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Representations of p'-valenced Schurian schemes

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Abstract. Let p be a prime number. We consider representations of p'-valenced Schurian schemes over a field of characteristic p, especially the case that the cardinality of the underlying set can be divided by p and not by p^2. A typical example of such scheme is obtained by the following way. Let G be a finite group of order pq, where q is prime to p, and let H be a p'-subgroup of G. Define the scheme by the action of G on $H \setminus G$. In this case, we will show that the adjacency algebra is a direct sum of some Brauer tree algebras and simple algebras, and hence it has finite representation type.
Also we give some examples of the case that G is the symmetric group of degree p and H is its Young subgroup.

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§1. Introduction

Let (X, S) be an association scheme in the sense in [14], and let F be an algebraically closed field of positive characteristic p. It is natural to consider the problem: "Determine the representation type of the adjacency algebra". We say a finite dimensional F-algebra A has *finite representation type* if the cardinality of isomorphism classes of finite dimensional indecomposable A-modules is finite. It is well-known that a group algebra has finite representation type if and only if its Sylow p-subgroup is cyclic ([7]). We want to consider a generalization of this fact to association schemes. But an association scheme does not have something like a Sylow subgroup. So we consider the case that $|X|$, the cardinality of the underlying set X, can be divided by p and not by p^2. But this assumption is not enough to our problem.

Example 1.1. Let (X, S) be the group association scheme of the symmetric group $\mathfrak{S}_3$ of degree 3, and let $p = 3$. The adjacency algbera FS is isomorphic to the center of the group algebra $F\mathfrak{S}_3$, and so FS is isomorphic to $F[x,y]/(x^2,y^2,xy)$. This algebra has infinite representation type (see [4, I.4.3.1]).

So we strengthen our hypothesis. Suppose the scheme is p'-valenced, namely the valency of every relation in S is prime to p.

Question 1.2. Let (X,S) be a p'-valenced scheme and F be an algebraically closed field of positive characteristic p. Suppose $|X|$ can be divided by p and not by p^2. Is it true that FS has finite representation type?

In this article, we will give a partial result to this question. A typical example of a scheme satisfying the conditions in Question 1.2 is obtained as follows.

Example 1.3. Let G be a finite group with order pq, where q is prime to p, and H a p'-subgroup of G. Define a Schurian scheme (X,S) by the action of G on $H \setminus G$. Then (X,S) satisfies the assumption in Question 1.2.

We will denote the Schurian scheme defined by a finite group G and its subgroup H by $\mathfrak{X}(G,H)$. We call a scheme isomorphic to the Schurian scheme $\mathfrak{X}(G,H)$ defined by a p'-subgroup H a *strongly p'-valenced Schurian scheme*. For example, the Schurian scheme $\mathfrak{X}(G,H)$ defined by the way in Example 1.3 is a strongly p'-valenced. Also we write $\mathfrak{S}_n$, $\mathfrak{A}_n$, and $\mathfrak{C}_n$ for the symmetric group of degree n, the alternating group of degree n, and the cyclic group of order n, respectively.

Example 1.4. Let $G = \mathfrak{S}_4$, $H = \mathfrak{S}_3$, and let $p = 2$. Then $\mathfrak{X}(G,H)$ seems to be not strongly $2'$-valenced Schurian, since H is not a $2'$-subgroup. But easily we can see that $\mathfrak{X}(G,H)$ is isomorphic to $\mathfrak{X}(\mathfrak{A}_4,\mathfrak{C}_3)$, and $\mathfrak{X}(\mathfrak{A}_4,\mathfrak{C}_3)$ is strongly $2'$-valenced Schurian.

Our main result is as follows, though it is an easy corollary to the results in [5] and [10]. This is a partial answer to Question 1.2.

Theorem 1.5. *Let (X,S) be a strongly p'-valenced Schurian scheme with $|X| = pq$, where q is prime to p, and let F be an algebraically closed field of characteristic p. Then the adjacency algebra FS is a direct sum of some Brauer tree algebras and simple algebras, especially its representation type is finite.*

We note that the result is valid even for a non-Schurian scheme, if it is algebraically isomorphic to a Schurian scheme.

