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On weakly Ricci symmetric lightlike hypersurfaces of indefinite Sasakian manifolds

Fortuné Massamba

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Abstract. This paper deals with weakly Ricci-symmetric lightlike hypersurfaces of indefinite Sasakian manifolds, tangent to the structure vector field. We prove that, in a weakly Ricci symmetric lightlike η-Einstein (or Einstein) hypersurface of an indefinite Sasakian manifold, the associated 1-forms α and β satisfy $\alpha + \beta = 0$ (Theorem 4). Also, we show that there exist no weakly Ricci symmetric screen locally (or globally) conformal lightlike hypersurfaces of indefinite Sasakian manifolds with cyclic parallel Ricci tensor if $\alpha + \beta + \gamma$ is not everywhere zero (Theorem 5). A particular case of weakly Ricci symmetric condition is studied and we prove that a special weakly Ricci symmetric screen locally (or globally) conformal lightlike hypersurface cannot be η-Einstein (or Einstein) and under certain condition, it cannot be $(D \perp \langle\xi\rangle, D')$-mixed-totally geodesic (Theorem 7).

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§1. Introduction

The notion of weakly Ricci symmetric manifolds was considered in [10], [9] and others references therein. A non-flat semi-Riemannian manifold $\overline{M}$ is called *weakly Ricci-symmetric* if the Ricci tensor $\overline{Ric}$ is non-zero and satisfies the following condition, for any vector fields $\overline{X}$, $\overline{Y}$ and $\overline{Z}$ in $\overline{M}$,

$$\begin{aligned}(\overline{\nabla}_{\overline{X}}\overline{Ric})(\overline{Y},\overline{Z}) &= \overline{\alpha}(\overline{X})\overline{Ric}(\overline{Y},\overline{Z}) + \overline{\beta}(\overline{Y})\overline{Ric}(\overline{X},\overline{Z}) \\ &+ \overline{\gamma}(\overline{Z})\overline{Ric}(\overline{Y},\overline{X}),\end{aligned} \tag{1.1}$$

where $\overline{\alpha}$, $\overline{\beta}$ and $\overline{\gamma}$ defined respectively by , $\overline{g}(\overline{X},\overline{\rho}) = \overline{\alpha}(\overline{X})$, $\overline{g}(\overline{X},\overline{\delta}) = \overline{\beta}(\overline{X})$, $\overline{g}(\overline{X},\overline{\kappa}) = \overline{\gamma}(\overline{X})$, are 1-forms called the *associated 1-forms* which are not zero simultaneously and $\overline{\nabla}$ is the Levi-Civita connection for a semi-Riemannian metric $\overline{g}$. In such case, $\overline{\rho}$, $\overline{\delta}$ and $\overline{\kappa}$ are called *associated vector fields* corresponding to the 1-forms $\overline{\alpha}$, $\overline{\beta}$ and $\overline{\gamma}$ respectively. If in (1.1) the 1-form $\overline{\alpha}$ is replaced by $2\overline{\alpha}$, then the semi-Riemannian manifold is called a *generalized pseudo Ricci symmetric* introduced by Chaky and Koley in [3]. So the defining condition of weakly Ricci symmetric manifold is weaker than the generalized pseudo Ricci symmetric manifold. If in (1.1) the 1-form $\overline{\alpha}$ is replaced by $2\overline{\alpha}$ and $\overline{\beta}$ and $\overline{\gamma}$ are equal to $\overline{\alpha}$, then the semi-Riemannian manifold is called a *special weakly Ricci symmetric* and investigated by Singh and Kahan [9].

The purpose of this paper is to investigate the effect of weakly Ricci symmetric condition on the lightlike geometry of hypersurfaces of an indefinite Sasakian manifold, tangent to the structure vector field ξ. Especially, we pay attention to lightlike hypersurfaces with symmetric Ricci tensor. This, because of the geometric point of view and also, physically, Ricci tensor symmetric is essential (see [5] for details and references therein). In the next paragraph, we summarize basic formulae concerning geometric objects on lightlike hypersurfaces, using notations of [4]. In the last part of the paper, we consider a weakly Ricci symmetric lightlike hypersurface of an indefinite Sasakian manifold. We prove that, in a weakly Ricci symmetric lightlike η-Einstein (or Einstein) hypersurface of an indefinite Sasakian manifold, the associated 1-forms α and β satisfy $\alpha + \beta = 0$. We also prove that there exist no weakly Ricci symmetric screen locally (or globally) conformal lightlike hypersurfaces of indefinite Sasakian manifolds with cyclic parallel Ricci tensor if $\alpha + \beta + \gamma$ is not everywhere zero. Finally, we prove that a special weakly Ricci symmetric screen locally (or globally) conformal lightlike hypersurface cannot be η-Einstein (or Einstein) and if the trace of A_N satisfies the partial differential equation $\xi \cdot trA_N - \tau(\xi)trA_N = 0$, it cannot be $(D \perp \langle\xi\rangle, D')$-mixed-totally geodesic.

§2. Preliminaries

A $(2n+1)$-dimensional semi-Riemannian manifold $(\overline{M}, \overline{g})$ is said to be an *indefinite Sasakian manifold* if it admits an almost contact structure $(\overline{\phi}, \xi, \eta)$, i.e. $\overline{\phi}$ is a tensor field of type $(1,1)$ of rank $2n$, ξ is a vector field, and η is a 1-form, satisfying

$$\begin{aligned}
&\overline{\phi}^2 = -\mathbb{I} + \eta \otimes \xi,\ \eta(\xi) = 1,\ \eta \circ \overline{\phi} = 0,\ \overline{\phi}\xi = 0,\\
&\eta(\overline{X}) = \overline{g}(\xi, \overline{X}),\ \ \overline{g}(\overline{\phi}\,\overline{X}, \overline{\phi}\,\overline{Y}) = \overline{g}(\overline{X}, \overline{Y}) - \eta(\overline{X})\,\eta(\overline{Y}),\\
&(\overline{\nabla}_{\overline{X}}\eta)\overline{Y} = -\overline{g}(\overline{\phi}\,\overline{X}, \overline{Y}), (\overline{\nabla}_{\overline{X}}\overline{\phi})\overline{Y} = \overline{g}(\overline{X}, \overline{Y})\xi - \eta(\overline{Y})\overline{X},\\
&\overline{\nabla}_{\overline{X}}\xi = -\overline{\phi}(\overline{X}),\ \forall\, \overline{X}, \overline{Y} \in \Gamma(T\overline{M}))
\end{aligned} \tag{2.1}$$

where $\overline{\nabla}$ is the Levi-Civita connection for a semi-Riemannian metric $\overline{g}$.

A plane section σ in $T_p\overline{M}$ is called a $\overline{\phi}$-section if it is spanned by $\overline{X}$ and $\overline{\phi}\,\overline{X}$, where $\overline{X}$ is a unit tangent vector field orthogonal to ξ. The sectional curvature of a $\overline{\phi}$-section σ is called a $\overline{\phi}$-sectional curvature. A Sasakian manifold $\overline{M}$ with constant $\overline{\phi}$-sectional curvature c, $\overline{M}$ is said to be a *Sasakian space form* and is denoted by $\overline{M}(c)$. The curvature tensor $\overline{R}$ of a Sasakian space form $\overline{M}(c)$ is given by [11]

$$\begin{aligned}
&\overline{R}(\overline{X},\overline{Y})\overline{Z} = \frac{c+3}{4}\left(\overline{g}(\overline{Y},\overline{Z})\overline{X} - \overline{g}(\overline{X},\overline{Z})\overline{Y}\right)\\
&+\frac{c-1}{4}\left(\eta(\overline{X})\eta(\overline{Z})\overline{Y} - \eta(\overline{Y})\eta(\overline{Z})\overline{X} + \overline{g}(\overline{X},\overline{Z})\eta(\overline{Y})\xi - \overline{g}(\overline{Y},\overline{Z})\eta(\overline{X})\xi\right.\\
(2.2)\ &\left. + \overline{g}(\overline{\phi}\,\overline{Y},\overline{Z})\overline{\phi}\,\overline{X} - \overline{g}(\overline{\phi}\,\overline{X},\overline{Z})\overline{\phi}\,\overline{Y} - 2\overline{g}(\overline{\phi}\,\overline{X},\overline{Y})\overline{\phi}\,\overline{Z}\right),\ \overline{X},\ \overline{Y},\ \overline{Z}\in\Gamma(T\overline{M}).
\end{aligned}$$

Let $(\overline{M},\overline{g})$ be a $(2n+1)$-dimensional semi-Riemannian manifold with index s, $0 < s < 2n+1$ and let (M,g) be a hypersurface of $\overline{M}$, with $g = \overline{g}_{|M}$. M is a lightlike hypersurface of $\overline{M}$ if g is of constant rank $2n-1$ and the normal bundle $TM^{\perp}$ is a distribution of rank 1 on M [4]. A complementary bundle of $TM^{\perp}$ in TM is a rank $2n-1$ non-degenerate distribution over M. It is called a *screen distribution* and is often denoted by $S(TM)$. A lightlike hypersurface endowed with a specific screen distribution is denoted by the triple $(M,g,S(TM))$. As $TM^{\perp}$ lies in the tangent bundle, the following result has an important role in studying the geometry of a lightlike hypersurface.

