$  be a lasso for  $L$  and  $N(\omega)$  be the subgroup normally generated by  $\omega$ . Then the following sequence is exact:*

$$1 \rightarrow N(\omega) \rightarrow \pi_1(\mathbb{C}_L^2 - C_L) \rightarrow \pi_1(\mathbb{P}^2 - C) \rightarrow 1.$$

Moreover, if  $L$  is generic with respect to  $C$ , then

- $\omega$  is in the center of  $\pi_1(\mathbb{C}_L^2 - C_L)$  and  $N(\omega) \cong \mathbb{Z}$ , and
- the equality  $D(\pi_1(\mathbb{C}_L^2 - C_L)) = D(\pi_1(\mathbb{P}^2 - C))$  holds, where  $D(\star)$  denotes the commutator group of a group  $\star$ .

Note that we assume a base point  $b_0$  is chosen suitably. In the following, we omit the base points unless we need it explicitly.

By Proposition 1.1, when  $L$  is generic with respect to  $C$ , then  $\pi_1(\mathbb{P}^2 - C)$  is abelian if and only if  $\pi_1(\mathbb{C}_L^2 - C_L)$  is abelian. On the other hand, for a non-generic line  $L$ ,  $\pi_1(\mathbb{C}_L^2 - C_L)$  may be non-abelian. For example, when  $C$  is the quartic defined by  $\{X^3Z + Y^4 = 0\}$  which has an  $e_6$  singularity (= (3, 4)-cusp) at  $[0 : 0 : 1]$ , and  $L_\infty = \{Z = 0\}$  is the tangent line with multiplicity 4 at  $[1:0:0]$ , a presentation of the fundamental group  $\pi_1(\mathbb{C}_L^2 - C_L)$  is given by

$$\pi_1(\mathbb{C}_L^2 - C_L) \cong \langle a, b, c \mid (abca)a = b(abca), c(abca) = (abca)b \rangle$$

and its Alexander polynomial is

$$\Delta_C(t, L) = (t^2 - t + 1)(t^4 - t^2 + 1).$$

Hence  $\pi_1(\mathbb{P}^2 - C) \cong \mathbb{Z}/4\mathbb{Z}$ , while  $\pi_1(\mathbb{C}_L^2 - C_L)$  is non-abelian. More such examples can be found in [O3].

No.	Singularities	$L_\infty$	No.	Singularities	$L_\infty$
(1)	$2a_2$	(i)	(11)	$e_6$	(i)
(2)	$2a_2$	(iv)	(12)	$e_6$	(iv)
(3)	$2a_2 + a_1$	(i)	(13)	$a_3 + a_2 + a_1$	(ii), $a_3$
(4)	$2a_2 + a_1$	(iv)	(14)	$a_5 + a_1$	(i)
(5)	$3a_2$	(i)	(15)	$a_4 + a_2$	(ii), $a_2$
(6)	$a_2 + a_3$	(ii), $a_3$	(16)	$a_5 + a_2$	(iii), $a_2, a_5$
(7)	$a_5$	(i)	(17)	$2a_3$	(iii), $2a_3$
(8)	$a_5$	(iv)	(18)	$a_7$	(v), $a_7$
(9)	$a_6$	(ii), $a_6$	(19)	$2a_3 + a_1$	(iii), $2a_3$
(10)	$a_4 + a_2$	(v), $a_4$			

Table 1

Our purpose of this article is to study such phenomena for the case when  $C$  is a quartic and  $L$  is a non-generic line. For simplicity, we call such a configuration a *QL-configuration*. In [T], Tokunaga gave a list of *QL*-configurations which can be the branch loci of  $D_{2p}$ -covers, where  $D_{2p}$  is the dihedral group of order  $2p$  with  $p$  an odd prime number. In particular, the fundamental groups of the complements of such configurations are non-abelian. In this article, we

give explicit models for  $QL$ -configurations in [T], presentations of  $\pi_1(\mathbb{C}_L^2 - C_L)$  and the Alexander polynomials of them. We remark that all intersection points of  $C$  and  $L$  are not transversal by [T, Corollary 3.5].

We introduce some notations in order to state our results. By a suitable projective transformation, we may regard  $L$  as the line at infinity  $L_\infty$ .

Thus we only have to study the fundamental group  $\pi_1(\mathbb{C}^2 - C^a)$ , where  $C^a$  denotes the affine part of  $C$  which is defined by  $C^a := C - (C \cap L_\infty)$ . Table 1 is the list of the possible  $QL$ -configurations given in [T].

The numbers (i), ..., (v) in the column of  $L_\infty$  explain how  $C$  intersects  $L_\infty$  as follows:

- (i)  $L_\infty$  is bi-tangent to  $C$  at two distinct smooth points.
- (ii)  $L_\infty$  is tangent to a smooth point and passes through a singular point of  $C$ .
- (iii)  $L_\infty$  passes through two distinct singular points.
- (iv)  $L_\infty$  is tangent to  $C$  at a smooth point with intersection multiplicity 4.
- (v)  $L_\infty$  intersects  $C$  at a singular point with intersection multiplicity 4.

For notations for singularities, we use those in [M-P]. We first give an explicit model for each  $QL$ -configuration in Table 1. Note that  $L = L_\infty$  and quartics whose affine parts  $C^a$  are given by  $f(x, y) = 0$  in Table 2. Here we put  $x = X/Z$ ,  $y = Y/Z$ .

No.	$f(x, y)$
(1)	$-6y^3 + x^4 - 8x^2 + 16 - 2y^2x^2 + 8y^2 + y^4$
(2)	$-y^3 + x^4 - 2x^2 + 1$
(3)	$\frac{27}{4}y^4 - 16y^3 + 12y^2 - 3y + \frac{1}{4} - \frac{27}{2}x^2y^2 + \frac{27}{4}x^4$
(4)	$\frac{16}{9}x^3 - 2x^2 - 6x + \frac{15}{2} - 6y^2x - 9y^2 - \frac{1}{2}y^4$
(5)	$18x^2 + 18y^2 + 24xy^2 - 8x^3 + 2x^2y^2 + y^4 + x^4 - 27$
(6)	$1 + 3y - 4y^2x^2 + 3y^2 - 4y^3x + y^3 - y^4$
(7)	$36x^4 - 33x^3 + 10x^2 - x - 12y^2x^2 + 10y^2x - 2y^2 + y^4$
(8)	$-x^3 + x^2 - 2xy^2 + y^4$
(9)	$1 - 2xy - 4y + y^2x^2 + 4xy^2 + 6y^2 - 2xy^3 - 3y^3 + y^4$
(10)	$x^3 - (x^2 - y + 1)^2$
(11)	$(x - 1)^3 + y^4$
(12)	$x^3 + (y - x)^2(y + x)^2$

(13)	$y(81x + 54 + 81yx^2 - 54yx - 54y + 90y^2x + 18y^2 + 25y^3)$
(14)	$(y^2 - 2yx - x + x^2)(y^2 + 2yx - x + x^2)$
(15)	$256y^4 - 256y^3 + 96y^2 - 16y + 1 + 32xy^3 - 32xy^2 + 10xy - x + x^2y^2$
(16)	$(y - x)(-y^2x + 1 + y^3)$
(17)	$(y^2 + yx + 2)(y^2 + yx + 1)$
(18)	$(-x - 2 + y^2)(-x - 1 + y^2)$
(19)	$(-2y + 1 + 2x)(2y + 1 + 2x)(-y^2 + x^2 + x)$

Table 2: Defining equations of  $Q$ 

Now we are ready to state our result.

**Theorem 1.2** *For the  $QL$ -configurations given by the quartic polynomials as above and  $L_\infty$ , we have presentations of the fundamental groups  $\pi_1(\mathbb{C}_L^2 - C_L)$  and Alexander polynomials  $\Delta_C(t, L)$  as follows:*

No.	presentation of $\pi_1(\mathbb{C}_L^2 - C_L)$	$\Delta_C(t, L)$
(1)	$\langle a, b \mid aba = bab \rangle$	$t^2 - t + 1$
(2)	$\langle a, b \mid aba = bab \rangle$	$t^2 - t + 1$
(3)	$\langle a, b \mid aba = bab \rangle$	$t^2 - t + 1$
(4)	$\langle a, b, c \mid aca = cac, cbc = bcb, ab = ba \rangle$	$t^2 - t + 1$
(5)	$\langle a, b, c \mid aba = bab, bcb = cbc, \\ c(b^{-1}ab)c = (b^{-1}ab)c(b^{-1}ab) \rangle$	$(t^2 - t + 1)^2$
(6)	$\langle a, b \mid aba = bab \rangle$	$t^2 - t + 1$
(7)	$\langle a, b \mid aba = bab \rangle$	$t^2 - t + 1$
(8)	$\langle a, b \mid aba = bab \rangle$	$t^2 - t + 1$
(9)	$\langle a, b \mid aba = bab \rangle$	$t^2 - t + 1$
(10)	$\langle a, b \mid aba = bab \rangle$	$t^2 - t + 1$
(11)	$\langle a, b \mid aba = bab \rangle$	$t^2 - t + 1$
(12)	$\langle a, b, c \mid b(acba) = (acba)a, c(acba) = (acba)b \rangle$	$(t^2 - t + 1)(t^4 - t^2 + 1)$
(13)	$\langle a, b, c \mid bcb = cbc, ac = ba, accb = cbac \rangle$	$(t - 1)(t^2 - t + 1)$
(14)	$\langle a, b \mid (ab)^3 = (ba)^3, babba = abbab \rangle$	$(t - 1)(t^2 - t + 1)$
(15)	$\langle a, b \mid ababa = babab, abb = bba \rangle$	1
(16)	$\langle a, b \mid bba = abb \rangle$	$t^2 - 1$
(17)	$\langle a, b, c \mid ab = bc \rangle$	1
(18)	$\langle a, b, c \mid ab = bc \rangle$	1
(19)	$\langle a, b, c \mid bc = cb, bac = cab \rangle$	$(t - 1)^2(t + 1)$

Table 3

*Remark.* Some Alexander polynomials in the above list are computed in [O3] by two ways. They are mainly obtained by the *line degeneration method* and

some are done by the *Fox calculus*. And the group represented by  $\langle a, b \mid aba = bab \rangle$  is isomorphic to the braid group  $\mathbb{B}_3$  of 3 strings.