In section 4, we will give some examples of the adjacency algebra for the case $G = \mathfrak{S}_p$. For example, we consider the case that H is a Young subgroup of $\mathfrak{S}_p$. In this case, the principal block is the only one non-semisimple block of the adjacency algebra, and we can determine the Brauer tree of it. We note that, if $H = \mathfrak{S}_t \times \mathfrak{S}_{p-t}$, then the scheme is the Johnson scheme. This example is related to some results in [6] and [13].

§2. Preliminaries

Let (X, S) be an *association scheme*, namely, X is a finite set, S is a collection of non-empty subsets of $X \times X$ and they satisfy the following conditions:

(1) $X \times X = \bigcup_{s \in S} s$ (disjoint),

(2) $1 := \{(x, x) | x \in X\} \in S$,

(3) if $s \in S$ then $s^* := \{(y, x) | (x, y) \in s\} \in S$,

(4) and $\sigma_s \sigma_t = \sum_{u \in S} p_{st}^u \sigma_u$ for some $p_{st}^u \in \mathbb{Z}$, where $\sigma_s \in Mat_{|X|}(\mathbb{Z})$ for $s \in S$ is the *adjacency matrix*, i.e., $\sigma_s \in Mat_{|X|}(\mathbb{Z})$ by $(\sigma_s)_{xy} = 1$ if $(x, y) \in s$ and 0 otherwise.

Hence every row or column of σ_s contains exactly $n_s := p_{ss^*}^1$ ones. We call n_s the *valency* of $s \in S$. An association scheme (X, S) is said to be *p'-valenced* if every valency is prime to p. Also from the condition (4) $\mathbb{Z}S := \oplus_{s \in S} \mathbb{Z}\sigma_s \subset Mat_{|X|}(\mathbb{Z})$ is a $\mathbb{Z}$-algebra. Then for any commutative ring R with unity, we can define an R-algebra $RS := R \otimes_{\mathbb{Z}} \mathbb{Z}S$ and call it the *adjacency algebra* of (X, S) over R.

Let G be a finite group and H a subgroup of G. We know that the adjacency algebra of the *Schurian* (*association*) *scheme* $\mathfrak{X}(G, H)$ is isomorphic to the Hecke algebra $\mathrm{End}_{RG}(R[H \setminus G])$ as an R-algebra. Also, for $s \in S$, $n_s = |H : H \cap H^g|$ for some $g \in G$. So $\mathfrak{X}(G, H)$ is p'-valenced if and only if $|H : H \cap H^g|$ is prime to p for all $g \in G$. In particular, if H is a p'-subgroup of G then $\mathfrak{X}(G, H)$ is p'-valenced.

We recall a *strongly p'-valenced Schurian scheme* (X, S), which is isomorphic to a Schurian scheme $\mathfrak{X}(G, H)$, where H is a p'-subgroup of G.

From now on we prepare some terminologies and basic facts from the representation theory of algebras and the symmetric groups for later use. We refer to [3] or [11], and [8] or [9].

First we assume that all algebras and their (right) modules are finitely generated over the coefficient rings under consideration. If A is a ring with unity, then $\mathrm{IRR}(A)$ denotes a full set of non-isomorphic irreducible A-modules and $\mathrm{mod}(A)$ denotes the category of (finitely generated right) A-modules. Moreover, we fix the following notations: let F be an algebraically closed field of characteristic p and (K, R, F) be a (splitting) *p-modular system*, that is, R is a complete discrete valuation ring with F as residue field and K is the quotient field of R of characteristic 0. Here "splitting" means that K is a splitting field for all K-algebras considered here (of course F is so).

If A is an R-algebra then we also consider k-algebra $A^k := k \otimes_R A$, where k is K or F, and $A \subset A^K$ via $a \in A$ identifies with $1_K \otimes a \in A^K$. For an A^K-module M we have an R-free A-submodule M_0, called an *R-form* of M, such that $M \simeq K \otimes_R M_0$ and write ${M_0}^* = F \otimes_R M_0$, called a *modular reduction* of M.

Remark. ([11, Chapter 2 Theorem 1.6, 1.9]) An R-form M_0 of M always exists and is not unique in general, not even up to isomorphism. Then modular reductions of M are not isomorphic. However, the set of irreducible constituents of ${M_0}^*$ is uniquely determaind by M.