Theorem 1. **[4]** *Let $(M,g,S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold $(\overline{M},\overline{g})$. Then, there exists a unique vector bundle $N(TM)$ of rank 1 over M such that for any non-zero section E of $TM^{\perp}$ on a coordinate neighborhood $\mathcal{U}\subset M$, there exist a unique section N of $N(TM)$ on $\mathcal{U}$ satisfying*

(2.3) $\overline{g}(N,E) = 1$ *and* $\overline{g}(N,N) = \overline{g}(N,W) = 0,\ \forall\, W\in\Gamma(S(TM)|_{\mathcal{U}})$.

Throughout the paper, all manifolds are supposed to be paracompact and smooth. We denote Γ(E) the smooth sections of the vector bundle E. Also by $\perp$ and $\oplus$ we denote the orthogonal and nonorthogonal direct sum of two vector bundles. By Theorem 1 we may write down the following decomposition

$$\begin{aligned}
(2.4)\qquad TM &= S(TM)\perp TM^{\perp},\\
T\overline{M} &= TM\oplus N(TM) = S(TM)\perp (TM^{\perp}\oplus N(TM))
\end{aligned}$$

Let $\overline{\nabla}$ be the Levi-Civita connection on $(\overline{M},\overline{g})$, then by using the second decomposition of (2.4), we have Gauss and Weingarten formulae in the form

$$\begin{aligned}
(2.5)\qquad \overline{\nabla}_X Y &= \nabla_X Y + h(X,Y),\quad \forall X,\, Y\in\Gamma(TM),\\
(2.6)\text{ and}\qquad \overline{\nabla}_X V &= -A_V X + \nabla^{\perp}_X V,\quad \forall X\in\Gamma(TM),\, V\in\Gamma(N(TM)),
\end{aligned}$$

where $\nabla_X Y$, $A_V X \in \Gamma(TM)$ and $h(X,Y)$, $\nabla^{\perp}_X V \in \Gamma(N(TM))$. ∇ is a symmetric linear connection on M called an induced linear connection, $\nabla^{\perp}$ is a linear connection on the vector bundle $N(TM)$. h is a $\Gamma(N(TM))$-valued symmetric bilinear form and A_V is the shape operator of M concerning V.

Equivalently, consider a normalizing pair $\{E, N\}$ as in Theorem 1. Then (2.5) and (2.6) take the form, for any X, $Y \in \Gamma(TM|_{\mathcal{U}})$,

$$\overline{\nabla}_X Y = \nabla_X Y + B(X,Y)\, N \quad \text{and} \quad \overline{\nabla}_X N = -A_N X + \tau(X) N. \tag{2.7}$$

It is important to mention that the second fundamental form B is independent of the choice of screen distribution, in fact, from (2.7), we obtain

$$B(X,Y) = \overline{g}(\overline{\nabla}_X Y, E), \ \ \forall\, X,\, Y \in \Gamma(TM|_{\mathcal{U}}), \tag{2.8}$$
$$\tau(X) = \overline{g}(\nabla^{\perp}_X N, E), \ \ \forall\, X \in \Gamma(TM|_{\mathcal{U}}). \tag{2.9}$$

Let P be the projection morphism of TM on $S(TM)$ with respect to the orthogonal decomposition of TM. We have

$$\nabla_X PY = \nabla^*_X PY + C(X, PY) E \text{ and } \nabla_X E = -A^*_E X - \tau(X) E, \tag{2.10}$$

where $\nabla^*_X PY$ and $A^*_E X$ belong to $\Gamma(S(TM))$. C, A^*_E and ∇^* are called the local second fundamental form, the local shape operator and the induced connection on $S(TM)$. In general, the induced linear connection ∇ is not a metric connection and we have

$$(\nabla_X g)(Y,Z) = B(X,Y)\theta(Z) + B(X,Z)\theta(Y),$$

where θ is a differential 1-form locally defined on M by $\theta(\cdot) := \overline{g}(N, \cdot)$. Also, we have the following identities,

$$g(A^*_E X, PY) = B(X, PY), \ g(A^*_E X, N) = 0, \ B(X, E) = 0. \tag{2.11}$$

Finally, using (2.7), $\overline{R}$ and R are the curvature tensor fields of $\overline{M}$ and M are related as

$$\begin{aligned} \overline{R}(X,Y)Z &= R(X,Y)Z + B(X,Z)A_N Y - B(Y,Z)A_N X \\ &+ \{(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + \tau(X)B(Y,Z) - \tau(Y)B(X,Z)\}\, N, \end{aligned} \tag{2.12}$$

where $(\nabla_X B)(Y,Z) = X.B(Y,Z) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z)$. (2.13)

§3. Main Results

Let $(\overline{M}, \overline{\phi}, \xi, \eta, \overline{g})$ be an indefinite Sasakian manifold and (M, g) be its lightlike hypersurface, *tangent to the structure vector field ξ ($\xi \in TM$)*. If E is a local section of $TM^\perp$, then $\overline{g}(\overline{\phi}E, E) = 0$, and $\overline{\phi}E$ is tangent to M. Thus $\overline{\phi}(TM^\perp)$ is a distribution on M of rank 1 such that $\overline{\phi}(TM^\perp) \cap TM^\perp = \{0\}$. This enables us to choose a screen distribution $S(TM)$ such that it contains $\overline{\phi}(TM^\perp)$ as vector subbundle. We consider a local section N of $N(TM)$. Since $\overline{g}(\overline{\phi}\,N, E) = -\overline{g}(N, \overline{\phi}\,E) = 0$, we deduce that $\overline{\phi}\,E$ is also tangent to M and belongs to $S(TM)$. On the other hand, since $\overline{g}(\overline{\phi}\,N, N) = 0$, we see that the component of $\overline{\phi}\,N$ with respect to E vanishes. Thus $\overline{\phi}\,N \in \Gamma(S(TM))$. From (2.1), we have $\overline{g}(\overline{\phi}\,N, \overline{\phi}E) = 1$. Therefore, $\overline{\phi}(TM^\perp) \oplus \overline{\phi}(N(TM))$ (direct sum but not orthogonal) is a nondegenerate vector subbundle of $S(TM)$ of rank 2.

It is known [2] that if M is tangent to the structure vector field ξ, then, ξ *belongs to* $S(TM)$. Using this, and since $\overline{g}(\overline{\phi}E, \xi) = \overline{g}(\overline{\phi}N, \xi) = 0$, there exists a nondegenerate distribution D_0 of rank $2n-4$ on M such that

$$S(TM) = \left\{\overline{\phi}(TM^\perp) \oplus \overline{\phi}(N(TM))\right\} \perp D_0 \perp \langle\xi\rangle, \tag{3.1}$$

where $\langle\xi\rangle$ is the distribution spanned by ξ, that is, $\langle\xi\rangle = Span\{\xi\}$. It is easy to check that the distribution D_0 is invariant under $\overline{\phi}$, i.e. $\overline{\phi}(D_0) = D_0$.

Example 1. Let $\mathbb{R}^7$ be the 7-dimensional real number space. We consider $\{x_i\}_{1\leq i\leq 7}$ as cartesian coordinates on $\mathbb{R}^7$ and define with respect to the natural field of frames $\left\{\frac{\partial}{\partial x_i}\right\}$ a tensor field $\overline{\phi}$ of type $(1,1)$ by its matrix:

$$\begin{aligned}
&\overline{\phi}(\frac{\partial}{\partial x_1}) = -\frac{\partial}{\partial x_2}, \quad \overline{\phi}(\frac{\partial}{\partial x_2}) = \frac{\partial}{\partial x_1} + x_4\frac{\partial}{\partial x_7}, \quad \overline{\phi}(\frac{\partial}{\partial x_3}) = -\frac{\partial}{\partial x_4},\\
&\overline{\phi}(\frac{\partial}{\partial x_4}) = \frac{\partial}{\partial x_3} + x_6\frac{\partial}{\partial x_7}, \quad \overline{\phi}(\frac{\partial}{\partial x_5}) = -\frac{\partial}{\partial x_6}, \quad \overline{\phi}(\frac{\partial}{\partial x_6}) = \frac{\partial}{\partial x_5},\\
&\overline{\phi}(\frac{\partial}{\partial x_7}) = 0.
\end{aligned} \tag{3.2}$$

The differential 1-form η is defined by

$$\eta = dx_7 - x_4dx_1 - x_6dx_3. \tag{3.3}$$

The vector field ξ is defined by $\xi = \frac{\partial}{\partial x_7}$. It is easy to check (2.1) and thus $(\overline{\phi}, \xi, \eta)$ is an almost contact structure on $\mathbb{R}^7$. Finally we define metric $\overline{g}$ on $\mathbb{R}^7$ by

$$\begin{aligned}
\overline{g} \;&=\; (x_4^2-1)dx_1^2 - dx_2^2 + (x_6^2+1)dx_3^2 + dx_4^2 - dx_5^2 - dx_6^2 + dx_7^2\\
&-\; x_4dx_1\otimes dx_7 - x_4dx_7\otimes dx_1 + x_4x_6dx_1\otimes dx_3 + x_4x_6dx_3\otimes dx_1\\
&-\; x_6dx_3\otimes dx_7 - x_6dx_7\otimes dx_3,
\end{aligned} \tag{3.4}$$

with respect to the natural field of frames. It is easy to check that $\overline{g}$ is a semi-Riemannian metric and $(\overline{\phi}, \xi, \eta, \overline{g})$ given by (3.2)-(3.4) is a Sasakian structure on $\mathbb{R}^7$.