## §2. Preliminaries

### 2.1. Zariski-van Kampen's method

In this subsection, we briefly summarize Zariski-van Kampen's method in computing the fundamental group. For details, see [O3], [S] and [T-S]. Let  $C^a$  be a reduced affine curve defined by a polynomial  $f(x, y)$  of degree  $d$ . By a suitable linear transformation of coordinate system, we may assume that the coefficient of  $y^d$  is a non-zero constant.

Let  $p : \mathbb{C}^2 - C^a \rightarrow \mathbb{C}$  be a map given by  $(x, y) \mapsto x$ . For  $s \in \mathbb{C}$ ,  $y$ -coordinates of the intersection points of the affine line  $\{x = s\}$  and  $C$  are corresponding to roots of the equation  $f(s, y) = 0$ . We denote by  $D_{(f,y)}(s)$  the discriminant with respect to  $y$  and put

$$\Sigma := \{s \in \mathbb{C} \mid D_{(f,y)}(s) = 0\}.$$

We call lines  $\{x = s \mid s \in \Sigma\}$  *singular lines*. Since  $C^a$  is reduced,  $\Sigma$  is a finite set. For all  $t \in \mathbb{C} - \Sigma$ ,  $p^{-1}(t)$  is isomorphic to the  $d$ -punctured affine line and restriction

$$p|_{p^{-1}(\mathbb{C}-\Sigma)} : p^{-1}(\mathbb{C} - \Sigma) \rightarrow \mathbb{C} - \Sigma$$

is the local trivial fibration.

We choose a sufficiently large positive real number  $S$  such that a disc  $B_S := \{s \in \mathbb{C} \mid |s| < S\}$  contains  $\Sigma$ . Since the inclusion  $p^{-1}(B_S) \hookrightarrow \mathbb{C}^2 - C$  gives a homotopy equivalence, we only have to compute  $\pi_1(p^{-1}(B_S))$ . Since the coefficients of  $y^{d-\nu}$ , ( $\nu > 0$ ) in  $f(x, y)$  are polynomials in  $x$  and they are bounded on  $B_S$ , we can take a base point  $b'_0 = (b_0, \tilde{b}_0) \in \mathbb{C}^2 - C^a$  where  $p(b'_0) = b_0$ ,  $b_0 \in B_S$  and  $(B_S \times \{\tilde{b}_0\}) \cap C = \emptyset$ .

In this setting, the restriction of  $p$

$$p|_{p^{-1}(B_S)} : p^{-1}(B_S) \rightarrow B_S$$

has the holomorphic section  $s \mapsto (s, \tilde{b}_0)$  which passes through  $b'_0$ . Using this section, one can define an action of  $\pi_1(B_S - \Sigma, b_0)$  on  $\pi_1(p^{-1}(b_0), b'_0)$ . We call this action *the monodromy action*. Let  $r$  be a number of points of  $\Sigma$ . Then  $\pi_1(B_S - \Sigma, b_0)$  is the free group generated by loops  $\gamma_1, \gamma_2, \dots, \gamma_r$ , and  $\pi_1(p^{-1}(b_0), b'_0)$  is generated by loops  $g_1, g_2, \dots, g_d$ . We denote the action of  $\gamma_j$  on  $g_i$  by  $g_i^{\gamma_j}$ . Now, Zariski-van Kampen's theorem can be stated as follows.

**Theorem 2.1** ([T-S], [S]) *The inclusion map  $p^{-1}(b_0) \hookrightarrow \mathbb{C}^2 - C^a$  induces an isomorphism*

$$\pi_1(p^{-1}(b_0), b'_0)/N \rightarrow \pi_1(\mathbb{C}^2 - C^a, b'_0),$$

where  $N$  is the minimal normal subgroup of  $\pi_1(p^{-1}(b_0), b'_0)$  which contains

$$\{g_i^{-1}g_i^{\gamma_j} \mid i = 1, 2, \dots, d, j = 1, 2, \dots, r\}.$$

Thus the presentation of  $\pi_1(\mathbb{C}^2 - C^a, b'_0)$  is

$$\langle g_1, g_2, \dots, g_d \mid g_i = g_i^{\gamma_j} \rangle_{i=1,2,\dots,d, j=1,2,\dots,r}.$$

In particular, we call relations  $g_i = g_i^{\gamma_j}$  *monodromy relations*.

## 2.2. Some basic monodromy actions

In this subsection, we recall some basic monodromy actions. We consider curves whose local equations at  $(0, 0)$  given by

$$(i) h_1 = x - y^2, \quad (ii) h_2 = x^2 - y^2, \quad (iii) h_3 = x^3 - y^2.$$

Since the line  $x = 0$  is a singular line for all cases, we take a base point  $x = \epsilon$  and its general fiber  $F_\epsilon = p^{-1}(\epsilon)$ , where  $\epsilon$  is a sufficiently small positive number. Since each  $h_i$  have degree 2 with respect to  $y$ ,  $F_\epsilon$  is isomorphic to the 2-punctured plane. We take meridians  $g_1$  and  $g_2$  as generators of  $\pi_1(F_\epsilon)$  as follows:

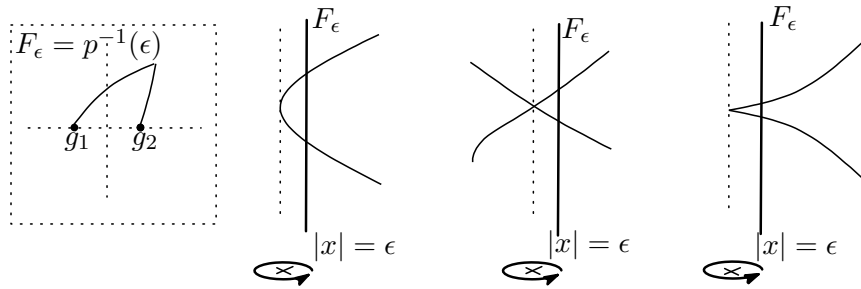


Figure 2.1

When  $x$  goes with counter clockwise direction along the circle  $|x| = \epsilon$  which is the generator of  $\pi_1(\mathbb{C} - \{0\})$ ,  $g_1$  and  $g_2$  are moved as following figures by the monodromy action.

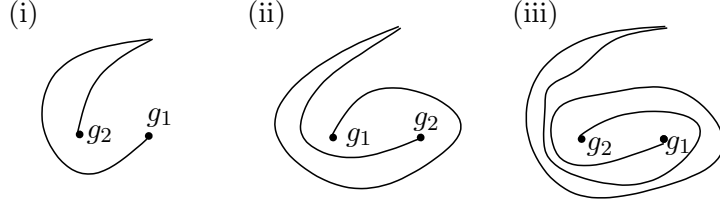


Figure 2.2

By Theorem 2.1, we have monodromy relations of each cases:

$$(i) \ g_1 = g_2 \quad (ii) \ g_1 g_2 = g_2 g_1 \quad (iii) \ g_1 g_2 g_1 = g_2 g_1 g_2.$$

We call these relations *the tangential relation*, *the nodal relation* and *the cuspidal relation* respectively.

### 2.3. Alexander polynomial

In this subsection, we briefly summarize Fox calculus in computing the Alexander polynomial. For details, see [C-F, 119p]. Suppose that  $G := \pi_1(\mathbb{C}^2 - C^a, b'_0)$  is given by the following finite representation:

$$G = \langle g_1, \dots, g_n \mid R_1, \dots, R_m \rangle,$$

where  $g_i$  are generators of  $\pi_1(p^{-1}(b_0))$  and  $R_i$  denotes the monodromy relations. Let  $F(n)$  be a free group of rank  $= n$  which is generated by  $g_1, \dots, g_n$ . Moreover we consider the group ring  $\mathbb{C}[F(n)]$  of  $F(n)$  with  $\mathbb{C}$ -coefficient. The Fox differentials  $\frac{\partial}{\partial g_i} : \mathbb{C}[F(n)] \rightarrow \mathbb{C}[F(n)]$  are a  $\mathbb{C}$ -linear operator which satisfies the following two properties:

$$(i) \ \frac{\partial}{\partial g_j}(g_i) = \delta_{ij}, \quad (ii) \ \frac{\partial}{\partial g_j}(uv) = \frac{\partial u}{\partial g_j} + u \frac{\partial v}{\partial g_j}, \quad u, v \in \mathbb{C}[F(n)].$$

Let  $\gamma : \mathbb{C}[F(n)] \rightarrow \mathbb{C}[t, t^{-1}]$  be a ring homomorphism defined by  $g_i, g_i^{-1} \mapsto t, t^{-1}$  for all  $i$ . Now we get  $(m \times n)$ -matrix whose elements are in  $\mathbb{C}[t, t^{-1}]$ . We put

$$A := \left( \gamma \left( \frac{\partial R_i}{\partial g_j} \right) \right)$$

and call  $A$  the *Alexander matrix*. Then *Alexander polynomial*  $\Delta_C(t)$  is given by the greatest common divisor of the determinants of all  $(n-1) \times (n-1)$  submatrices of the Alexander matrix  $A$  if  $m \geq n-1$ , and it is understood that

$$\begin{aligned} \Delta_C(t) &= 0 & \text{if } n-1 > m, \\ \Delta_C(t) &= 1 & \text{if } n-1 \leq 0. \end{aligned}$$

**Example 2.2** For the group presented by  $\langle a, b \mid aba = bab \rangle$ , we put the monodromy relation as  $R = abab^{-1}a^{-1}b^{-1}$ . Then we have

$$\frac{\partial R}{\partial a} = 1 + ab - abab^{-1}a^{-1}, \quad \frac{\partial R}{\partial b} = a - abab^{-1} - abab^{-1}a^{-1}b^{-1}.$$

Moreover the Alexander matrix  $A$  and Alexander polynomial  $\Delta(t)$  are given as follows.

$$A = \begin{bmatrix} 1 + t^2 - t & t - t^2 - 1 \end{bmatrix}, \quad \Delta(t) = t^2 - t + 1.$$

### §3. Proof of Theorem 1.2

In this section, we give a proof of Theorem 1.2. Since our proof is done by case by case computation and most of them are similar, we give complete proofs for the cases (5) and (9), and rough sketches for the remaining cases.