Then for $M \in \mathrm{IRR}(A^K)$ and $S \in \mathrm{IRR}(A^F)$, we can write $d_{M,S}$, called the *decomposition number*, the composition multiplicity of S in ${M_0}^*$, i.e., the number of factors isomorphic to S in any composition series of ${M_0}^*$. Also matrix $(d_{M,S})$ is called the *decomposition matrix*.

Furthermore, for $U, V \in \mathrm{mod}(A^F)$, write $U \leftrightarrow V$ if they have the same composition factors with multiplicities, and $U \mid V$ means that U is isomorphic to a direct summand of V.

A *partition* of the positive integer n is a non-increasing sequence $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_d)$ of non-negative integers whose sum is n. The *Young diagram* $[\lambda]$ associated with λ is the set of the ordered pairs (i, j) of integers, called the *nodes* of $[\lambda]$, with $1 \leq i \leq d$ and $1 \leq j \leq \lambda_i$, where d denotes the largest number such that $\lambda_d \neq 0$, called the *depth* of λ. They are illustrated as arrays of squares. So for example the partition $(n-m, 1^m)$, we use exponential expressions to indicate repeating terms in the sequence, is called *hook partition* from its shape. A partition λ is said to be *p-singular* if there is an integer $i \geq 0$ such that $\lambda_{i+1} = \lambda_{i+2} = \cdots = \lambda_{i+p}$, and is *p-regular* otherwise. We denote by $P(n)$ and $P(n)^0$ the sets of the partitions and p-regular partitions of n, respectively. The *dominance order* $\trianglelefteq$ on $P(n)$ is defined as follows: given $\lambda, \mu \in P(n), \lambda \trianglelefteq \mu$ if and only if $\sum_{1 \leq i \leq j} \lambda_i \leq \sum_{1 \leq i \leq j} \mu_i$ for all $j \geq 1$.

Given $\lambda \in P(n)$, we have a $k\mathfrak{S}_n$-module S_k^λ called the *Specht module* corresponding to λ over k, where k is K or F. We know that $\mathrm{IRR}(K\mathfrak{S}_n) = \{S_K^\lambda \mid \lambda \in P(n)\}$ and $\mathrm{IRR}(F\mathfrak{S}_n) = \{D^\lambda \mid \lambda \in P(n)^0\}$, where D^λ denotes the *head* of S_F^λ. Namely, S_F^λ has the unique maximal submodule with quotient D^λ. Moreover, $\mathfrak{S}_\lambda := \mathfrak{S}_{\lambda_1} \times \mathfrak{S}_{\lambda_2} \times \cdots \times \mathfrak{S}_{\lambda_d}$ denotes the *Young subgroup* of $\mathfrak{S}_n$ corresponding to λ. Then in closing of this section we introduce the following fact.

Proposition 2.1. ([9, Corrolary 2.2.2]) *Given* $\lambda, \mu \in P(n)$, $\lambda \trianglelefteq \mu$ *if and only if* S_K^μ *appears as direct summands of* $K_{\mathfrak{S}_\lambda}\uparrow^{\mathfrak{S}_n} := K_{\mathfrak{S}_\lambda} \otimes_{K\mathfrak{S}_\lambda} K\mathfrak{S}_n \simeq K[\mathfrak{S}_\lambda \backslash \mathfrak{S}_n]$, *where* $K_{\mathfrak{S}_\lambda}$ *is the trivial module of* $K\mathfrak{S}_\lambda$.

§3. Brauer tree algebras and Schur functors

First we recall the definition of a *Brauer tree algebra* with a *Brauer tree* T according to [1] and [2]. So let k be an arbitrary field in this section.

A *Brauer tree* is a tree, namely, a finite undirected simple graph without cycles, which has the following informations and property:

(1) an anticlockwise cyclic ordering of the edges incident to each vertex,

(2) a positive integer, called the *multiplicity*, of each vertex,

(3) and at most one vertex with multiplicity greater than one.

If there exists the vertex whose multiplicity greater than one, it is called the *exceptional vertex*, and its multiplicity is called the *exceptional multiplicity*.