We now define a hypersurface M of $(\mathbb{R}^7, \overline{\phi}, \xi, \eta, \overline{g})$ as

$$M = \left\{(x_1, ..., x_7) \in \mathbb{R}^7 : x_5 = x_4\right\}. \tag{3.5}$$

Thus the tangent space TM is spanned by $\{U_i\}_{1\leq i\leq 6}$, where $U_1 = \frac{\partial}{\partial x_1}$, $U_2 = \frac{\partial}{\partial x_2}$, $U_3 = \frac{\partial}{\partial x_3}$, $U_4 = \frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_5}$, $U_5 = \frac{\partial}{\partial x_6}$, $U_6 = \xi$ and the 1-dimensional distribution $TM^\perp$ of rank 1 is spanned by E, where $E = \frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_5}$. It follows that $TM^\perp \subset TM$. Then M is a 6-dimensional lightlike hypersurface of $\mathbb{R}^7$. Also, the transversal bundle $N(TM)$ is spanned by $N = \frac{1}{2}\left(\frac{\partial}{\partial x_4} - \frac{\partial}{\partial x_5}\right)$. On the other hand, by using the almost contact structure of $\mathbb{R}^7$ and also by taking into account of the decomposition (3.1), the distribution D_0 is spanned by $\left\{F, \overline{\phi}F\right\}$, where $F = U_2$, $\overline{\phi}F = U_1 + x_4\xi$ and the distributions $\langle\xi\rangle$, $\overline{\phi}(TM^\perp)$ and $\overline{\phi}(N(TM))$ are spanned, respectively, by ξ, $\overline{\phi}E = U_3 - U_5 + x_6\xi$ and $\overline{\phi}N = \frac{1}{2}(U_3 + U_5 + x_6\xi)$. Hence M is a lightlike hypersurface of $\mathbb{R}^7$.

Moreover, from (2.4) and (3.1) we obtain the decomposition

$$\begin{aligned} TM &= \left\{\overline{\phi}(TM^\perp) \oplus \overline{\phi}(N(TM))\right\} \perp D_0 \perp <\xi> \perp TM^\perp, \\ T\overline{M} &= \left\{\overline{\phi}(TM^\perp) \oplus \overline{\phi}(N(TM))\right\} \perp D_0 \perp <\xi> \perp (TM^\perp \oplus N(TM)). \end{aligned} \tag{3.6}$$

Now, we consider the distributions on M,

$$D := TM^\perp \perp \overline{\phi}(TM^\perp) \perp D_0, \quad D' := \overline{\phi}(N(TM)). \tag{3.7}$$

Then D is invariant under $\overline{\phi}$ and

$$TM = D \oplus D' \perp \langle\xi\rangle. \tag{3.8}$$

Let us consider the local lightlike vector fields $U := -\overline{\phi}\,N$, $V := -\overline{\phi}\,E$. Then, from (3.8), any $X \in \Gamma(TM)$ is written as $X = RX + QX + \eta(X)\xi$, $QX = u(X)\,U$, where R and Q are the projection morphisms of TM into D and D', respectively, and u is a differential 1-form locally defined on M by $u(\cdot) := g(V, \cdot)$. Applying $\overline{\phi}$ to X and (2.1) (note that $\overline{\phi}^2\,N = -N$), we obtain $\overline{\phi}\,X = \phi\,X + u(X)\,N$, where ϕ is a tensor field of type $(1,1)$ defined on M by $\phi\,X := \overline{\phi}\,RX$ and we also have $\phi^2\,X = -X + \eta(X)\xi + u(X)\,U$, $\forall\, X \in \Gamma(TM)$. Now applying ϕ to $\phi^2\,X$ and since $\phi U = 0$, we obtain $\phi^3 + \phi = 0$, which shows that ϕ is an f-structure [11] of constant rank. We have the following useful identities

$$\nabla_X\xi = -\phi X, \tag{3.9}$$
$$B(X,\xi) = -u(X), \tag{3.10}$$
$$B(X,U) = C(X,V) \tag{3.11}$$
$$(\nabla_X u)Y = -B(X,\phi Y) - u(Y)\tau(X). \tag{3.12}$$

Lemma 1. *Let M be a lightlike hypersurface of an indefinite Sasakian manifold $\overline{M}$ with $\xi \in TM$. Then, M is $(D \perp \langle\xi\rangle, D')$-mixed totally geodesic if and only if, for any $X \in \Gamma(D \perp \langle\xi\rangle)$,*

$$A_N X \in \Gamma(\overline{\phi}(TM^{\perp}) \perp D_0 \perp \langle\xi\rangle). \tag{3.13}$$

Proof. By the definition, M is $(D \perp \langle\xi\rangle, D')$-mixed totally geodesic if and only if, for any $X \in \Gamma(D \perp \langle\xi\rangle)$, $B(X, U) = 0$, From (3.11) we obtain $u(A_N X) = \overline{g}(A_N X, V) = 0$. i.e. $A_N X \in \Gamma(D \perp \langle\xi\rangle)$. Given that $A_N X$ has no component in $\Gamma(TM^{\perp})$, then $A_N X \in \Gamma(\overline{\phi}(TM^{\perp}) \perp D_0 \perp \langle\xi\rangle)$. The converse is obvious by using (3.11). □

From (3.10), we have $B(\xi, U) = -1$. This means that the lightlike hypersurface M of an indefinite Sasakian manifold $\overline{M}$, with $\xi \in TM$, cannot be $(\langle\xi\rangle, D')$-mixed totally geodesic.

Lemma 2. *Let M be a lightlike hypersurface of an indefinite Sasakian space $\overline{M}^{2n+1}$ with $\xi \in TM$. Then, for any $X \in \Gamma(TM)$,*

$$\nabla_X U = -\sum_{i=1}^{2n-4} \frac{C(X, \overline{\phi}F_i)}{g(F_i, F_i)} F_i - \theta(X)\xi + C(X, U)E + \tau(X)U \tag{3.14}$$

Proof. From the definition of lightlike hypersurface of an indefinite Sasakian manifold through the local field of frames $\left\{\overline{\phi}E, \overline{\phi}N, \xi, E, F_i\right\}_{1 \le i \le 2n-4}$ on $\mathcal{U} \subset M$, where $\{F_i\}_{1 \le i \le 2n-4}$ is an orthonormal field of frames of D_0, we have, for any $X \in \Gamma(TM)$, $\nabla_X U = \sum_{i=1}^{2n-4} \lambda_i F_i + \varphi_1 \xi + \varphi_2 E + \varphi_3 V + \varphi_4 U$. From (2.7) and (2.10), we obtain $\lambda_i g(F_i, F_i) = g(\nabla_X U, F_i) = -\overline{g}(A_N X, \overline{\phi}F_i) = -C(X, \overline{\phi}F_i)$, $\varphi_1 = g(\nabla_X U, \xi) = -\overline{g}(\overline{\nabla}_X \overline{\phi}N, \xi) = -\theta(X)$, $\varphi_2 = g(\nabla_X U, N) = C(X, U)$, $\varphi_3 = g(\nabla_X U, U) = 0$ and $\varphi_4 = g(\nabla_X U, V) = \tau(X)$ which prove (3.14). □

We are now concerned with the structure equations of the immersions of a lightlike hypersurface in a semi-Riemannian manifold. Let us consider the pair $\{E, N\}$ on $\mathcal{U} \subset M$ (see Theorem 1) and by using (2.2) and (2.12), we have

$$\begin{aligned} &(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) = \tau(Y)B(X, Z) - \tau(X)B(Y, Z) \\ &+\frac{c-1}{4}\left(\overline{g}(\overline{\phi}Y, Z)u(X) - \overline{g}(\overline{\phi}X, Z)u(Y) - 2\overline{g}(\overline{\phi}X, Y)u(Z)\right). \end{aligned} \tag{3.15}$$

In the sequel, we need the following proposition.