**Case (5):** In this case,  $Q$  is a quartic with  $3a_2$  and  $L$  is its unique bi-tangent line. As we have seen in the introduction, such an example is given by an equation

$$F(X, Y, Z) := 18X^2Z^2 + 18Y^2Z^2 + 24XY^2Z - 8X^3Z + 2X^2Y^2 + Y^4 + X^4 - 27Z^4.$$

When we put  $C = \{F(X, Y, Z) = 0\}$  and its affine part  $C^a = \{f(x, y) = F(x, y, 1) = 0\}$ , then

- (i)  $C$  has three cusps at  $[3 : 0 : 1]$  and  $[-3/2 : \pm 3\sqrt{3}/2 : 1]$ ,
- (ii)  $L_\infty = \{Z = 0\}$  is bi-tangent to  $C$ , and
- (iii) the discriminant of  $f$  with respect to  $y$  is

$$4096(x + 1)(x - 3)^3(2x + 3)^6.$$

Figure 3.1 shows the graph of the real part of  $C^a$ . Put  $\Sigma = \{x \in \mathbb{C} \mid x = -3/2, -1, 3\}$  and take the base point  $b_0 := -1 - \epsilon$ , where  $\epsilon$  is a small positive real number. We denote by  $F_0$  the fiber  $p^{-1}(b_0)$  which is isomorphic to 4-punctured plane.

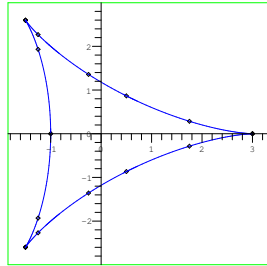


Figure 3.1

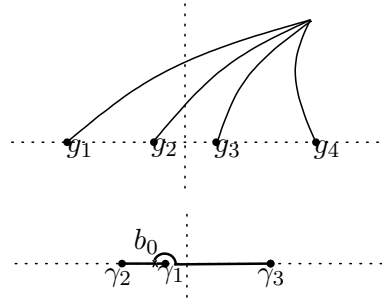


Figure 3.2

To compute the monodromy action by the fundamental group of base space  $\pi_1(\mathbb{C} - \Sigma, b_0)$ , we fix  $g_1, g_2, g_3, g_4$  as generators of  $\pi_1(F_0, b'_0)$ , and meridians  $\gamma_1, \gamma_2, \gamma_3$  as generators of  $\pi_1(\mathbb{C} - \Sigma, b_0)$ , where  $\gamma_1, \gamma_2$  and  $\gamma_3$  are given by the following:

- $\gamma_1$ : the meridian of  $x = -1$  from  $b_0$
- $\gamma_2$ : the meridian of  $x = -3/2$  from  $b_0$
- $\gamma_3$ : the meridian of  $x = 3$  from  $b_0$  stepping aside  $x = -1$ .

Figure 3.2 shows our choice of  $g_1, g_2, g_3, g_4$  and  $\gamma_1, \gamma_2, \gamma_3$ . Note that  $\bullet$  denotes a circle. In this setting, we see each monodromy actions explicitly.

Monodromy action of  $\gamma_1$ : Since  $g_2$  and  $g_3$  have the tangential relation, we obtain  $g_2 = g_3$ .

Monodromy action of  $\gamma_2$ : Since  $g_1, g_2$  and  $g_3, g_4$  have the cuspidal relations each other, we obtain  $g_1 g_2 g_1 = g_2 g_1 g_2$  and  $g_3 g_4 g_3 = g_4 g_3 g_4$ .

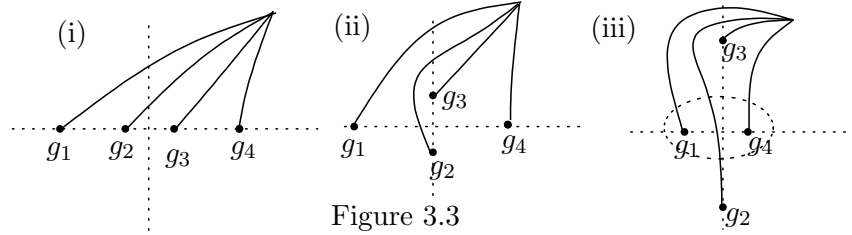


Figure 3.3

Monodromy action of  $\gamma_3$ : Figure 3.3 corresponds each steps of the monodromy action with respect to  $\gamma_2$  as follows:

- start position,
- $z$  comes close to  $-1$  and steps aside  $-1$ ,
- $z$  comes close to  $3$ .

By Figure 3.3, we can observe  $g_1$  and  $g_4$  have the cuspidal relation.

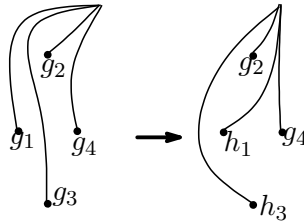


Figure 3.4

For simplicity, we replace two generators  $g_1$  and  $g_3$  by  $h_1$  and  $h_3$  as in Figure 3.4. Namely we put

$$h_1 = g_2^{-1}g_1g_2, \quad h_3 = g_1g_3g_1^{-1}.$$

$h_1$  and  $g_4$  have the cuspidal relation, we just only obtain  $g_4h_1g_4 = h_1g_4h_1$ . By Theorem 2.1,

$$\pi_1(\mathbb{C}_L^2 - C_L) \cong \langle g_1, g_2, g_4, h_1 \mid g_1g_2g_1 = g_2g_1g_2, g_2g_4g_2 = g_4g_2g_4, g_4h_1g_4 = h_1g_4h_1, h_1 = g_2^{-1}g_1g_2 \rangle.$$

Now the Alexander matrix  $A$  and its Alexander polynomial are

$$A = \begin{bmatrix} t^2 - t + 1 & -t^2 + t - 1 & 0 \\ 0 & t^2 - t + 1 & -t^2 + t - 1 \\ t - t^2 + 1 & (1 - t)(t^2 - t + 1) & t^2 - t + 1 \end{bmatrix}, \quad \Delta_C(t) = (t^2 - t + 1)^2.$$

**Case (9):** In this case,  $Q$  is a quartic with an  $a_6$  singularity and  $L$  is its unique tangent line which intersects  $C$  at the  $a_6$  singular point. As we have seen in the introduction, such an example is given by an equation

$$F := X^4 - 2Z^2YX - 4Z^3Y + Y^2X^2 + 4ZY^2X + 6Y^2Z^2 - 2XY^3 - 3ZY^3 + Y^4.$$

When we put  $C = \{F(X, Y, Z) = 0\}$  and its affine part  $C^a = \{f(x, y) = F(x, y, 1) = 0\}$ , then

- (i)  $C$  has an  $a_6$  singularity at  $[1 : 0 : 0]$ ,
- (ii)  $L_\infty = \{Z = 0\}$  is a simple tangent line which intersects  $C$  at the  $a_6$  singular point, and
- (iii) The discriminant of  $f$  with respect to  $y$  is

$$D_{(f,y)}(x) = 16x^4 + 124x^3 + 232x^2 + 96x + 229.$$

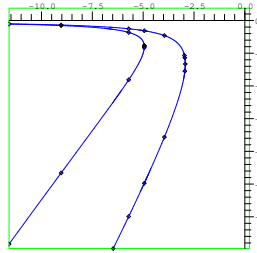


Figure 3.5

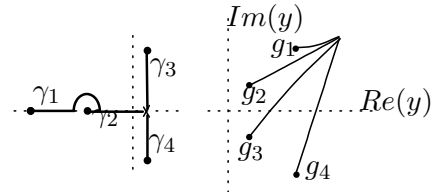


Figure 3.6

Figure 3.5 shows the graph of the real part of  $C^a$ . We put  $\Sigma := \{\alpha_1, \alpha_2, \beta, \bar{\beta}\}$ , where  $\alpha_i$  are real roots of  $D_{(f,y)}(x) = 0$  and  $\beta, \bar{\beta}$  are complex roots. We assume that  $\alpha_1 < \alpha_2$ . Take the base point  $b_0$  of  $\mathbb{C} - \Sigma$  at  $b_0 = \text{Re}(\beta) = \text{Re}(\bar{\beta}) \in \mathbb{C}$

and meridians  $\gamma_1, \gamma_2, \gamma_3, \gamma_4$  as generators of  $\pi_1(\mathbb{C} - \Sigma, b_0)$ . For  $F_0 = p^{-1}(b_0)$ , we also fix meridians  $g_1, g_2, g_3, g_4$  as generators of  $\pi_1(F_0, b'_0)$ . Figure 3.6 shows our setting of generators.

Monodromy action of  $\gamma_3$ : By Figure 3.7, we have  $g_2 = g_4$  immediately. Similarly, we have monodromy relation  $g_1 = g_3$  with  $\gamma_4$ .

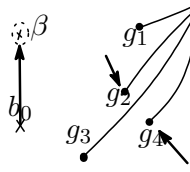


Figure 3.7

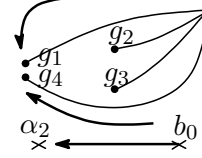


Figure 3.8

Monodromy action of  $\gamma_2$ : By Figure 3.8,  $g_1$  and  $g_4$  have the tangential relation. Considering the homotopy equivalence of roops, we have the monodromy relation,

$$g_1 g_2 g_3 = g_2 g_3 g_4.$$

By using previous relations  $g_2 = g_4$  and  $g_1 = g_3$ , this relation implies  $g_1 g_2 g_1 = g_2 g_1 g_2$ .

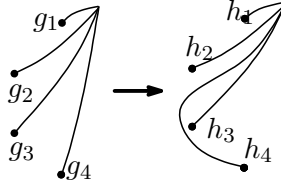


Figure 3.9

Monodromy action of  $\gamma_1$ : For simplicity, we replace generators as follows:

$$h_1 = g_1, \quad h_2 = g_2, \quad h_3 = g_3, \quad h_4 = g_3 g_4 g_3^{-1}.$$

Figure 3.9 explains how to replace the base position of generators, and Figure 3.10 corresponds each steps of the monodromy action with respect to  $\gamma_1$ :

- (i)  $z$  comes close to  $\alpha_2$ , (ii)  $z$  steps aside  $\alpha_2$ , (iii)  $z$  rotates around  $\alpha_1$ .