Moreover, a (finite dimensional) k-algebra A is called a *Brauer tree algebra* for a Brauer tree T, if there is a one-to-one corrspondence between the edges i of T and the irreducible A-modules $S_i \in \mathrm{IRR}(A)$ which has the following properties:

(1) $P_i/\mathrm{rad}(P_i) \simeq \mathrm{soc}(P_i) \simeq S_i$, where P_i is the projective cover of S_i,

(2) $\mathrm{rad}(P_i)/\mathrm{soc}(P_i)$, called the *heart* of P_i, is the direct sum of two (possibly zero) uniserial modules U_i, V_i corresponding to the two vertices u, v at the end of the edge i, respectively,

(3) and if the edges around u are cyclically ordered $i, i_1, i_2, \cdots, i_r, i$ in anticlockwise direction and the multiplicity of u is m_u, then the corresponding uniserial module U_i has composition factors (from the top)

$$S_{i_1}, S_{i_2}, \cdots, S_{i_r}, S_i, S_{i_1}, S_{i_2}, \cdots, S_{i_r}, S_i, \cdots, \cdots, S_i, S_{i_1}, S_{i_2}, \cdots, S_{i_r}$$

so that $S_{i_1}, S_{i_2}, \cdots, S_{i_r}$ appear m_u times and S_i appears $m_u - 1$ times.

Example 3.1. Let G be a finite group and B a p-block of the group algebra FG whose *defect group* is cyclic. Then B is a Brauer tree algebra with the Brauer tree T_B: the vertex u at the end of the edge i corresponding to the *p-conjugate class* of irreducible KG-module V_u such that the decomposition number $d_{V_u,S_i} \neq 0$. In fact, at most one p-conjugate class has the size m greater than one. So if there exists a such p-conjugate class, the vertex corresponding to this class is the exceptional vertex and m is the exceptional multiplicity.

In the rest of this article if a k-algebra A is a direct sum of some Brauer tree algebras and simple algebras, then we call A an *extended Brauer tree algebra* as simple algebra is presented by one vertex. Hence A has finite representation type.

Let A be a k-algebra and e be a non-zero idempotent of A. Also $J(A)$ is the Jacobson radical of A. According to [5] and [10], we consider the *Schur functor* $f = f_{A,e}$ from $\mathrm{mod}(A)$ to $\mathrm{mod}(eAe)$, namely, for $V, V' \in \mathrm{mod}(A)$ and A-map $\alpha : V \to V'$, $f(V) := Ve$ and $f(\alpha) : Ve \to V'e$ is the eAe-map given by the restriction of α to Ve. Then the following holds.

Theorem 3.2. (see [5] and [10]) *We use the above notations.*

(1) $f(VJ(A)) = f(V)J(eAe)$ *for any A-module V.*

(2) *f is exact. In particular, if V is an A-module and W is an A-submodule of V, then $f(V/W) \simeq f(V)/f(W)$ as eAe-modules.*

(3) *If V is an irreducible A-module then $f(V)$ is either zero or irreducible eAe-module. Moreover, f induces the bijection from* $\mathrm{IRR}(A)^e := \{V \in \mathrm{IRR}(A) | f(V) \neq 0\}$ *to* $\mathrm{IRR}(eAe)$.

(4) *If P is the projective cover of $S \in \mathrm{IRR}(A)^e$, then $f(P)$ is the projective cover of $f(S) \in \mathrm{IRR}(eAe)$.*

(5) *Put $k = K$. Let e be an idempotent of A_0, an R-form of A, satisfying the condition $e^* \neq 0$. Then $d_{V,S} = d_{f(V),f^*(S)}$ for $V \in \mathrm{IRR}(A)^e$ and $S \in \mathrm{IRR}({A_0}^*)^{e^*}$, where $f^* := f_{{A_0}^*,e^*}$. Therefore the decomposition matrix of eAe is the submatrix of the decomposition matrix of A, where the row (column resp.) indices are restricted to* $\mathrm{IRR}(A)^e$ $(\mathrm{IRR}({A_0}^*)^{e^*}$ *resp.).*

From this theorem, we have

Corollary 3.3. *Let G be a finite group, H a p'-subgroup of G and $e := \frac{1}{|H|}\sum_{h\in H} h \in RG$. If G has a cyclic Sylow p-subgroup, then the Hecke algebra e^*FGe^* is an extended Brauer tree algebra, and the decomposition matrix of $eKGe$ is the submatrix of the decomposition matrix of G, where the row (column resp.) indices are restricted to* $\mathrm{IRR}(KG)^e$ $(\mathrm{IRR}(FG)^{e^*}$ *resp.).*

Proof. The second half follows from the above theorem (5). So we need only prove the first half.