Proposition 3. *Let M be a lightlike hypersurface of an indefinite Sasakian space form $\overline{M}(c)$ of constant curvature c with $\xi \in TM$. Then, the Lie derivative of the second fundamental form B with respect to ξ is given by*

$$(L_\xi B)(X, Y) = -\tau(\xi)B(X, Y), \ \forall\, X,\ Y \in \Gamma(TM). \tag{3.16}$$

Moreover, if $\tau(\xi) \neq 0$, then ξ is a Killing vector field with respect to the second fundamental form B if and only if M is totally geodesic

Proof. By replacing Z with ξ into (2.13) and using (3.9), we obtain

$$(\nabla_\xi B)(X,Y) = (L_\xi B)(X,Y) + B(\phi X, Y) + B(X, \phi Y). \tag{3.17}$$

Likewise, by replacing Z with X and X with ξ into (2.13) and also using (3.9) and (3.10), we have

$$(\nabla_X B)(\xi, Y) = -X.u(Y) + B(\phi X, Y) + u(\nabla_X Y). \tag{3.18}$$

Substracting (3.17) and (3.18), and using (3.12) we obtain

$$(\nabla_\xi B)(X,Y) - (\nabla_X B)(\xi,Y) = (L_\xi B)(X,Y) - u(Y)\tau(X). \tag{3.19}$$

From (3.15), the left hand side of (3.19) becomes

$$(\nabla_\xi B)(X,Y) - (\nabla_X B)(\xi,Y) = -u(Y)\tau(X) - \tau(\xi)B(X,Y). \tag{3.20}$$

The expressions (3.19) and (3.20) implies $(L_\xi B)(X,Y) = -\tau(\xi)B(X,Y)$. If $\tau(\xi) \neq 0$, the equivalence follows. $\square$

Note that the 1-form τ in (2.9) depends on the vector field E and it is easy to see that if $\overline{E} = \lambda E$ with λ a positive smooth function on M, the associated 1-form $\overline{\tau}$ is related to τ by

$$\tau(X) = \overline{\tau}(X) + X(\ln\lambda), \;\; \forall\, X \in \Gamma(TM|_{\mathcal{U}}). \tag{3.21}$$

The induced connection ∇ on the lightlike hypersurface M is not metric in general and the Ricci tensor associated is not symmetric, contrary to the case of semi-Riemannian manifolds. However, for η-Einstein (or Einstein) lightlike hypersurfaces, that is, the Ricci tensor Ric tensor satisfies $Ric(X,Y) = k_1 g(X,Y) + k_2\eta(X)\eta(Y)$ (or $Ric(X,Y) = kg(X,Y)$), due to the symmetric of the induced degenerate metric g and $\eta\otimes\eta$, the Ricci tensor is symmetric, and the notion of η-Einstein (respectively Einstein) manifold does not depend on the choice of the screen distribution $S(TM)$. Consequently

Proposition 4. *On a lightlike η-Einstein (respectively, Einstein) hypersurface, the 1-form τ in (2.9) is closed.*

Proof. Define the Ricci tensor Ric as

$$Ric(X,Y) = \mathrm{trace}(Z \longrightarrow R(X,Y)Z), \;\; \forall\, X,\, Y \in \Gamma(TM)$$

where R is the curvature tensor of the induced connection ∇.

Consider a local quasi-orthogonal frame field $\{X_0, N, X_i\}_{i=1,\dots,2n-1}$ on $\overline{M}$ where $\{X_0, X_i\}$ is a local frame field on M with respect to the decomposition (3.7) with N, the unique section of transversal bundle $N(TM)$ satisfying

(2.3), and $E = X_0$. Put $R_{ls} := Ric(X_s, X_l)$ and $R_{0k} := Ric(X_k, X_0)$. Using the frame field $\{X_0, N, X_i\}$, a direct calculation gives locally $R_{ls} - R_{sl} = 2d\tau(X_l, X_s)$ and $R_{0k} - R_{k0} = 2d\tau(X_0, X_k)$. Since the Ricci tensor is symmetric on M which is η-Einstein (respectively, Einstein), we have $d\tau = 0$. □

By definition $Ric(X,Y) = \mathrm{trace}(Z \longrightarrow R(X,Y)Z)$, we have, for any X, $Y \in \Gamma(TM)$,

$$\begin{aligned} Ric(X,Y) &= \sum_{i=1}^{2n-4} \varepsilon_i \overline{g}(R(F_i,X)Y,F_i) + \overline{g}(R(\xi,X)Y,\xi) + \overline{g}(R(E,X)Y,N) \\ &+ \overline{g}(R(\overline{\phi}E,X)Y,\overline{\phi}N) + \overline{g}(R(\overline{\phi}N,X)Y,\overline{\phi}E), \end{aligned} \tag{3.22}$$

where $\{F_i\}_{1\leq i\leq 2n-4}$ is an orthogonal basis of D_0 and $\varepsilon_i = g(F_i,F_i) \neq 0$, since the distribution D_0 is non-degenerate. From Gauss and Codazzi equations, we obtain

$$\begin{aligned} \overline{g}(R(F_i,X)Y,F_i) &= \frac{c+3}{4}\{\varepsilon_i\overline{g}(X,Y) - \overline{g}(X,F_i)\overline{g}(Y,F_i)\} \\ &+ \frac{c-1}{4}\big\{-\varepsilon_i\eta(X)\eta(Y) + \overline{g}(F_i,\overline{\phi}Y)\overline{g}(\phi X,F_i) \\ &+ \overline{g}(\overline{\phi}X,Y)\overline{g}(\phi F_i,F_i) + 2\overline{g}(\overline{\phi}X,F_i)\overline{g}(\phi Y,F_i)\big\} \\ &+ B(X,Y)C(F_i,F_i) - B(F_i,Y)C(X,F_i), \end{aligned} \tag{3.23}$$

$$\begin{aligned} \overline{g}(R(\xi,X)Y,\xi) &= \frac{c+3}{4}\{-\eta(Y)\eta(X) + \overline{g}(X,Y)\} \\ &+ \frac{c-1}{4}\{-\overline{g}(X,Y) + \eta(X)\eta(Y)\} + B(X,Y))C(\xi,\xi) \\ &- B(\xi,Y)C(X,\xi), \end{aligned} \tag{3.24}$$

$$\begin{aligned} \overline{g}(R(E,X)Y,N) &= \frac{c+3}{4}\overline{g}(X,Y) + \frac{c-1}{4}\{-\eta(X)\eta(Y) + u(Y)\theta(\phi X) \\ &- 2u(X)\theta(\phi Y)\}, \end{aligned} \tag{3.25}$$

$$\begin{aligned} \overline{g}(R(\overline{\phi}E,X)Y,\overline{\phi}N) &= \frac{c+3}{4}\{-u(Y)v(X) + \overline{g}(X,Y)\} + \frac{c-1}{4}\{-\eta(X)\eta(Y) \\ &+ 2u(\overline{\phi}X)v(\phi Y)\big\} + B(X,Y)C(\overline{\phi}E,\overline{\phi}N) \\ &- B(\overline{\phi}E,Y)C(X,\overline{\phi}N), \end{aligned} \tag{3.26}$$

$$\begin{aligned} \overline{g}(R(\overline{\phi}N,X)Y,\overline{\phi}E) &= \frac{c+3}{4}\{\overline{g}(X,Y) - u(X)v(Y)\} \\ &+ \frac{c-1}{4}\big\{-\eta(X)\eta(Y) - \theta(Y)u(\phi X) + 2v(\overline{\phi}X)u(\phi Y)\big\} \\ &+ B(X,Y)C(\overline{\phi}N,\overline{\phi}E) - B(\overline{\phi}N,Y)C(X,\overline{\phi}E). \end{aligned} \tag{3.27}$$

So substituting (3.23), (3.24), (3.25), (3.26) and (3.27) in (3.22) and by regrouping like terms, we have the following result.

Proposition 5. *Let M be a lightlike hypersurface of an indefinite Sasakian manifold $\overline{M}$ with $\xi \in TM$. Then the Ricci tensor Ric is given by, for any X, $Y \in \Gamma(TM)$,*

$$\begin{aligned} Ric(X,Y) &= ag(X,Y) - b\eta(X)\eta(Y) + B(X,Y)trA_N \\ &- B(A_N X, Y), \end{aligned} \tag{3.28}$$

where $a = \frac{(2n+1)(c+3)-8}{4}$, and $b = \frac{(2n+1)(c-1)}{4}$ and trace tr is written with respect to g restricted to $S(TM)$.

By Proposition 5 and using (3.10), we have the following useful identities

$$Ric(X,\xi) = 2(n-1)\eta(X) - u(X)trA_N + u(A_N X), \tag{3.29}$$

$$Ric(\xi,Y) = 2(n-1)\eta(Y) - u(Y)trA_N - B(A_N\xi, Y). \tag{3.30}$$

From (3.28), we have

$$Ric(X,Y) - Ric(Y,X) = B(A_N X, Y) - B(A_N Y, X). \tag{3.31}$$

This means that the Ricci tensor of a lightlike hypersurface M of an indefinite Sasakian space form $\overline{M}(c)$ is not symmetric in general. So, only some privileged conditions on the local second fundamental form of M may enable the Ricci tensor to be symmetric. It is easy to check, from (3.28), that the Ricci tensor of M is symmetric if and only if the shape operator of M is symmetric with respect to the second fundamental form of M. Also, the Ricci tensor of the induced connection ∇ of any totally geodesic lightlike hypersurface is symmetric.