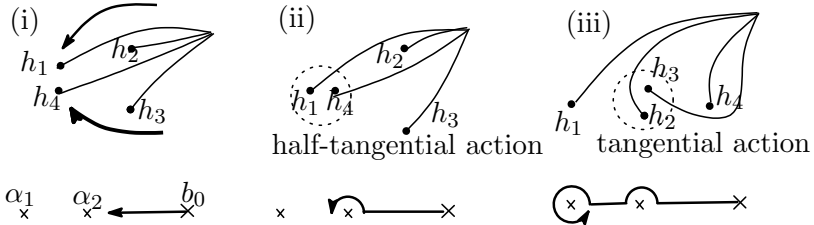


Figure 3.10

By Figure 3.10, we have just one monodromy relation with  $\gamma_1$ :

$$h_2 = h_4 h_3 h_4^{-1} \Leftrightarrow h_2 h_4 = h_4 h_3.$$

Rewriting this relation into  $g_1, g_2, g_3, g_4$ , and  $g_1 = g_3$ ,  $g_2 = g_4$ , we obtain

$$g_2 g_1 g_2 = g_1 g_2 g_1.$$

By Theorem 2.1, we have

$$\pi_1(\mathbb{C}_L^2 - C_L) \cong \langle g_1, g_2 \mid g_2 g_1 g_2 = g_1 g_2 g_1 \rangle.$$

Moreover the Alexander polynomial is equal to  $t^2 - t + 1$  by Example 2.2.

For simplicity, we sometimes denote the monodromy action of each  $\gamma_i$  and its monodromy relations by  $\gamma_i$ -action and  $\gamma_i$ -relation in the following sketch of proofs.

**Case (1):** We consider the following homogenized polynomial:

$$F(X, Y, Z) := 16Z^4 - 8X^2Z^2 + X^4 + 8Y^2Z^2 - 2Y^2X^2 - 6Y^3Z + Y^4.$$

When we put  $C = \{F(X, Y, Z) = 0\}$  and its affine part  $C^a = \{f(x, y) = F(x, y, 1) = 0\}$ , then

- (i)  $C$  has two cusps at  $[\pm 2 : 0 : 1]$ ,
- (ii)  $L_\infty = \{Z = 0\}$  is a bi-tangent line, and
- (iii) the discriminant of  $f$  with respect to  $y$  is

$$D_{(f,y)}(x) = -144(x-2)^4(x+2)^4(64x^2-13).$$

Figure 3.11 shows the real part of the affine curve  $C^a$  and the setting of generators.

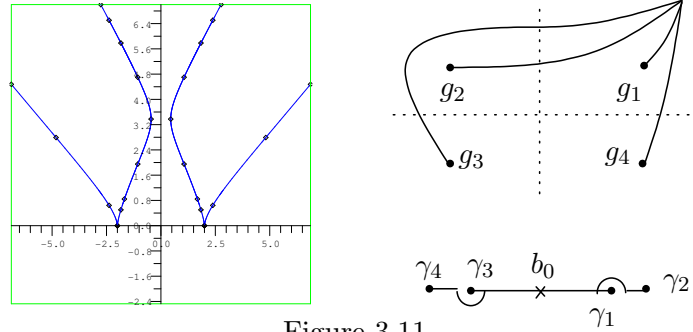


Figure 3.11

Put  $\Sigma := \{x \in \mathbb{C} \mid D_{(f,y)}(x) = 0\}$  and take a base point  $b_0 = 0 \in \mathbb{C} - \Sigma$ . Furthermore, we take generators of  $\pi_1(\mathbb{C} - \Sigma, b_0)$  and  $\pi_1(p^{-1}(b_0), b'_0)$  as Figure 3.12. Monodromy relations with each  $\gamma_i$  are

$$(\gamma_1) \quad g_1 = g_4, \quad (\gamma_2) \quad g_1 = g_3, \quad g_1 g_2 g_1 = g_2 g_1 g_2.$$

Since  $f(x, y) = f(-x, y)$ , monodromy relations with  $\gamma_3$  and  $\gamma_4$  are same to  $\gamma_1$ -relation and  $\gamma_2$ -relation. By Theorem 2.1 and Example 2.2, we have

$$\pi_1(\mathbb{C}_L^2 - C_L) \cong \langle g_1, g_2 \mid g_1 g_2 g_1 = g_2 g_1 g_2 \rangle, \quad \Delta_C(t, L) = t^2 - t + 1.$$

**Case (2):** We consider the following homogenized polynomial:

$$F(X, Y, Z) := -Y^3 Z + X^4 - 2X^2 Z^2 + Z^4.$$

When we put  $C = \{F(X, Y, Z) = 0\}$  and its affine part  $C^a = \{f(x, y) = F(x, y, 1) = 0\}$ , then

- (i)  $C$  has two cusps at  $[\pm 1 : 0 : 1]$ ,
- (ii)  $L_\infty = \{Z = 0\}$  is a tangent line with multiplicity 4, and
- (iii) the discriminant of  $f$  with respect to  $y$  is

$$D_{(f,y)}(x) = -27(x-1)^4(x+1)^4.$$

Figure 3.12 show the real part of the affine curve  $C^a$  and the setting of generators.

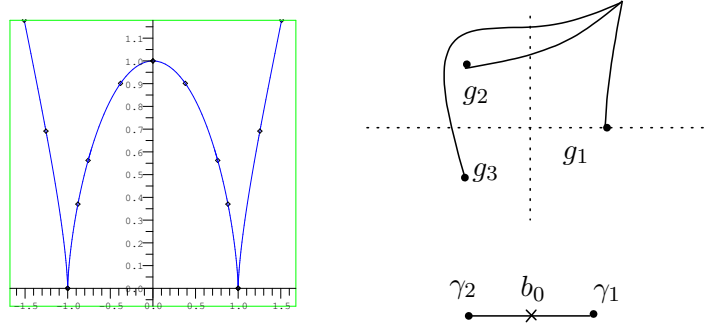


Figure 3.12

Put  $\Sigma := \{x \in \mathbb{C} \mid D_{(f,y)}(x) = 0\}$  and take a base point  $b_0 = 0 \in \mathbb{C} - \Sigma$ . Moreover, we take generators of  $\pi_1(\mathbb{C} - \Sigma, b_0)$  and  $\pi_1(p^{-1}(b_0), b'_0)$  as Figure 3.13. Monodromy relations with  $\gamma_1$  are

$$g_1 = g_3, \quad g_1 g_2 g_1 = g_2 g_1 g_2.$$

Since  $f(x, y) = f(-x, y)$ , the monodromy relation with  $\gamma_2$  is same to the  $\gamma_1$ -relation. By Theorem 2.1 and Example 2.2, we have

$$\pi_1(\mathbb{C}_L^2 - C_L) \cong \langle g_1, g_2 \mid g_1 g_2 g_1 = g_2 g_1 g_2 \rangle, \quad \Delta_C(t, L) = t^2 - t + 1.$$

**Case (3):** We consider the following homogenized polynomial:

$$F(X, Y, Z) := \frac{27}{4} Y^4 - 16 Y^3 Z + 12 Y^2 Z^2 - 3 Y Z^3 + \frac{1}{4} Z^4 - \frac{27}{2} X^2 Y^2 + \frac{27}{4} X^4.$$

When we put  $C = \{F(X, Y, Z) = 0\}$  and its affine part  $C^a = \{f(x, y) = F(x, y, 1) = 0\}$ , then

- (i)  $C$  has two cusps and  $a_1$  singular point at  $[\pm 1/4 : 1/4 : 1]$ ,  $[0 : 1 : 1]$  respectively,
- (ii)  $L_\infty = \{Z = 0\}$  is a bi-tangent line, and
- (iii) the discriminant of  $f$  with respect to  $y$  is

$$D_{(f,y)}(x) = -\frac{19683}{16} x^2 (27x^2 - 2) (4x - 1)^3 (4x + 1)^3.$$

Figure 3.13 shows the real part of the affine curve  $C^a$  and the setting of generators.

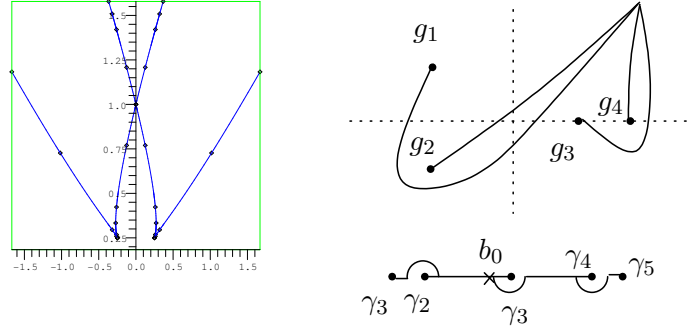


Figure 3.13

Put  $\Sigma := \{x \in \mathbb{C} \mid D_{(f,y)}(x) = 0\}$  and take a base point  $b_0 = -\epsilon \in \mathbb{C} - \Sigma$ , where  $\epsilon$  is a sufficiently small positive real number. Moreover, we take generators of  $\pi_1(\mathbb{C} - \Sigma, b_0)$  and  $\pi_1(p^{-1}(b_0), b'_0)$  as Figure 3.14. Monodromy relations with each  $\gamma_i$  are

$$\begin{aligned} (\gamma_1) \quad g_3 g_4 &= g_4 g_3, & (\gamma_2) \quad g_1 g_2 g_1 &= g_2 g_1 g_2, \\ (\gamma_3) \quad g_1 &= g_4 g_3 g_4^{-1}, & (\gamma_5) \quad g_1 &= g_4. \end{aligned}$$

Since  $f(x, y) = f(-x, y)$ ,  $\gamma_4$ -action is same to  $\gamma_2$ -action. By Theorem 2.1 and Example 2.2, we have

$$\pi_1(\mathbb{C}_L^2 - C_L) \cong \langle g_1, g_2 \mid g_1 g_2 g_1 = g_2 g_1 g_2 \rangle, \quad \Delta_C(t, L) = t^2 - t + 1.$$

**Case (4):** We consider the following homogenized polynomial:

$$F(X, Y, Z) := \frac{16}{9}X^3Z - 2X^2Z^2 - 6XZ^3 + \frac{15}{2}Z^4 - 6XY^2Z - 9Y^2Z^2 - \frac{1}{2}Y^4.$$

When we put  $C = \{F(X, Y, Z) = 0\}$  and its affine part  $C^a = \{f(x, y) = F(x, y, 1) = 0\}$ , then

- (i)  $C$  has two cusps and an  $a_1$  singular point at  $[-3 : \pm 3 : 1]$ ,  $[3/2 : 0 : 1]$ ,
- (ii)  $L_\infty = \{Z = 0\}$  is a tangent line with multiplicity 4, and
- (iii) the discriminant of  $f$  with respect to  $y$  is

$$D_{(f,y)}(x) = -\frac{4096}{729} (8x + 15) (2x - 3)^2 (x + 3)^6.$$

Figure 3.14 shows the real part of the affine curve  $C^a$  and the setting of generators.