As G has a cyclic Sylow p-subgroup, FG is an extended Brauer tree algebra, i.e., for any p-block B^* of FG, B^* is a Brauer tree algebra with the Brauer tree T_{B^*} or a simple algebra (see Example 3.1).
Case 1. B^* is a simple algebra (i.e., the defect of B^* is 0).

In this case $\mid \mathrm{IRR}(B) \mid = \mid \mathrm{IRR}(B^*) \mid = 1$. So let $\mathrm{IRR}(B) = \{V\}$. Then $e^*B^*e^*$ is 0 or a simple algebra accoding as V is in $\mathrm{IRR}(KG)^e$ or not.
Case 2. B^* is a Brauer tree algebra (i.e., the defect of B^* is not 0).

We will show that $e^*B^*e^*$ is a direct sum of some Brauer tree algebras. First we mention that $V \in \mathrm{IRR}(KG)^e$ if and only if $K_H \mid V{\downarrow_H}$, and $S \in \mathrm{IRR}(FG)^{e^*}$ if and only if $F_H \mid S{\downarrow_H}$ as e is the central primitive idempotent of KH correponding to K_H.

Let $f = f_{KG,e}$ and $f^* = f_{FG,e^*}$ be the Schur functors. Then we consider the map $f\!\!f := (f, f^*)$ from $(\mathrm{mod}(KG), \mathrm{mod}(FG))$ to $(\mathrm{mod}(eKGe), \mathrm{mod}(e^*FGe^*))$ via $f\!\!f(V,S) = (f(V), f^*(S))$. Here we identify the edge (vertex respectively) of the Brauer tree T_{B^*} with the correponding irreducible FG-module (the p-conjugate class of irreducible KG-modules respectively) (see Example 3.1). Hence the image $f\!\!f(T_{B^*})$ is $\emptyset$ or a disjoint union of some Brauer trees as follows: Put $V \in \mathrm{IRR}(B)$ and $S \in \mathrm{IRR}(B^*)$ with the decomposition number $d_{V,S} \neq 0$, i.e., the vertex V is at the end of the edge S $\circ\!\!-\!\!-$. As the above mention and H is a p'-subgroup of G, if $S \in \mathrm{IRR}(FG)^{e^*}$ then $V \in \mathrm{IRR}(KG)^e$ and $d_{f(V),f^*(S)} = d_{V,S} \neq 0$ from the above theorem, i.e., $f\!\!f$ preserve the branch $\circ\!\!-\!\!-$ in $f\!\!f(T_{B^*})$. By contraposition if $V \notin \mathrm{IRR}(KG)^e$ then $F_H \nmid S{\downarrow_H}$, i.e., $S \notin \mathrm{IRR}(FG)^{e^*}$. Namely, if f deletes the vertex V then $f\!\!f$ lopps off the all edges around V with the vertex V. On the other hand, if $S \notin \mathrm{IRR}(FG)^{e^*}$ and V is at the end of tree T_{B^*} then $V \notin \mathrm{IRR}(KG)^e$. Therefore,

$$f\!\!f(\circ\!\!-\!\!-) = \begin{cases} \circ\!\!-\!\!- & \text{if } S \in \mathrm{IRR}(FG)^{e^*} \\ \circ & \text{if } S \notin \mathrm{IRR}(FG)^{e^*} \text{ and } V \text{ is not at the end of tree } T_{B^*}. \\ \emptyset & \text{otherwise} \end{cases}$$

Furthermore, from the construction of T_{B^*} and $f\!\!f(T_{B^*})$ there is still an anticlockwise cyclic ordering of the edges incident to each vertex of each tree parts in $f\!\!f(T_{B^*})$, and if T_{B^*} has the exceptional vertex $V \in \mathrm{IRR}(KG)^e$ with multiplicity m, then $f(V)$ is the exceptional vertex and its multiplicity is m. Therefore, $f\!\!f(T_{B^*})$ is $\emptyset$ or a disjoint union of some Brauer trees accoding as $\mathrm{IRR}(B) \cap \mathrm{IRR}(KG)^e$ is $\emptyset$ or not.