Are there any others, with symmetric induced Ricci tensors, but not necessarily totally geodesic or shape operator symmetric with respect to the second fundamental form ? Here is one such class. First, we recall the definition of screen conformal lightlike hypersurfaces of a semi-Riemannian manifold $\overline{M}$.

A lightlike hypersurface $(M, g, S(TM))$ of a semi-Riemannian manifold is screen locally conformal if the shape operators A_N and A^*_E of M and its screen distribution $S(TM)$, respectively, are related by [5]

$$A_N = \varphi\, A^*_E \tag{3.32}$$

where φ is a non-vanshing smooth function on $\mathcal{U}$ in M. In case $\mathcal{U} = M$ the screen conformality is said to be global. Such a submanifold has some important and desirable properties, for instance, the integrability of its screen distribution (see [5] for details).

Theorem 2. *Let $(M, g, S(TM))$ be a locally (or globally) screen conformal lightlike hypersurface of an indefinite Sasakian manifold $(\overline{M}(c), \overline{g})$ of constant curvature c with $\xi \in TM$. Then the Ricci tensor of the induced connection ∇ is symmetric.*

Proof. From (3.31) and (3.32), we obtain

$$\begin{aligned} Ric(X,Y) - Ric(Y,X) &= B(A_N X, Y) - B(A_N Y, X) \\ &= \varphi\left(B(A_E^* X, Y) - B(A_E^* Y, X)\right) \\ &= \varphi g([A_E^*, A_E^*]Y, X) = 0. \end{aligned}$$

This complete the proof. ∎

Example 2. Let M be a hypersurface of $\mathbb{R}^7$, of Example 1, given by

$$x_5 = x_4,$$

where $(x_1, ..., x_7)$ is a local coordinate system for $\mathbb{R}^7$. As explained in Example 1, M is a lightlike hypersurface of $\mathbb{R}^7$ having a local quasi-orthogonal field of frames $\{U_1, U_2, U_3, U_4 = E, U_5, U_6 = \xi, N\}$ along M. Denote by $\overline{\nabla}$ the Levi-Civita connection on $\mathbb{R}^7$. Then, by straightforward calculations, we obtain

$$\begin{aligned} &\overline{\nabla}_{U_1} N = -\frac{1}{4}x_4 U_1 - \frac{1}{4}(x_4^2+1)\xi, \;\; \overline{\nabla}_{U_2} N = \overline{\nabla}_{U_4} N = \overline{\nabla}_{U_5} N = 0, \\ &\overline{\nabla}_{U_3} N = -\frac{1}{4}x_6 U_1 - \frac{1}{4}x_4 x_6 \xi, \;\; \overline{\nabla}_{\xi} N = \frac{1}{4}U_1 + \frac{1}{4}x_4\xi, \end{aligned} \tag{3.33}$$

$$\begin{aligned} &\nabla_{U_1} E = -\frac{1}{2}x_4 U_1 - \frac{1}{2}(x_4^2+1)\xi, \;\; \nabla_{U_2} E = \nabla_{U_4} E = \nabla_{U_5} E = 0, \\ &\nabla_{U_3} E = -\frac{1}{2}x_6 U_1 - \frac{1}{2}x_4 x_6 \xi, \;\; \nabla_{\xi} E = \frac{1}{2}U_1 + \frac{1}{2}x_4\xi. \end{aligned} \tag{3.34}$$

Using these equations above, the differential 1-form τ vanishes i.e. $\tau(X) = 0$, for any $X \in \Gamma(TM)$. So, from the Gauss and Weingarten formulae we infer

$$\begin{aligned} &A_N U_1 = \frac{1}{4}x_4 U_1 + \frac{1}{4}(x_4^2+1)\xi, \;\; A_N U_2 = A_N U_4 = A_N U_5 = 0, \\ &A_N U_3 = \frac{1}{4}x_6 U_1 + \frac{1}{4}x_4 x_6 \xi, \;\; A_N \xi = -\frac{1}{4}U_1 - \frac{1}{4}x_4\xi, \end{aligned} \tag{3.35}$$

$$\begin{aligned} &A_E^* U_1 = \frac{1}{2}x_4 U_1 + \frac{1}{2}(x_4^2+1)\xi, \;\; A_E^* U_2 = A_E^* U_4 = A_E^* U_5 = 0, \\ &A_E^* U_3 = \frac{1}{2}x_6 U_1 + \frac{1}{2}x_4 x_6 \xi, \;\; A_E^* \xi = -\frac{1}{2}U_1 - \frac{1}{2}x_4\xi. \end{aligned} \tag{3.36}$$

From (3.35) and (3.36), $A_N X = \frac{1}{2}A_E^* X$, for any $X \in \Gamma(TM)$ and $trA_N = 0$, i.e. the shape operator A_N is trace-free. Therefore, the hypersurface M of $\mathbb{R}^7$ is screen conformal and minimal. So, its screen distribution is integrable. The non-vanishing components of the second fundamental form B are given by

$$B(U_1, U_1) = -x_4, \;\; B(U_1, U_3) = -\frac{1}{2}x_6, \;\; B(U_1, U_6) = \frac{1}{2}. \tag{3.37}$$

From the above equations, it is easy to check that, $B(A_N U_i, U_j) = B(A_N U_j, U_i)$, for any $i \neq j$ and $i, j = 1, ..., 6$. Consequently, Ricci tensor of the induced connection ∇ on the hypersurface M of $\mathbb{R}^7$ is symmetric.

Also, we have

Theorem 3. *Let $(M, g, S(TM))$ be a totally contact geodesic lightlike hypersurface of an indefinite Sasakian manifold $(\overline{M}, \overline{g})$ with $\xi \in TM$. If the second fundamental form B of M is parallel, then the Ricci tensor of the induced connection ∇ is symmetric.*

Proof. M is said to be totally contact geodesic lightlike hypersurface of an indefinite Sasakian manifold $(\overline{M}, \overline{g})$ if the local second fundamental form B of M satisfies

$$B(X, Y) = \eta(X)B(Y, \xi) + \eta(Y)B(X, \xi) = -\eta(X)u(Y) - \eta(Y)u(X),$$

for any X, $Y \in \Gamma(TM)$ and its covariant derivative is given by

$$\begin{aligned}(\nabla_X B)(Y, Z) &= (u(X)\theta(Y) + g(\phi X, Y))u(Z)\\ &+ (u(X)\theta(Z) + g(\phi X, Z))u(Y)\\ &+ (\tau(X)u(Y) + B(X, \phi Y))\eta(Z)\\ &+ (\tau(X)u(Z) + B(X, \phi Z))\eta(Y).\end{aligned} \tag{3.38}$$

B is parallel if $(\nabla_Z B)(X, Y) = 0$, for any X, Y, $Z \in \Gamma(TM)$. Using (3.38), we have, for any $X \in \Gamma(TM)$, $0 = (\nabla_X B)(\xi, U) = \tau(X)$.

Proceed as in the proof of Proposition 4. Consider a local quasi-orthogonal frame field $\{X_0, N, X_i\}_{i=1,\dots,2n-1}$ on $\overline{M}$ where $\{X_0, X_i\}$ is a local frame field on M with respect to the decomposition (3.7) with N, the unique section of transversal bundle $N(TM)$ satisfying (2.3), and $E = X_0$. Put $R_{ls} := Ric(X_s, X_l)$ and $R_{0k} := Ric(X_k, X_0)$. Using the frame field $\{X_0, N, X_i\}$, we have locally $R_{ls} - R_{sl} = 2d\tau(X_l, X_s) = 0$ and $R_{0k} - R_{k0} = 2d\tau(X_0, X_k) = 0$ which complete the proof. □

Example 3. Let $(\mathbb{R}^5, \overline{g})$ be the 5-dimensional semi-Riemmannian manifold, where the metric $\overline{g}$ is given, with respect to the cartesian coordinates $\{x_i\}_{1\leq i\leq 5}$ on $\mathbb{R}^5$ and the natural field of frames $\{\frac{\partial}{\partial x_i}\}$, by

$$\begin{aligned}\overline{g} &= (x_3^2 - 1)dx_1^2 - dx_2^2 + dx_3^2 + dx_4^2 + dx_5^2 - x_3 dx_1 \otimes dx_5\\ &- x_3 dx_5 \otimes dx_1,\end{aligned} \tag{3.39}$$

We define with respect to the natural field of frames $\left\{\frac{\partial}{\partial x_i}\right\}$ a tensor field $\overline{\phi}$ of type $(1, 1)$ by its matrix:

$$\begin{aligned}&\overline{\phi}(\frac{\partial}{\partial x_1}) = -\frac{\partial}{\partial x_2}, \quad \overline{\phi}(\frac{\partial}{\partial x_2}) = \frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_5}, \quad \overline{\phi}(\frac{\partial}{\partial x_3}) = -\frac{\partial}{\partial x_4},\\ &\overline{\phi}(\frac{\partial}{\partial x_4}) = \frac{\partial}{\partial x_3} \text{ and } \overline{\phi}(\frac{\partial}{\partial x_5}) = 0.\end{aligned} \tag{3.40}$$

The differential 1-form η and the vector field ξ are defined, respectively, by

$$\eta = dx_5 - x_3 dx_1 \text{ and } \xi = \frac{\partial}{\partial x_5}. \tag{3.41}$$

It is easy to check $(\overline{\phi}, \xi, \eta, \overline{g})$ given by (3.39)-(3.41) is a Sasakian structure on $\mathbb{R}^5$.