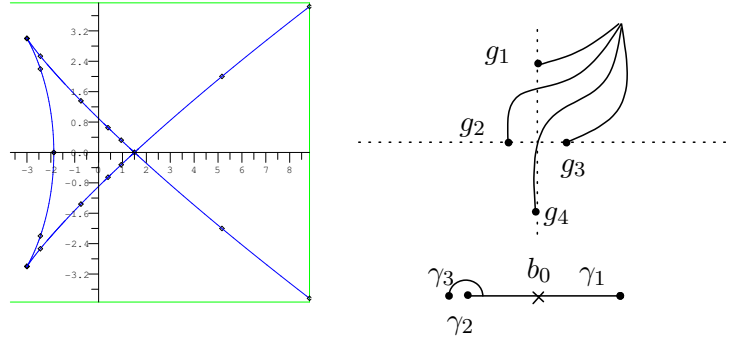


Figure 3.14

Put  $\Sigma := \{x \in \mathbb{C} \mid D_{(f,y)}(x) = 0\}$  and take a base point  $b_0 = 3/2 - \epsilon \in \mathbb{C} - \Sigma$ , where  $\epsilon$  is a sufficiently small positive real number. We also take generators of  $\pi_1(\mathbb{C} - \Sigma, b_0)$  and  $\pi_1(p^{-1}(b_0), b'_0)$  as Figure 3.15. Monodromy relations with  $\gamma_i$  are

$$(\gamma_1) g_2 g_3 = g_3 g_2, (\gamma_2) g_1 = g_4, (\gamma_3) g_1 g_2 g_1 = g_2 g_1 g_2, g_3 g_4 g_3 = g_4 g_3 g_4.$$

By Theorem 2.1, we have

$$\pi_1(\mathbb{C}_L^2 - C_L) \cong \langle g_1, g_2, g_3 \mid g_1 g_2 g_1 = g_2 g_1 g_2, g_1 g_3 g_1 = g_3 g_1 g_3, g_2 g_3 = g_3 g_2 \rangle.$$

By this presentation, the Alexander matrix  $A$  and its Alexander polynomial are given by

$$A = \begin{bmatrix} t^2 - t + 1 & -t^2 + t - 1 & 0 \\ t^2 - t + 1 & 0 & -t^2 + t - 1 \\ 0 & 1 - t & t - 1 \end{bmatrix}, \quad \Delta_C(t, L) = t^2 - t + 1.$$

**Case (6):** We consider the following homogenized polynomial:

$$F(X, Y, Z) := Z^4 + 3YZ^3 - 4X^2Y^2 + 3Y^2Z^2 - 4XY^3 + Y^3Z - Y^4.$$

When we put  $C = \{F(X, Y, Z) = 0\}$  and its affine part  $C^a = \{f(x, y) = F(x, y, 1) = 0\}$ , then

- (i)  $C$  has one  $a_3$  and one  $a_2$  singular points at  $[1 : 0 : 0]$ ,  $[1/2, -1, 1]$  respectively,
- (ii)  $L_\infty = \{Z = 0\}$  is a tangent line which passes through  $a_3$ , and
- (iii) the discriminant of  $f$  with respect to  $y$  is

$$D_{(f,y)}(x) = -(256x^4 - 32x^3 - 480x^2 + 822x - 283)(2x - 1)^3.$$

Figure 3.15 shows the real part of the affine curve  $C^a$  and the setting of generators.

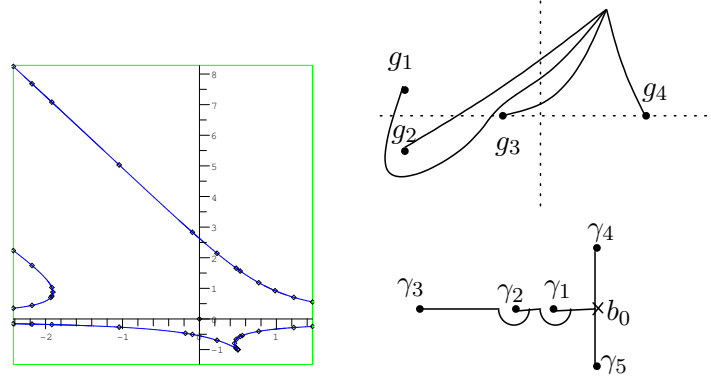


Figure 3.15

Put  $\Sigma := \{x \in \mathbb{C} \mid D_{(f,y)}(x) = 0\}$  and take a base point  $b_0 = \alpha \in \mathbb{C} - \Sigma$ , where  $\alpha$  is the real part of the complex root of  $256x^4 - 32x^3 - 480x^2 + 822x - 283 = 0$ . We take generators of  $\pi_1(\mathbb{C} - \Sigma, b_0)$  and  $\pi_1(p^{-1}(b_0), b'_0)$  as Figure 3.16. Monodromy relations with each  $\gamma_i$  are

$$\begin{aligned} (\gamma_1) \quad g_1 g_2 g_1 &= g_2 g_1 g_2, & (\gamma_2) \quad g_1 &= g_3, \\ (\gamma_3) \quad g_1 &= g_3, & (\gamma_4) \quad g_2 &= g_3 g_4 g_3^{-1}, \\ (\gamma_5) \quad g_4 &= g_2 g_1 g_2^{-1}. \end{aligned}$$

Computing these relations, we have  $g_1 g_2 g_1 = g_2 g_1 g_2$ . By Theorem 2.1 and Example 2.2, we have

$$\pi_1(\mathbb{C}_L^2 - C_L) \cong \langle g_1, g_2 \mid g_1 g_2 g_1 = g_2 g_1 g_2 \rangle, \quad \Delta_C(t, L) = t^2 - t + 1.$$

**Case (7):** We consider the following homogenized polynomial:

$$F(X, Y, Z) := 36X^4 - 33X^3Z + 10X^2Y^2 - XZ^3 - 12X^2Y^2 + 10XY^2Z - 2Y^2Z^2 + Y^4.$$

When we put  $C = \{F(X, Y, Z) = 0\}$  and its affine part  $C^a = \{f(x, y) = F(x, y, 1) = 0\}$ , then

- (i)  $C$  has an  $a_5$  singular point at  $[1/3 : 0 : 1]$ ,
- (ii)  $L_\infty = \{Z = 0\}$  is a bi-tangent line, and
- (iii) the discriminant of  $f$  with respect to  $y$  is

$$D_{(f,y)}(x) = 256x(4x - 1)(-1 + 3x)^8.$$

The Figure 3.16 shows the real part of the affine curve  $C^a$  and the setting of generators.

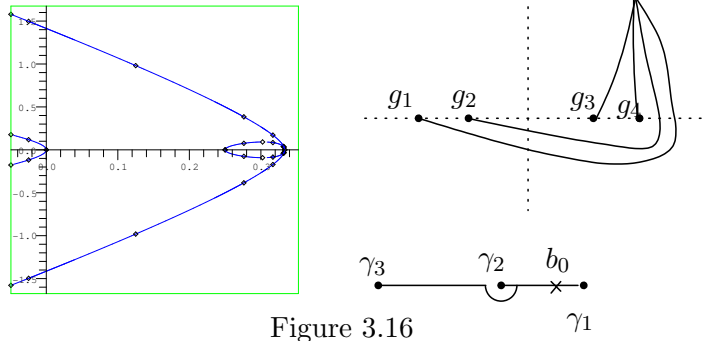


Figure 3.16

Put  $\Sigma := \{x \in \mathbb{C} \mid D_{(f,y)}(x) = 0\}$  and take a base point  $b_0 = 1/3 - \epsilon \in \mathbb{C} - \Sigma$ , where  $\epsilon$  is a sufficiently small positive real number. Moreover, we take generators of  $\pi_1(\mathbb{C} - \Sigma, b_0)$  and  $\pi_1(p^{-1}(b_0), b'_0)$  as Figure 3.17. Monodromy relations with each  $\gamma_i$  are

$$\begin{aligned} (\gamma_1) \quad g_1 &= g_3, \quad g_2 = g_3 g_4 g_3^{-1}, \quad g_4 = (g_3 g_4 g_2 g_1) g_2 (g_3 g_4 g_2 g_1)^{-1}, \\ (\gamma_2) \quad g_3 &= g_3 g_4 g_2 (g_3 g_4)^{-1}. \end{aligned}$$

Since the relative position of  $g_1$  and  $g_4$  about  $\gamma_3$  is fixed on real axis, the  $\gamma_3$ -relation is same to that of  $\gamma_2$ -relation. Computing these relations, we have  $g_1 g_2 g_1 = g_2 g_1 g_2$ . By Theorem 2.1 and Example 2.2, we have

$$\pi_1(\mathbb{C}_L^2 - C_L) \cong \langle g_1, g_2 \mid g_1 g_2 g_1 = g_2 g_1 g_2 \rangle, \quad \Delta_C(t, L) = t^2 - t + 1.$$

**Case (8):** We consider the following homogenized polynomial:

$$F(X, Y, Z) := -X^3Z + X^2Z^2 - 2XY^2Z + Y^4.$$

When we put  $C = \{F(X, Y, Z) = 0\}$  and its affine part  $C^a = \{f(x, y) = F(x, y, 1) = 0\}$ , then

- (i)  $C$  has an  $a_5$  singular point at  $[0 : 0 : 1]$ ,
- (ii)  $L_\infty = \{Z = 0\}$  is a tangent line with multiplicity 4, and
- (iii) the discriminant of  $f$  with respect to  $y$  is  $D_{(f,y)}(x) = -256x^8(x-1)$ .