So we need only consider the case $\mathrm{IRR}(B) \cap \mathrm{IRR}(KG)^e \neq \emptyset$ and each Brauer tree parts $\widetilde{T}$ in $f\!\!f(T_{B^*})$. Let $\beta^*_{\widetilde{T}}$ be the block of $e^*B^*e^*$ corresponding to $\widetilde{T}$. Then there is a one-to-one corrspondence $f\!\!f$ between the edges of $\widetilde{T}$ and the irreducible FG-modules in $\mathrm{IRR}\left(\beta^*_{\widetilde{T}}\right)$ which has the properties (1)$\sim$(3) in the definition of the Brauer tree algebras as follows:

(1) is clear since e^*FGe^* is symmetric algebra.

(2) and (3) : Let $f^*(S_i) \in \mathrm{IRR}\left(\beta^*_{\widetilde{T}}\right)$. So we use the same notation in the definition of the Brauer tree algebras. As f^* preserves inclusion (in particular the unique maximal submodule and the simple socle) and direct sum by the above theorem,

$$\begin{array}{ccccccc} P_i & & & & f^*(P_i) & & \\ | & \simeq S_i & & & | & \simeq f^*(S_i) & \\ \mathrm{rad}(P_i) & & f^* & & f^*(\mathrm{rad}(P_i)) = \mathrm{rad}(f^*(P_i)) & & \\ | & \simeq U_i \oplus V_i & \longrightarrow & & | & \simeq f^*(U_i) \oplus f^*(V_i) & , \\ \mathrm{soc}(P_i) = S_i & & & & f^*(S_i) = \mathrm{soc}(f^*(P_i)) & & \\ | & & & & | & & \\ 0 & & & & 0 & & \end{array}$$

possibly $f^*(U_i) = 0$ or $f^*(V_i) = 0$. Moreover, if U_i is a uniserial FG-module then $f^*(U_i)$ is either zero or a uniserial e^*FGe^*-module by the above theorem (1) and (2). In fact, if the edges around $f^*(U_i)$ are remaining cyclically ordered $i, i_{j_1}, i_{j_2}, \cdots, i_{j_n}, i$ in anticlockwise direction and the multiplicity of $f^*(U_i)$ is the same m_u, then the corresponding uniserial module $f^*(U_i)$ has composition factors (from the top)

$$S_{i_{j_1}}, S_{i_{j_2}}, \cdots, S_{i_{j_n}}, S_i, S_{i_{j_1}}, S_{i_{j_2}}, \cdots, S_{i_{j_n}}, S_i, \cdots, \cdots, S_i, S_{i_{j_1}}, S_{i_{j_2}}, \cdots, S_{i_{j_n}}$$

so that $S_{i_{j_1}}, S_{i_{j_2}}, \cdots, S_{i_{j_n}}$ appear m_u times and S_i appears $m_u - 1$ times.

Therefore $\beta^*_{\tilde{T}}$ is a Brauer tree algebra, namely, the assertion holds.

In particular, we get the next theorem:

Theorem 3.4. *Let (X, S) be a strongly p'-valenced Schurian scheme with $|X| = pq$, where q is prime to p, and let F be an algebraically closed field of characteristic p. Then the adjacency algebra FS is an extended Brauer tree algebra, especially its representation type is finite.*

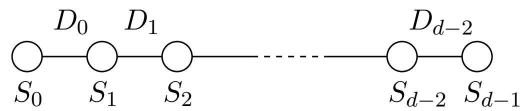
Proof. By the assumption $(X, S) \simeq \mathfrak{X}(G, H)$ as association schemes for some finite group G and its p'-subgroup H. Hence FG is an extended Brauer tree algebra as $\mid G \mid$ is devided by p and not by p^2. So put $e := \frac{1}{|H|}\sum_{h \in H} h \in RG$. Then $FS \simeq e^*FGe^*$ and FS is also an extended Brauer tree algebra by the above corollary.