Consider a hypersurface (M, g) in $\mathbb{R}^5$ given by the equation $x_4 = x_2$, where $(x_1, ..., x_5)$ is a local coordinate system for $\mathbb{R}^5$. The tangent space TM is spanned by $\{U_1, U_2, U_3, U_4\}$, where $U_1 = \frac{\partial}{\partial x_1}$, $U_2 = \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_4}$, $U_3 = \frac{\partial}{\partial x_3}$, $U_4 = \xi$, and the 1-dimensional distribution $TM^{\perp}$ of rank 1 is spanned by E with $E = \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_4}$. Also, the transversal bundle $N(TM)$ is spanned by $N = \frac{1}{2}\left(-\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_4}\right)$. It follows that $TM^{\perp} \subset TM$. Then M is a 4-dimensional lightlike hypersurface of $\mathbb{R}^5$ having a local quasi-orthogonal field of frames $\{U_1,\, U_2 = E,\, U_3,\, U_4 = \xi,\, N\}$ along M. Denote by $\overline{\nabla}$ the Levi-Civita connection on $\mathbb{R}^5$. Then, by straightforward calculations, we obtain

$$\overline{\nabla}_X N = 0, \quad \forall\, X \in \Gamma(TM).$$

Using these equations above, the differential 1-form τ vanishes i.e. $\tau(X) = 0$, for any $X \in \Gamma(TM)$. So, from the Gauss and Weingarten formulae we have

$$A_N X = 0, \;\; A^*_E X = 0 \text{ and } \nabla_X E = 0, \quad \forall\, X \in \Gamma(TM).$$

Therefore, by Duggal-Bejancu theorems (Theorem 2.2 and Theorem 2.7) in [4] the lightlike hypersurface M of $\mathbb{R}^5$ is totally geodesic and its distribution is parallel. Also, from the above equations, it is easy to check that $\eta(X)B(Y, \xi) + \eta(Y)B(X, \xi) = 0 = B(X, Y)$, for any $X,\, Y \in \Gamma(TM)$. So M is totally contact geodesic, parallel and admits a symmetric induced Ricci tensor.

On the other hand, by considering again the Example 1, since $B(U_1, U_6) = \frac{1}{2} \neq -\eta(U_1)u(U_6) - \eta(U_6)u(U_1) = 0$, the hypersurface M of $\mathbb{R}^7$ defined in the example 1 is not totally contact geodesic.

Based on discussion so far it would be appropriate to say that from the geometric point of view alone, the induced tensor *Ric* on M must be symmetric, as without this property one only obtains tensorial relations. Physically, *Ric* symmetric is essential. Consequently, as per above note, it is desirable to assume that only $d\tau$ vanishes locally (or globally) on M. Luckily, we have so far seen that there are large classes of hypersurfaces with symmetric Ricci tensor.

In particular, symmetric induced Ricci tensor has been useful in finding several good properties of lightlike hypersurfaces [5]. For these reasons, only symmetric induced Ricci tensor will be considered in the sequel of this paper. In this case, the weakly Ricci-symmetric notion in lightlike hypersurfaces becomes valid.

Next, we investigate the effect of weakly Ricci symmetric condition on the geometry of lightlike hypersurfaces in an indefinite Sasakian manifold.

Suppose that Ricci tensor of a lightlike hypersurface M of an indefinite Sasakian manifold $(\overline{M}, \overline{g})$ with $\xi \in TM$ is symmetric. A submanifold M is called a weakly Ricci symmetric if

$$\begin{aligned}(\nabla_X Ric)(Y,Z) &= \alpha(X)Ric(Y,Z) + \beta(Y)Ric(X,Z)\\ &+ \gamma(Z)Ric(Y,X),\end{aligned} \tag{3.42}$$

where α, β and γ are defined respectively by , for any $X \in \Gamma(TM)$, $g(X,\rho) = \alpha(X)$, $g(X,\delta) = \beta(X)$, $g(X,\kappa) = \gamma(X)$, are 1-forms called the *associated 1-forms* which are not zero simultaneously. We denote this kind of $2n$-dimensional submanifold by $(WRS)_{2n}$.

Note that the covariant derivative of the induced Ricci tensor on M (3.28) is given by, for any X, Y, $Z \in \Gamma(TM)$,

$$\begin{aligned}(\nabla_X Ric)(Y,Z) &= a\,(B(X,Y)\theta(Z) + B(X,Z)\theta(Y))\\ &+ b\left(\eta(Y)\overline{g}(\overline{\phi}X,Z) + \eta(Z)\overline{g}(\overline{\phi}X,Y)\right) + (\nabla_X B)(Y,Z)trA_N\\ &+ B(Y,Z)(X.trA_N) - (\nabla_X B)(A_N Y, Z).\end{aligned} \tag{3.43}$$

Also, for a lightlike η-Einstein hypersurface M, that is, the Ricci tensor Ric tensor satisfies $Ric(X,Y) = k_1 g(X,Y) + k_2\eta(X)\eta(Y)$, the functions k_1 and k_2 are not necessarily constant on M. Since M is tangent the structure vector field ξ in an indefinite Sasakian manifold, k_1 and k_2 satisfy

$$k_1 + k_2 = 2(n-1). \tag{3.44}$$

Theorem 4. *Let M be weakly Ricci symmetric lightlike η-Einstein (or Einstein) hypersurface of an indefinite Sasakian manifold $\overline{M}^{2n+1}(n > 1)$ with $\xi \in TM$. Then the 1-forms α and β satisfy $\alpha + \beta = 0$.*

Proof. Suppose that M is a $(WRS)_{2n}$ lightlike η-Einstein hypersurfaces of an indefinite Sasakian manifold $\overline{M}^{2n+1}(n > 1)$ with $\xi \in TM$. Since M is η-Einstein, $Ric(Y,Z) = k_1 g(X,Y) + k_2\eta(X)\eta(Y)$. So, from (3.42) and using (2.11), we obtain

$$\begin{aligned}&k_1\,(B(X,Y)\theta(Z) + B(X,Z)\theta(Y)) + k_2\,(\eta(Z)(\nabla_X\eta)Y + \eta(Y)(\nabla_X\eta)Z)\\ &+(\nabla_X k_1)g(Y,Z) + (\nabla_X k_2)\eta(Y)\eta(Z) = \alpha(X)\,(k_1 g(Y,Z) + k_2\eta(Y)\eta(Z))\\ &+\beta(Y)\,(k_1 g(X,Z) + k_2\eta(X)\eta(Z)) + \gamma(Z)\,(k_1 g(Y,X) + k_2\eta(Y)\eta(X))\,.\end{aligned} \tag{3.45}$$

Putting $Z=\xi$ in (3.45) and using (3.44), we have

$$
\begin{aligned}
-k_1u(X)\theta(Y)+k_2(\nabla_X\eta)Y &= (k_1+k_2)\alpha(X)\eta(Y)+(k_1+k_2)\beta(Y)\eta(X)\\
&+ \gamma(\xi)\left(k_1g(Y,X)+k_2\eta(Y)\eta(X)\right). \qquad (3.46)
\end{aligned}
$$