Figure 3.17 shows the real part of the affine curve  $C^a$  and the setting of generators.

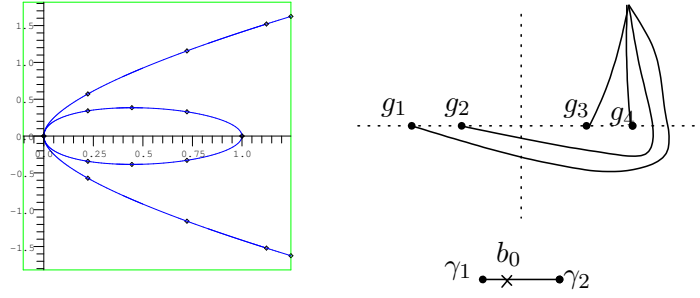


Figure 3.17

Put  $\Sigma := \{x \in \mathbb{C} \mid D_{(f,y)}(x) = 0\}$  and take a base point  $b_0 = \epsilon \in \mathbb{C} - \Sigma$ , where  $\epsilon$  is a sufficiently small positive real number. We take generators of  $\pi_1(\mathbb{C} - \Sigma, b_0)$  and  $\pi_1(p^{-1}(b_0), b'_0)$  as Figure 3.17. Monodromy relations with each  $\gamma_i$  are

$$\begin{aligned} (\gamma_1) \quad g_1 &= g_3, \quad g_2 = g_3 g_4 g_3^{-1}, \quad g_4 = (g_3 g_4 g_2 g_1) g_2 (g_3 g_4 g_2 g_1)^{-1}, \\ (\gamma_2) \quad g_3 &= g_3 g_4 g_2 (g_3 g_4)^{-1}. \end{aligned}$$

Computing these relations, we have  $g_1 g_2 g_1 = g_2 g_1 g_2$ . By Theorem 2.1 and Example 2.2, we have

$$\pi_1(\mathbb{C}_L^2 - C_L) \cong \langle g_1, g_2 \mid g_1 g_2 g_1 = g_2 g_1 g_2 \rangle, \quad \Delta_C(t, L) = t^2 - t + 1.$$

**Case (10):** We consider the following homogenized polynomial:

$$F(X, Y, Z) := X^3 Z - (X^2 - YZ + Z^2)^2.$$

When we put  $C = \{F(X, Y, Z) = 0\}$  and its affine part  $C^a = \{f(x, y) = F(x, y, 1) = 0\}$ , then

- (i)  $C$  has one  $a_4$  and one  $a_2$  at  $[0 : 1 : 0]$ ,  $[0, 1, 1]$  respectively,
- (ii)  $L_\infty = \{Z = 0\}$  is a line which intersects at the  $a_4$  singular, and
- (iii) The discriminant of  $f$  with respect to  $y$  is  $D_{(f,y)}(x) = 4x^3$ .

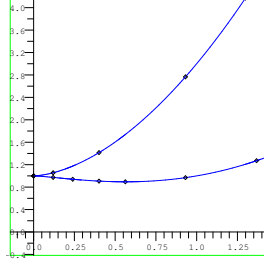


Figure 3.18

Figure 3.18 shows the real part of the affine curve  $C^a$ . By the graph and the discriminant with respect to  $y$ , we can clearly see that all monodromy relations are only one cuspidal relation. By Theorem 2.1 and Example 2.2, we have

$$\pi_1(\mathbb{C}_L^2 - C_L) \cong \langle g_1, g_2 \mid g_1 g_2 g_1 = g_2 g_1 g_2 \rangle,$$

$$\Delta_C(t, L) = t^2 - t + 1.$$

**Case (11):** We consider the following homogenized polynomial:

$$F(X, Y, Z) := X^3 Z + (Y - X)^2 (Y + X)^2.$$

When we put  $C = \{F(X, Y, Z) = 0\}$  and its affine part  $C^a = \{f(x, y) = F(x, y, 1) = 0\}$ , then

- (i)  $C$  has an  $e_6$  singular point at  $[0 : 0 : 1]$ ,
- (ii)  $L_\infty = \{Z = 0\}$  is a bi-tangent line, and
- (iii) the discriminant of  $f$  with respect to  $y$  is  $D_{(f,y)}(x) = 256x^9(1+x)$ .

Figure 3.19 shows the real part of the affine curve  $C^a$  and the setting of generators.

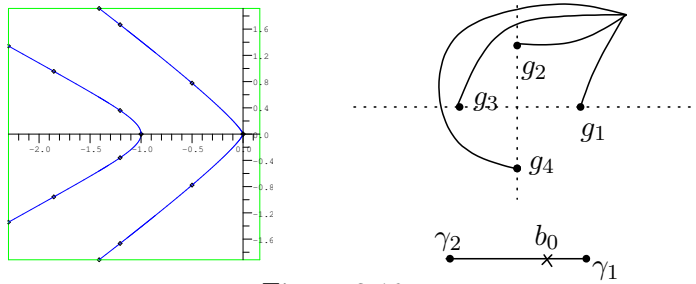


Figure 3.19

Put  $\Sigma := \{x \in \mathbb{C} \mid D_{(f,y)}(x) = 0\}$  and take a base point  $b_0 = -1 + \epsilon \in \mathbb{C} - \Sigma$ , where  $\epsilon$  is a small number. Moreover, we take generators of  $\pi_1(\mathbb{C} - \Sigma, b_0)$  and

$\pi_1(p^{-1}(b_0), b'_0)$  as Figure 3.19. Monodromy relations with each  $\gamma_i$  are

$$\begin{aligned} (\gamma_1) \quad g_1 &= g_4, \quad g_2 = (g_4 g_3 g_2) g_1 (g_4 g_3 g_2)^{-1}, \quad g_3 = (g_4 g_3 g_2 g_1) g_2 (g_4 g_3 g_2 g_1)^{-1}, \\ (\gamma_2) \quad g_2 &= (g_3 g_2)^{-1} g_4 (g_3 g_2). \end{aligned}$$

Computing these relations, we have  $g_1 g_3 g_1 = g_3 g_1 g_3$ . By Theorem 2.1 and Example 2.2, we have

$$\pi_1(\mathbb{C}_L^2 - C_L) \cong \langle g_1, g_3 \mid g_1 g_3 g_1 = g_3 g_1 g_3 \rangle, \quad \Delta_C(t, L) = t^2 - t + 1.$$

**Case (12):** We consider the following homogenized polynomial:

$$F(X, Y, Z) := (X - Z)^3 Z + Y^4.$$

When we put  $C = \{F(X, Y, Z) = 0\}$  and its affine part  $C^a = \{f(x, y) = F(x, y, 1) = 0\}$ , then

- (i)  $C$  has an  $e_6$  singular point at  $[1 : 0 : 1]$ ,
- (ii)  $L_\infty = \{Z = 0\}$  is a tangent line with multiplicity 4, and
- (iii) The discriminant of  $f$  with respect to  $y$  is  $256 (x - 1)^9$ .

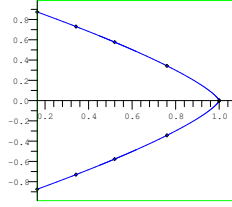


Figure 3.20

Figure 3.20 shows the real part of the affine curve  $C^a$ . As we can observe clearly by the graph and the discriminant with respect to  $y$ , all monodromy relations are given by the  $e_6$ -action of  $\gamma_1$ . Monodromy relations are

$$g_1 = g_4, \quad g_2 = (g_4 g_3 g_2) g_1 (g_4 g_3 g_2)^{-1}, \quad g_3 = (g_4 g_3 g_2 g_1) g_2 (g_4 g_3 g_2 g_1)^{-1}.$$

Computing these relations and replacing  $w = g_1 g_3 g_2 g_1$ . By Theorem 2.1, we have

$$\langle g_1, g_2, g_3 \mid w g_1 = g_2 w, w g_2 = g_3 w \rangle.$$

Then the Alexander matrix and the Alexander polynomial are given by

$$A = \begin{bmatrix} t - t^3 - 1 & t^3 - t^2 + 1 & t(t - 1) \\ t^4 - t^3 + t - 1 & -t^2(t^2 - t + 1) & t^2 - t + 1 \end{bmatrix}$$

$$\Delta_C(t, L) = (t^2 - t + 1)(t^4 - t^2 + 1).$$

**Case (13):** We consider the following homogenized polynomial:

$$F(X, Y, Z) := Y(81XZ^2 + 54Z^4 + 81X^2Y - 54XYZ - 54YZ^2 + 90XY^2 + 18Y^2Z + 25Y^3).$$

When we put  $C = \{F(X, Y, Z) = 0\}$  and its affine part  $C^a = \{f(x, y) = F(x, y, 1) = 0\}$ , then

- (i)  $C$  has one  $a_1$ , one  $a_2$  and one  $a_3$  singular points at  $[-2/3 : 0 : 1]$ ,  $[-3/4 : 3/4 : 1]$ ,  $[1 : 0 : 0]$  respectively,
- (ii)  $L_\infty = \{Z = 0\}$  is a tangent line which intersects  $C$  at the  $a_3$  singular point, and
- (iii) the discriminant of  $f$  with respect to  $y$  is

$$D_{(f,y)}(x) = 243(3x + 2)^2(5x^2 - 12x - 12)(4x + 3)^3.$$

Figure 3.21 shows the real part of the affine curve  $C^a$  and the setting of generators.