§4. Examples

Example 4.1. Let $\mu := (\mu_1, \mu_2, \ldots, \mu_d)$ be a partition of p whose depth is d $(d \geq 2)$. And let $G := \mathfrak{S}_p$, H be a Young subgroup $\mathfrak{S}_\mu$ of G, and let (X, S) be the Schurian scheme $\mathfrak{X}(G, H)$. Also let B_0 (β_0 resp.) be the principal p-block of RG (RS resp.). Then the following holds.

(1) $\mathrm{IRR}(\beta_0) = \{S_i \mid 0 \leq i \leq d-1\}$ and $\mathrm{IRR}({\beta_0}^*) = \{D_i \mid 0 \leq i \leq d-2\}$, where S_i (D_i resp.) denotes the irreducible KS-module (FS-module resp.) corresponding to the partition $(p - i, 1^i)$.

(2) ${\beta_0}^*$ is the Brauer tree algebra with tree being the following straight line:

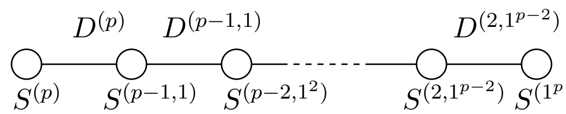


So the decomposition matrix of ${\beta_0}^*$ is the following form:

$$\begin{array}{c} \\ S_0 \\ S_1 \\ S_2 \\ \vdots \\ S_{d-2} \\ S_{d-1} \end{array} \begin{array}{c} \begin{array}{ccccc} D_0 & D_1 & \cdots & \cdots & D_{d-2} \end{array} \\ \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 1 & 1 & \cdots & \cdots & 0 \\ 0 & 1 & 1\cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots 1 & 1 \\ 0 & 0 & \cdots & \cdots & 1 \end{pmatrix} \end{array}.$$

Proof. First we may assume that $p > 2$. As H is a p'-subgroup of G, we may identify FS with e^*FGe^*, where $e := \frac{1}{|H|}\sum_{h\in H} h \in RG$.

We know that $\mathrm{IRR}(B_0) = \{S_K^\lambda \mid \lambda \text{ is a hook partition}\}$ and $\mathrm{IRR}({B_0}^*) = \{D^\lambda \mid \lambda \text{ is a } p\text{-regular hook partition}\}$. Moreover, ${B_0}^*$ is the Brauer tree algebra with tree being the following straight line:



Here $\mathrm{IRR}(KG)^e \cap \mathrm{IRR}(B_0) = \{S_K^{(p-i,1^i)} \mid 0 \le i \le d-1\}$ since $(p) \rhd (p-1,1) \rhd \cdots \rhd (p-d+1, 1^{d-1}) \trianglerighteq \mu \ntrianglelefteq (p-d, 1^d)$ and Proposition 2.1([9, Corollary 2.2.2]). Then $D^{(p-i,1^i)} \notin \mathrm{IRR}(FG)^{e^*}$ for $d-1 \le i \le p-2$ and $D^{(p-(d-2),1^{d-2})} \in \mathrm{IRR}(FG)^{e^*}$ from the above Brauer tree. So we need only prove the first half of (2), namely, $\mathrm{IRR}(FG)^{e^*} \cap \mathrm{IRR}({B_0}^*) = \{D^{(p-i,1^i)} \mid 0 \le i \le d-2\}$ by Corollary 3.3.

Now we use the induction on d. (i) $d = 2$, i.e., $\mu = (p-j, j)$ for some $1 \le j < p$. In this case $\mathrm{IRR}(FG)^{e^*} \cap \mathrm{IRR}({B_0}^*) = \{D^{(p)}\}$. (ii) $d \ge 3$. There exists $\widetilde{\mu} \in P(p)$ such that the depth of $\widetilde{\mu}$ is $d-1$ and $\widetilde{H} := \mathfrak{S}_{\widetilde{\mu}} \ge \mathfrak{S}_\mu = H$. So by the hypothesis $F_{\widetilde{H}} \mid D_i{\downarrow_{\widetilde{H}}}$ for $0 \le i \le d-3$. Then $F_H \mid D_i{\downarrow_H}$, i.e., $D^{(p-i,1^i)} \in \mathrm{IRR}(FG)^{e^*}$ for $0 \le i \le d-3$ and the assertion holds.

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