Again, taking $X=\xi$ in (3.46) and using the fact that $k_1+k_2\neq 0$ $(n>1)$, we get $\alpha(\xi)\eta(Y)+\beta(Y)+\gamma(\xi)\eta(Y)=0$, that is,

$$
\beta(Y)=-(\alpha(\xi)+\gamma(\xi))\eta(Y). \qquad (3.47)
$$

On the other hand, by taking $X=V$ in (3.46), we have $k_2(\nabla_V\eta)Y=(k_1+k_2)\alpha(V)\eta(Y)+k_1\gamma(\xi)u(Y)$ which implies, by taking $Y=U$, $\gamma(\xi)=0$. Use this and (3.47) in (3.46), we get $-k_1u(X)\theta(Y)+k_2(\nabla_X\eta)Y=(k_1+k_2)\left(\alpha(X)-\alpha(\xi)\eta(X)\right)\eta(Y)$, that is

$$
-k_1u(X)\theta(Y)+k_2(\nabla_X\eta)Y=(k_1+k_2)\left(\alpha(X)+\beta(X)\right)\eta(Y)
$$

which implies, for $Y=\xi$, $\alpha(X)=-\beta(X)$ and the proof is complete. □

Example 4. Let M be a hypersurface of $\mathbb{R}^5$ (indefinite Sasakian manifold defined in the Example 3) given by

$$
x_4=x_2,\quad x_3>0,
$$

where $(x_1,\dots,x_5)$ is a local coordinate system for $\mathbb{R}^5$. By proceeding as in Example 3, M is a totally geodesic lightlike hypersurface of $\mathbb{R}^5$ having a local quasi-orthogonal field of frames $\{U_1,\,U_2=E,\,U_3,\,\xi,\,N\}$, where $U_1=\frac{\partial}{\partial x_1}$, $E=\frac{\partial}{\partial x_2}+\frac{\partial}{\partial x_4}$, $U_3=\frac{\partial}{\partial x_3}$, $\xi=\frac{\partial}{\partial x_5}$, $N=\frac{1}{2}\left(-\frac{\partial}{\partial x_2}+\frac{\partial}{\partial x_4}\right)$ along M. Using (3.28), M is η-Einstein. This means that the induced Ricci tensor on M is symmetric and it is given by $Ric(X,Y)=ag(X,Y)-b\eta(X)\eta(Y)$, where nonzero constants a and b satisfy $a-b=2$. The non-vanishing components of the induced Ricci tensor on M are given by

$$
\begin{aligned}
&Ric(U_1,U_1)=ax_3^2-a,\quad Ric(U_3,U_3)=a,\\
&Ric(\xi,\xi)=2,\quad Ric(U_1,\xi)=-ax_3. \qquad (3.48)
\end{aligned}
$$

Using (3.43) and a direct calculation, it is easy to check that, for any X, Y, $Z\in\Gamma(TM)$,

$$
\begin{aligned}
(\nabla_X Ric)(Y,Z) &= \alpha(X)Ric(Y,Z)+\beta(Y)Ric(X,Z)\\
&+ \gamma(Z)Ric(Y,X), \qquad (3.49)
\end{aligned}
$$

where the associated 1-forms α, β and γ are defined explicitly by

$$
\begin{aligned}
&\alpha(\xi)=\beta(\xi)=\gamma(\xi)=0,\quad \alpha(U_1)=\beta(U_1)=\gamma(U_1)=0,\\
&\alpha(U_3)=\beta(U_3)=\gamma(U_3)=0,\quad \alpha(E)=-\beta(E)=\frac{b}{ax_3},\quad \gamma(E)=0. \qquad (3.50)
\end{aligned}
$$

The associated 1-forms α, β and γ are not zero simultaneously and $\alpha+\beta=0$.

Note that a lightlike Einstein hypersurface of a 3-dimensional indefinite Sasakian manifold, tangent to the structure vector field ξ, is Ricci flat. So, that hypersurface cannot be $(WRS)_2$.

A non-zero Ricci tensor Ric of lightlike hypersurface M is said to be cyclic parallel if $C\nabla Ric = 0$, namely, for any $X,\ Y,\ Z \in \Gamma(TM)$,

$$(\nabla_X Ric)(Y,Z) + (\nabla_Y Ric)(Z,X) + (\nabla_Z Ric)(X,Y) = 0. \tag{3.51}$$

Let M admits a cyclic parallel Ricci tensor. Then, we have

$$\begin{aligned} 0 &= (\nabla_X Ric)(Y,Z) + (\nabla_Y Ric)(Z,X) + (\nabla_Z Ric)(X,Y) \\ &= \alpha(X)Ric(Y,Z) + (\beta(X) + \gamma(X))Ric(Z,Y) \\ &+ \alpha(Y)Ric(Z,X) + (\gamma(Y) + \beta(Y))Ric(X,Z) \\ &+ \alpha(Z)Ric(X,Y) + (\gamma(Z) + \beta(Z))Ric(Y,X). \end{aligned} \tag{3.52}$$

Taking $Z = \xi$ in (3.52) and using (3.29) and (3.31)

$$\begin{aligned} &\alpha(X)\{2(n-1)\eta(Y) - u(Y)trA_N + u(A_NY)\} \\ &+(\beta(X) + \gamma(X))\{2(n-1)\eta(Y) - u(Y)trA_N - B(A_N\xi, Y)\} \\ &+\alpha(Y)\{2(n-1)\eta(X) - u(X)trA_N - B(A_N\xi, X)\} \\ &+(\gamma(Y) + \beta(Y))\{2(n-1)\eta(X) - u(X)trA_N + u(A_NX)\} \\ &+\alpha(\xi)Ric(X,Y) + (\gamma(\xi) + \beta(\xi))Ric(Y,X) = 0. \end{aligned} \tag{3.53}$$

Again, taking $Y = \xi$ in (3.53), using (3.11), (3.29) and (3.31), we have

$$\begin{aligned} &(2n-3)(\alpha(X) + \beta(X) + \gamma(X)) + (\alpha(\xi) + \beta(\xi) + \gamma(\xi))\{4(n-1)\eta(X) \\ &-2u(X)trA_N + u(A_NX) - B(A_N\xi, X)\} = 0. \end{aligned} \tag{3.54}$$

Putting $X = \xi$ in (3.54) and using (3.10), we get $3(2n-3)(\alpha(\xi)+\beta(\xi)+\gamma(\xi)) = 0$, that is

$$\alpha(\xi) + \beta(\xi) + \gamma(\xi) = 0. \tag{3.55}$$

Using (3.55) in (3.54), we have,

$$\alpha(X) + \beta(X) + \gamma(X) = 0,\ \ \forall\, X \in \Gamma(TM). \tag{3.56}$$

Therefore, we have

Theorem 5. *There exist no weakly Ricci symmetric screen locally (or globally) conformal lightlike hypersurfaces M of indefinite Sasakian manifolds $\overline{M}^{2n+1}$ with $\xi \in TM$ and cyclic parallel Ricci tensor if $\alpha + \beta + \gamma$ is not everywhere zero.*

By Theorem 4 and definition $(WRS)_{2n}$, it is easy to see that there are no weakly Ricci symmetric lightlike Einstein hypersurfaces, tangent to the structure vector field ξ, with cyclic parallel Ricci tensor.

If in (3.42) the 1-form α is replaced by 2α and β and γ are equal to α, then we have

$$\begin{aligned}(\nabla_X Ric)(Y,Z) &= 2\alpha(X)Ric(Y,Z)+\alpha(Y)Ric(X,Z)\\ &+ \alpha(Z)Ric(Y,X),\end{aligned} \tag{3.57}$$

where α is a non-zero 1-form defined by $\alpha(X) = g(X,\rho)$. A submanifold which satisfies (3.57) is called a *special weakly Ricci symmetric* submanifold and denoted by $(SWRS)_{2n}$.

Theorem 6. *There exist no special weakly Ricci symmetric screen locally (or globally) conformal (or Einstein) lightlike hypersurfaces M of an indefinite Sasakian manifold $\overline{M}^{2n+1}$ with $\xi \in TM$ and cyclic parallel Ricci tensor.*

Proof. Suppose that M is a special weakly Ricci symmetric screen locally (or globally) conformal (or Einstein) lightlike hypersurface M of an indefinite Sasakian manifold $\overline{M}^{2n+1}$ with $\xi \in TM$. If M admits a cyclic parallel Ricci tensor, then, from (3.56), we have $2\alpha(X)+\alpha(X)+\alpha(X) = 0$, $\forall\, X \in \Gamma(TM)$, that is $\alpha(X) = 0$ which contradicts the definition of $(SWRS)_{2n}$. $\square$

From the differential geometry of lightlike hypersurfaces, we recall the following result. The submanifold M is $(D \perp \langle\xi\rangle, D')$-mixed totally geodesic if for any $X \in \Gamma(D \perp \langle\xi\rangle)$, $Y \in \Gamma(D')$, $B(X,Y) = 0$. The Latter reduces to $B(X,U) = 0$, since the distribution D is of rank 1 and spanned by U.

Theorem 7. *Let M be a special weakly Ricci symmetric screen locally (or globally) conformal lightlike hypersurface of an indefinite Sasakian space $(\overline{M}^{2n+1}(c)$, $n > 1)$ of constant curvature c, with $\xi \in TM$. Then, M cannot be η-Einstein (or Einstein). Moreover, if the trace of A_N satisfies the partial differential equation $\xi \cdot trA_N - \tau(\xi)trA_N = 0$, M cannot be $(D \perp \langle\xi\rangle, D')$-mixed totally geodesic.*

Proof. Suppose that special weakly Ricci symmetric screen locally (or globally) conformal lightlike hypersurface M is η-Einstein (or Einstein). Then, from Theorem 4, for any $M \in \Gamma(TM)$, $2\alpha(X) = -\alpha(X)$, that is $\alpha(X) = 0$ which is inadmissible by the definition of $(SWRS)_{2n}$. So M cannot be Einstein.