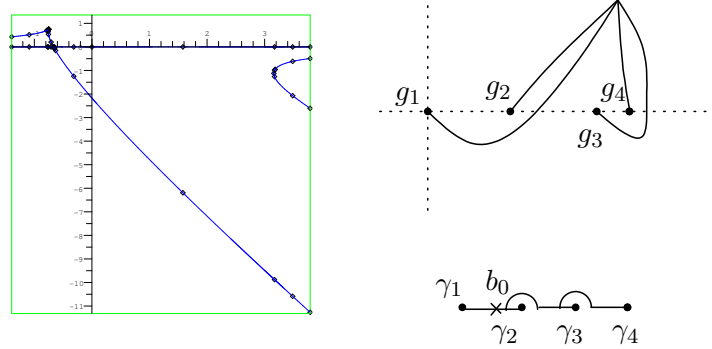


Figure 3.21

Put  $\Sigma := \{x \in \mathbb{C} \mid D_{(f,y)}(x) = 0\}$  and take a base point  $b_0 = -3/4 - \epsilon \in \mathbb{C} - \Sigma$ , where  $\epsilon$  is a sufficiently small positive real number. We take generators of  $\pi_1(\mathbb{C} - \Sigma, b_0)$  and  $\pi_1(p^{-1}(b_0), b'_0)$  as Figure 3.21. Monodromy relations with each  $\gamma_i$  are

$$\begin{aligned} (\gamma_1) \quad g_4 g_3 g_4^{-1} &= g_2, & (\gamma_2) \quad g_3 g_4 g_3 &= g_4 g_3 g_4, \\ (\gamma_3) \quad g_1 g_2 &= g_2 g_1, & (\gamma_4) \quad g_1 g_4 &= g_3 g_1. \end{aligned}$$

By Theorem 2.1, we have

$$\langle g_1, g_3, g_4 \mid g_3 g_4 g_3 = g_4 g_3 g_4, \quad g_1 g_4 = g_3 g_1, \quad (g_1 g_4)(g_4 g_3) = (g_4 g_3)(g_1 g_4) \rangle.$$

Then the Alexander matrix and the Alexander polynomial are

$$A = \begin{bmatrix} 0 & t^2 - t + 1 & -t^2 + t - 1 \\ 1 - t^2 & t(t^2 - 1) & -t^3 + t^2 + t - 1 \\ 1 - t & -1 & t \end{bmatrix}, \quad \Delta_C(t, L) = (t^2 - t + 1)(t - 1).$$

**Case (14):** We consider the following homogenized polynomial:

$$F(X, Y, Z) := (Y^2 - 2XY - XZ + X^2) (Y^2 + 2XY - XZ + X^2).$$

When we put  $C = \{F(X, Y, Z) = 0\}$  and its affine part  $C^a = \{f(x, y) = F(x, y, 1) = 0\}$ , then

- (i)  $C$  has one  $a_5$  and one  $a_1$  singular points at  $[0 : 0 : 1]$ ,  $[1 : 0 : 1]$ ,
- (ii)  $L_\infty = \{Z = 0\}$  is a bi-tangent line, and
- (iii) the discriminant of  $f$  with respect to  $y$  is

$$D_{(f,y)}(x) = 4096x^8(-1+x)^2.$$

Figure 3.22 shows the real part of the affine curve  $C^a$  and the setting of generators.

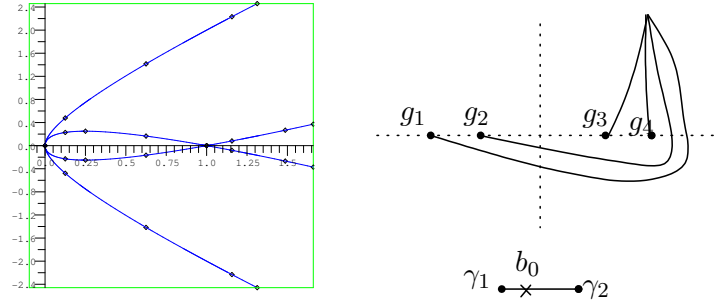


Figure 3.22

Put  $\Sigma := \{x \in \mathbb{C} \mid D_{(f,y)}(x) = 0\}$  and take a base point  $b_0 = \epsilon \in \mathbb{C} - \Sigma$ , where  $\epsilon$  is a sufficiently small positive real number. We also take generators of  $\pi_1(\mathbb{C} - \Sigma, b_0)$  and  $\pi_1(p^{-1}(b_0), b'_0)$  as Figure 3.22. Monodromy relations with each  $\gamma_i$  are

$$\begin{aligned} (\gamma_1) \quad g_1 &= g_3, \quad g_2 = g_3 g_4 g_3^{-1}, \quad g_4 = (g_3 g_4 g_2 g_1) g_2 (g_3 g_4 g_2 g_1)^{-1}, \\ (\gamma_2) \quad g_3 g_3 g_4 g_2 (g_3 g_4)^{-1} &= g_3 g_4 g_2 (g_3 g_4)^{-1} g_3. \end{aligned}$$

Computing these relations, we have  $g_2 g_1 g_2 g_2 g_1 = g_1 g_2 g_2 g_1 g_2$  and  $(g_1 g_2)^3 = (g_2 g_1)^3$ . By theorem 2.1, we have

$$\pi_1(\mathbb{C}_L^2 - C_L) \cong \langle g_1, g_2 \mid (g_1 g_2)^3 = (g_2 g_1)^3, g_2 g_1 g_2 g_2 g_1 = g_1 g_2 g_2 g_1 g_2 \rangle.$$

Then the Alexander matrix and the Alexander polynomial are given by

$$A = \begin{bmatrix} -t^5 + t^4 - t^3 + t^2 - t + 1 & -(-t^5 + t^4 - t^3 + t^2 - t + 1) \\ t^4 - t^3 + t - 1 & -(t^4 - t^3 + t - 1) \end{bmatrix},$$

$$\Delta_C(t, L) = (t^2 - t + 1)(t - 1).$$

**Case (15):** We consider the following homogenized polynomial:

$$F(X, Y, Z) := 256Y^4 - 256Y^3Z + 96Y^2Z^2 - 16YZ^3 + 1 + 32XY^3 - 32XY^2Z + 10XYZ^2 - XZ^3 + X^2Y^2.$$

When we put  $C = \{F(X, Y, Z) = 0\}$  and its affine part  $C^a = \{f(x, y) = F(x, y, 1) = 0\}$ , then

- (i)  $C$  has one  $a_2$  and one  $a_4$  singular points at  $[1 : 0 : 0], [0 : 1/4 : 1]$ ,
- (ii)  $L_\infty = \{Z = 0\}$  is a tangent line which passes through a cusp, and
- (iii) the discriminant of  $f$  with respect to  $y$  is

$$D_{(f,y)}(x) = 65536x^7(x - 64).$$

Figure 3.23 shows the real part of the affine curve  $C^a$  and the setting of generators.

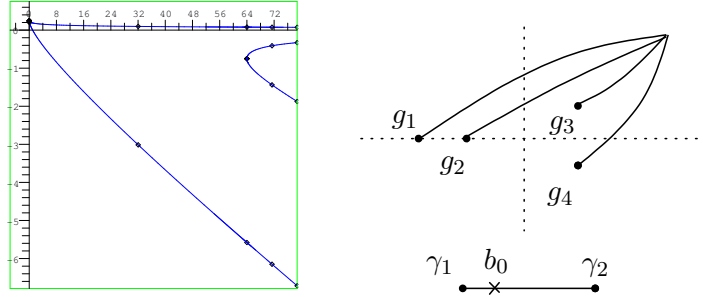


Figure 3.23

Put  $\Sigma := \{x \in \mathbb{C} \mid D_{(f,y)}(x) = 0\}$  and take a base point  $b_0 = \epsilon \in \mathbb{C} - \Sigma$ . We take generators of  $\pi_1(\mathbb{C} - \Sigma, b_0)$  and  $\pi_1(p^{-1}(b_0), b'_0)$  as Figure 3.23. Monodromy relations with each  $\gamma_i$  are

$$\begin{aligned} (\gamma_1) \quad g_1 &= g_4, \quad g_3 = g_1 g_2 g_1^{-1}, \quad g_2 = (g_1 g_2 g_3) g_4 (g_1 g_2 g_3)^{-1}, \\ (\gamma_2) \quad (g_3^{-1} g_2 g_3) g_4 (g_3^{-1} g_4 g_3)^{-1} &= g_3. \end{aligned}$$

Computing these relations, we have two relations  $g_2 g_1 g_2 g_1 g_2 = g_1 g_2 g_1 g_2 g_1$  and  $g_1 g_2 g_2 = g_2 g_2 g_1$ . By Theorem 2.1, we have

$$\langle g_1, g_2 \mid g_2 g_1 g_2 g_1 g_2 = g_1 g_2 g_1 g_2 g_1, \quad g_1 g_2 g_2 = g_2 g_2 g_1 \rangle.$$

The Alexander matrix and its Alexander polynomial are given by

$$A = \begin{bmatrix} -t^4 + t^3 - t^2 + t - 1 & t^4 - t^3 + t^2 - t + 1 \\ 1 - t^2 & t^2 - 1 \end{bmatrix}, \quad \Delta_C(t, L) = 1.$$

**Case (16):** We consider the following homogenized polynomial:

$$F(X, Y, Z) := (Y - X)(-XY^2 + Z^3 + Y^3).$$

When we put  $C = \{F(X, Y, Z) = 0\}$  and its affine part  $C^a = \{f(x, y) = F(x, y, 1) = 0\}$ , then

- (i)  $C$  has one  $a_5$  and one  $a_2$  singular points at  $[1 : 1 : 0]$ ,  $[1 : 0 : 0]$  respectively,
- (ii)  $L_\infty = \{Z = 0\}$  is a line which passes through each  $a_2$  and  $a_5$ , and
- (iii) the discriminant of  $f$  with respect to  $y$  is  $D_{(f,y)}(x) = 4x^3 - 27$ .

Figure 3.24 shows the real part of the affine curve  $C^a$  and the setting of generators.