Suppose now that M is $(D \perp \langle\xi\rangle, D')$-mixed totally geodesic.Since M is a special weakly Ricci lightlike hypersurface, so, by the use of of (3.57) we obtain

$$\begin{aligned}(\nabla_X Ric)(Y,\xi) &= 2\alpha(X)Ric(Y,\xi) + \alpha(Y)Ric(X,\xi) + \alpha(\xi)Ric(Y,X)\\ &= 2(2n-1)\alpha(X)\eta(Y) + (2n-1)\alpha(Y)\eta(X)\\ &- (2\alpha(X)u(Y) + \alpha(Y)u(X))\, trA_N + 2\alpha(X)u(A_NY)\\ &+ \alpha(Y)u(A_NX) + \alpha(\xi)Ric(Y,X). \end{aligned} \tag{3.58}$$

Replacing X with ξ and using (3.11), (3.58) becomes

$$\begin{aligned}(\nabla_\xi Ric)(Y,\xi) &= 2(2n-1)\alpha(\xi)\eta(Y) + 2(n-1)\alpha(Y)\\ &- 2\alpha(\xi)u(Y)trA_N + 2\alpha(\xi)u(A_NY)\\ &+ \alpha(\xi)\left((2n-1)\eta(Y) - u(Y)trA_N + u(A_NY)\right)\\ &= 3(2n-1)\alpha(\xi)\eta(Y) + 2(n-1)\alpha(Y)\\ &- 3\alpha(\xi)u(Y)trA_N + 3\alpha(\xi)u(A_NY). \end{aligned} \tag{3.59}$$

On the other hand, using $\overline{\phi}\xi = \phi\xi = 0$,

$$\begin{aligned}&(\nabla_\xi Ric)(Y,\xi) = \xi\cdot Ric(Y,\xi) - Ric(\nabla_\xi Y,\xi) - Ric(Y,\nabla_\xi\xi)\\ &= 2(n-1)\xi\cdot\eta(Y) - \xi\cdot u(Y)trA_N - u(Y)\xi\cdot trA_N\\ &+ \xi\cdot u(A_NY) - (2n-1)\eta(\nabla_\xi Y) + u(\nabla_\xi Y)trA_N - u(A_N\nabla_\xi Y)\\ &= \xi\cdot u(A_NY) - g(\nabla_\xi V,Y)trA_N - u(Y)\xi\cdot trA_N - u(A_N\nabla_\xi Y)\\ &= u(Y)(\tau(\xi)trA_N - \xi\cdot trA_N) + \xi\cdot u(A_NY) - u(A_N\nabla_\xi Y). \end{aligned} \tag{3.60}$$

From (3.59) and (3.60), we obtain

$$\begin{aligned}&3(2n-1)\alpha(\xi)\eta(Y) + 2(n-1)\alpha(Y) - 3\alpha(\xi)u(Y)trA_N + 3\alpha(\xi)u(A_NY)\\ &= u(Y)(\tau(\xi)trA_N - \xi\cdot trA_N) + \xi\cdot u(A_NY) - u(A_N\nabla_\xi Y). \end{aligned} \tag{3.61}$$

Substituting Y with ξ in (3.61), we obtain $8(n-1)\alpha(\xi) = 0$. Since $n > 1$, we have

$$\alpha(\xi) = 0. \tag{3.62}$$

Taking (3.62) in (3.61),

$$\begin{aligned}2(n-1)\alpha(Y) &= u(Y)(\tau(\xi)trA_N - \xi\cdot trA_N) + \xi\cdot u(A_NY)\\ &- u(A_N\nabla_\xi Y). \end{aligned} \tag{3.63}$$

Since M is a $(D\perp\langle\xi\rangle, D')$-mixed-totally geodesic, then, by Theorem 1, for any $Y\in\Gamma(D)$, $A_NY\in\Gamma(\overline{\phi}(TM^\perp)\perp D_0)$. Moreover, for any $Y\in\Gamma(D)$, $u(Y)=0$ and since the distribution D is invariant under $\overline{\phi}$, using (3.10), we have, $g(\nabla_\xi Y,V) = \overline{g}(A^*_E\xi,\overline{\phi}Y) = -u(\overline{\phi}Y) = 0$, that is, $\nabla_\xi Y\in\Gamma(D\perp\langle\xi\rangle)$.

As $g(\nabla_\xi Y, \xi) = 0$, then $\nabla_\xi Y \in \Gamma(D)$ and $A_N \nabla_\xi Y \in \Gamma(\overline{\phi}(TM^\perp) \perp D_0)$. So (3.63) becomes $2(n-1)\alpha(Y) = 0$, $\forall\, Y \in \Gamma(D)$ and since $n > 1$,

$$\alpha(Y) = 0, \quad \forall\, Y \in \Gamma(D). \tag{3.64}$$

Next, we compute $\alpha(U)$. Using (3.11) and (3.16), the right hand side of (3.63) is reduced to

$$\begin{aligned}
& u(Y)(\tau(\xi) trA_N - \xi \cdot trA_N) + \xi \cdot u(A_N Y) - u(A_N \nabla_\xi Y) \\
&= u(Y)(\tau(\xi) trA_N - \xi \cdot trA_N) + \xi \cdot C(Y, V) - C(\nabla_\xi Y, V) \\
&= u(Y)(\tau(\xi) trA_N - \xi \cdot trA_N) + \xi \cdot B(Y, U) - B(\nabla_\xi Y, U) \\
&= u(Y)(\tau(\xi) trA_N - \xi \cdot trA_N) + (L_\xi B)(Y, U) + B(\phi Y, U) + B(\nabla_\xi U, Y) \\
&= u(Y)(\tau(\xi) trA_N - \xi \cdot trA_N) - \tau(\xi) B(Y, U) + B(\phi Y, U) + B(\nabla_\xi U, Y).
\end{aligned} \tag{3.65}$$

From Lemma 2, we obtain

$$\begin{aligned}
\nabla_\xi U &= -\sum_{i=1}^{2n-4} \frac{C(\xi, \overline{\phi} F_i)}{g(F_i, F_i)} F_i - \theta(\xi)\xi + C(\xi, U)E + \tau(\xi)U \\
&= -\sum_{i=1}^{2n-4} \frac{g(A_N \xi, \overline{\phi} F_i)}{g(F_i, F_i)} F_i + C(\xi, U)E + \tau(\xi)U.
\end{aligned} \tag{3.66}$$

As M is a $(D \perp \langle\xi\rangle, D')$-mixed totally geodesic, again by Theorem 1, $A_N \xi \in \Gamma(\overline{\phi}(TM^\perp \perp D_0 \perp \langle\xi\rangle)$. Writing

$$A_N \xi = v(A_N \xi)V + \sum_{i=1}^{2n-4} \mu_i F_i + \eta(A_N \xi)\xi,$$

we have $\overline{g}(A_N \xi, \overline{\phi} F_i) = \mu_i \overline{g}(F_i, \overline{\phi} F_i) = 0$, since $\overline{g}(F_i, \overline{\phi} F_i) = -\overline{g}(\overline{\phi} F_i, F_i) = -\overline{g}(F_i, \overline{\phi} F_i)$, i.e. $2\overline{g}(F_i, \overline{\phi} F_i) = 0$. So (3.66) becomes

$$\nabla_\xi U = C(\xi, U)E + \tau(\xi)U \tag{3.67}$$

and if the trace trA_N satisfies the partial differential equation $\xi \cdot trA_N - \tau(\xi) trA_N = 0$, with the aid of (3.67) and $B(E, \cdot) = 0$, (3.65) becomes

$$2(n-1)\alpha(Y) = B(\phi Y, U), \forall\, Y \in \Gamma(TM). \tag{3.68}$$

As $n > 1$ and since $\phi U = 0$, so by taking $Y = U$ in (3.68), we obtain

$$\alpha(U) = 0, \tag{3.69}$$

From (3.62), (3.64) and (3.69), $\alpha(Y) = 0$, $\forall\, Y \in \Gamma(TM)$ which is inadmissible by the definition of $(SWRS)_{2n}$. Thus a special weakly Ricci symmetric lightlike hypersurface M cannot be a $(D \perp \langle\xi\rangle, D')$-mixed totally geodesic. ☐

Some particular cases of lightlike submanifolds of indefinite Sasakian manifolds have been studied by Duggal and Sahin in [6]. They showed that in a contact screen Cauchy-Riemann (SCR)-lightlike submanifolds or irrotational screen real lightlike submanifold of an indefinite Sasakian manifold, the minimality notion is equivalent to the trace-free of the shape operator A_N with respect to g restricted to $S(TM)$. Therefore, there exist lightlike hypersurfaces of indefinite Sasakian manifolds whose the trace of A_N satisfies the partial differential equation above.

Finally, we note that Theorems 5, 6 and 7 are also valid for any lightlike hypersurface M of an indefinite Sasakian manifold, tangent to the structure vector field ξ and whose $d\tau$ (or τ) vanishes locally (or globally) on M or shape operator A_N symmetric with respect to its second fundamental form.

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Fortuné Massamba
Department of Mathematics, University of Botswana
Private Bag 0022 Gaborone, Rep. of Botswana.
E-mail: massfort@yahoo.fr, massambaf@mopipi.ub.bw