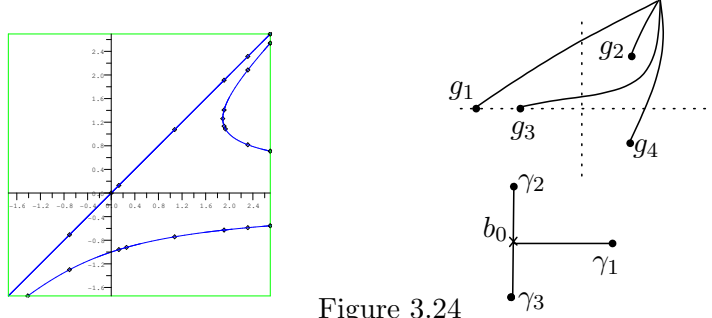


Figure 3.24

Put  $\Sigma := \{x \in \mathbb{C} \mid D_{(f,y)}(x) = 0\}$  and take a base point  $b_0 = \epsilon \in \mathbb{C} - \Sigma$ . Moreover, we take generators of  $\pi_1(\mathbb{C} - \Sigma, b_0)$  and  $\pi_1(p^{-1}(b_0), b'_0)$  as Figure 3.24. Monodromy relations with each  $\gamma_i$  are

$$(\gamma_1) g_2 = g_3 g_4 g_3^{-1}, \quad (\gamma_2) (g_2 g_3 g_2^{-1})^{-1} g_1 (g_2 g_3 g_2^{-1}) = g_2, \quad (\gamma_3) g_2^{-1} g_1 g_2 = g_4.$$

Computing these relations, we have  $g_3 g_3 g_2 = g_2 g_3 g_3$ . By Theorem 2.1, we have

$$\langle g_2, g_3 \mid g_3 g_3 g_2 = g_2 g_3 g_3 \rangle.$$

Then the Alexander matrix and its Alexander polynomial are

$$A = \begin{bmatrix} t^2 - 1 & 1 - t^2 \end{bmatrix}, \quad \Delta_C(t, L) = t^2 - 1.$$

**Case (17):** We consider the following homogenized polynomial:

$$F(X, Y, Z) := (Y^2 + XY + 2Z^2) (Y^2 + XY + Z^2)$$

When we put  $C = \{F(X, Y, Z) = 0\}$  and its affine part  $C^a = \{f(x, y) = F(x, y, 1) = 0\}$ , then

- (i)  $C$  has two  $a_3$  singular points at  $[1 : 0 : 0]$ ,  $[-1 : 1 : 0]$  respectively,
- (ii)  $L_\infty = \{Z = 0\}$  is a line which intersects  $C$  at two  $a_3$  singularities, and
- (iii) the discriminant of  $f$  with respect to  $y$  is

$$D_{(f,y)}(x) = (x - 2)(x + 2)(x^2 - 8).$$

Figure 3.25 shows the real part of the affine curve  $C^a$  and the setting of generators.

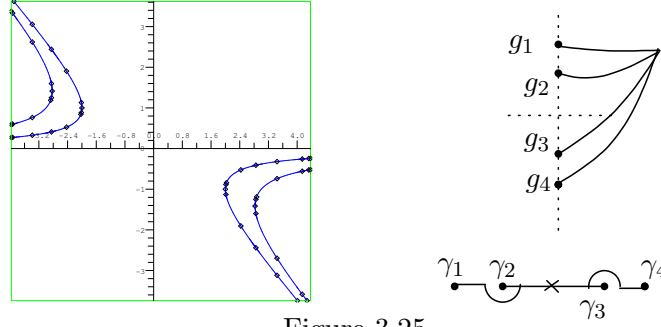


Figure 3.25

Put  $\Sigma := \{x \in \mathbb{C} \mid D_{(f,y)}(x) = 0\}$  and take a base point  $b_0 = 0 \in \mathbb{C} - \Sigma$ . We take generators of  $\pi_1(\mathbb{C} - \Sigma, b_0)$  and  $\pi_1(p^{-1}(b_0), b'_0)$  as Figure 3.25. Monodromy relations with each  $\gamma_i$  are

$$(\gamma_1 \text{ and } \gamma_3) \ g_2 = g_3, \ (\gamma_2 \text{ and } \gamma_4) \ g_3^{-1} g_4 g_3 = g_1.$$

Computing these relations, we have  $g_1 g_2 = g_2 g_4$ . By Theorem 2.1, we have

$$\langle g_1, g_2, g_4 \mid g_1 g_2 = g_2 g_4 \rangle.$$

The Alexander matrix and its Alexander polynomial are:

$$A = \begin{bmatrix} 1 & t-1 & -t \end{bmatrix}, \Delta_C(t, L) = 1.$$

**Case (18):** We consider the following homogenized polynomial:

$$F(X, Y, Z) := (-XZ - 2Z^2 + Y^2) (-XZ - Z^2 + Y^2).$$

When we put  $C = \{F(X, Y, Z) = 0\}$  and its affine part  $C^a = \{f(x, y) = F(x, y, 1) = 0\}$ , then

- (i)  $C$  has an  $a_7$  singular point at  $[1 : 0 : 0]$ ,
- (ii)  $L_\infty = \{Z = 0\}$  is a line which intersects  $C$  at the  $a_7$  singular point, and
- (iii) the discriminant of  $f$  with respect to  $y$  is

$$D_{(f,y)}(x) = 16(x+2)(x+1).$$

Figure 3.26 shows the real part of the affine curve  $C^a$  and the setting of generators.

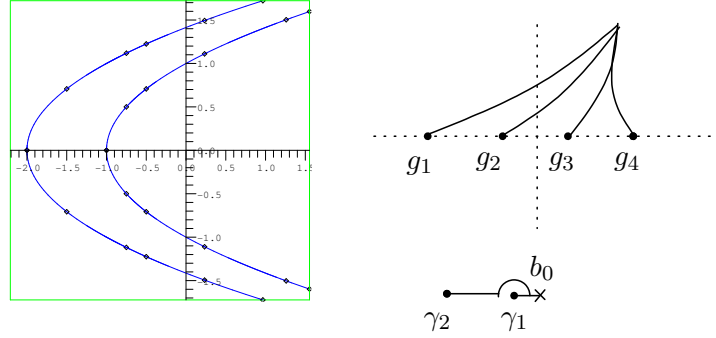


Figure 3.26

Put  $\Sigma := \{x \in \mathbb{C} \mid D_{(f,y)}(x) = 0\}$  and take a base point  $b_0 = -1 + \epsilon \in \mathbb{C} - \Sigma$ , where  $\epsilon$  is a small number. We take generators of  $\pi_1(\mathbb{C} - \Sigma, b_0)$  and  $\pi_1(p^{-1}(b_0), b'_0)$  as Figure 3.26. Monodromy relations with each  $\gamma_i$  are

$$(\gamma_1) g_2 = g_3, (\gamma_2) g_3^{-1} g_4 g_3 = g_1.$$

Computing these relations, we have  $g_1 g_2 = g_2 g_4$ . By Theorem 2.1, we have

$$\langle g_1, g_2, g_4 \mid g_1 g_2 = g_2 g_4 \rangle.$$

The Alexander matrix and its Alexander polynomial are:

$$A = \begin{bmatrix} 1 & t-1 & -t \end{bmatrix}, \Delta_C(t, L) = 1.$$

**Case (19):** We consider the following homogenized polynomial:

$$F(X, Y, Z) := (-2Y + Z + 2X)(2Y + Z + 2X)(-Y^2 + X^2 + XZ).$$

When we put  $C = \{F(X, Y, Z) = 0\}$  and its affine part  $C^a = \{f(x, y) = F(x, y, 1) = 0\}$ , then

- (i)  $C$  has two  $a_3$  and one  $a_1$  singular points at  $[1 : 1 : 0]$ ,  $[-1 : 1 : 0]$ ,  $[-\frac{1}{2} : 0 : 1]$  respectively,

- (ii)  $L_\infty = \{Z = 0\}$  is a line which intersects  $C$  at two  $a_3$ , and
- (iii) the discriminant of  $f$  with respect to  $y$  is

$$D_{(f,y)}(x) = 64x(x+1)(1+2x)^2.$$

Figure 3.27 shows the real part of  $C^a$  and the setting of generators.

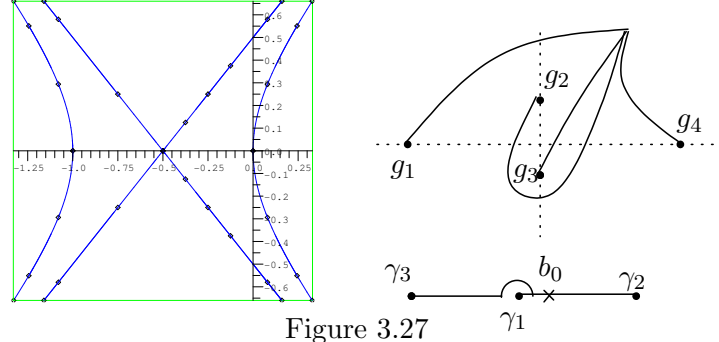


Figure 3.27

Put  $\Sigma := \{x \in \mathbb{C} \mid D_{(f,y)}(x) = 0\}$  and take a base point  $b_0 = -1/2 + \epsilon \in \mathbb{C} - \Sigma$ , where  $\epsilon$  is a small number. We take generators of  $\pi_1(\mathbb{C} - \Sigma, b_0)$  and  $\pi_1(p^{-1}(b_0), b'_0)$  as Figure 3.27. Monodromy relations with each  $\gamma_i$  are

$$(\gamma_1) g_2 g_3 = g_3 g_2, (\gamma_2) (g_3 g_2 g_3^{-1}) g_1 (g_3 g_2 g_3^{-1}) = g_4, (\gamma_3) g_1 g_2 = g_2 g_4.$$

Computing these relations, we have  $g_2 g_3 = g_3 g_2$  and  $g_2 g_1 g_3 = g_3 g_1 g_2$ . By Theorem 2.1, we have

$$\langle g_1, g_2, g_3 \mid g_2 g_3 = g_3 g_2, g_2 g_1 g_3 = g_3 g_1 g_2 \rangle.$$

Then the Alexander matrix and its Alexander polynomial are

$$A = \begin{bmatrix} 1-t & t-1 & 0 \\ 0 & 1-t^2 & t^2-1 \end{bmatrix}, \Delta_C(t, L) = (t-1)^2(t+1).$$

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