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$\mathfrak{n}$-adic p-basis and Regular semi-local ring

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Abstract. Let $(R, \mathfrak{m})$ be a regular local ring of prime characteristic p and R' a Noetherian subring of R such that $R^p \subset R'$. In the previous paper Furuya and Niitsuma [1], introducing the concept of $\mathfrak{m}$-adic p-basis, the first and the second authors proved the following theorem: The regular local ring R has an $\mathfrak{m}$-adic p-basis over the subring R' if and only if R' is a regular local ring. In this paper, we generalize this result to semi-local rings.

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§1. Introduction

Let $(R, \mathfrak{m})$ be a regular local ring of prime characteristic p and R' a Noetherian subring of R such that $R^p \subset R'$. In Kimura and Niitsuma [3], T. Kimura and the second author proved the following theorem, which had been called Kunz's conjecture. Under the assumption that R is finite as an R'-module, the following two conditions are equivalent: (1) The regular local ring R has a p-basis over the subring R'. (2) R' is a regular local ring. In the previous paper Furuya and Niitsuma [1], introducing the concept of $\mathfrak{m}$-adic p-basis, the first and the second authors generalized this result to the non-finite situations as follows. The following two conditions are equivalent: (1) The regular local ring R has an $\mathfrak{m}$-adic p-basis over the subring R'. (2) R' is a regular local ring.

In this paper, we generalize the result just above to the case of semi-local rings by introducing the concept of having locally adic p-bases (see Definition 3.1) and of property (P) (see Definition 3.2).

Let R be a Noetherian semi-local ring of prime characteristic p and R' a Noetherian subring of R such that $R^p \subset R'$. Let $\mathfrak{n}$ be the Jacobson radical of R. We show that if a semi-local ring R has an $\mathfrak{n}$-adic p-basis over R', R has locally adic p-bases over R' which have the same cardinal number (Theorem 3.4), and the converse statement holds (Theorem 3.5).

Finally, using these theorems, we prove the following main theorem.

Theorem 3.7. Let R be a regular semi-local ring of prime characteristic p and R' a Noetherian subring of R such that $R^p \subset R'$. Let $\mathfrak{n}$ be the Jacobson radical of R. Then the following conditions are equivalent:

(1) R/R' has an $\mathfrak{n}$-adic p-basis.

(2) R' is regular and R/R' satisfies property (P).

§2. Preliminaries

All rings in this paper are commutative rings with identity elements. We always denote by p a prime number and denote by $|X|$ the cardinal number of a set X.

Let P be a ring and R a P-algebra with $\mathrm{char}(R) = p$. Let $\mathfrak{a}$ be an ideal of R. Let R^p denote the subring $\{x^p \mid x \in R\}$ of R , $\mathfrak{a}^{(p)}$ the ideal $\{x^p \mid x \in \mathfrak{a}\}$ of R^p, and $P[R^p]$ the subring of R generated by the set $\{ax^p \mid a \in P, x \in R\}$.

Let $(\Omega_P(R), d_{R/P})$ be the module of differentials of R over P. For a subset W of R, we denote by $d_{R/P}(W)$ the set $\{d_{R/P}(w) \mid w \in W\}$. If an R-module M is generated by a subset B of M, then we write $M := R\langle B\rangle$.

Definition 2.1 ([1], Definition 2.1). A subset W of R is called an $\mathfrak{a}$-*adic* p-*basis* of R/P if the following conditions are satisfied :

(1) W is p-independent over $P[R^p]$.

(2) R is the closure of the subring $T := P[R^p][W]$ in R for the $\mathfrak{a}$-adic topology, that is, $R = \bigcap_{r=1}^{\infty}(T + \mathfrak{a}^r)$.

(3) $\mathfrak{a}^r \cap T = (\mathfrak{a}_T)^r$ for every $r \geq 1$, where $\mathfrak{a}_T := \mathfrak{a} \cap T$.

Definition 2.2 ([2], Definition 2.2). Let M be an R-module. We call M an $\mathfrak{a}$-*adic free* R-*module* if there exists an R-submodule N of M such that:

(1) N is a free R-module.

(2) M is the closure of N in M for the $\mathfrak{a}$-adic topology, that is, $M = \bigcap_{r=1}^{\infty}(N + \mathfrak{a}^r M)$.

(3) $\mathfrak{a}^r M \cap N = \mathfrak{a}^r N$ for every $r \geq 1$.

In this case, we call a free basis of N an $\mathfrak{a}$-*adic free basis* of M.

Let $(R, \mathfrak{m}, L)$ be a Noetherian local ring with $\mathrm{char}(R) = \mathrm{char}(L) = p$, and R' a subring of R such that $R^p \subset R'$. Then R' is a local ring with the maximal ideal $\mathfrak{m}' := \mathfrak{m} \cap R'$. Putting $L' := R'/\mathfrak{m}'$, then we have that $L \supset L' \supset L^p$. Furthermore, let $g : R \to R/\mathfrak{m} = L$ be the canonical mapping. Then we have the following:

Lemma 2.3. Suppose that R/R' has an $\mathfrak{m}$-adic p-basis W. Then there are two subsets W_1 and W_2 of W such that $W = W_1 \cup W_2$, $W_1 \cap W_2 = \emptyset$, $g(W_2)$ ($g : W_2 \to g(W_2)$ is bijective) is a p-basis of L/L' and $|W_1| = \dim_L \mathfrak{m}/(\mathfrak{m}^2 + \mathfrak{m}'R)$. More precisely, W_1 is expressed as $\{y_1+z_1, \dots, y_r+z_r\}$, where $z_i \in R'[W_2]$ $(i = 1, \dots, r)$ and $\{y_1, \dots, y_r\} \subset \mathfrak{m}$ such that $\{\bar{y}_1, \dots, \bar{y}_r\}$ $\big(\bar{y}_i := y_i + (\mathfrak{m}^2 + \mathfrak{m}'R)\big)$ is a basis of the L-vector space $\mathfrak{m}/(\mathfrak{m}^2 + \mathfrak{m}'R)$. Furthermore, $\{y_1, \dots, y_r\} \cup W_2$ is an $\mathfrak{m}$-adic p-basis of R/R'.

Proof. Putting $R'' := R'[W]$, then R'' is a local ring with the maximal ideal $\mathfrak{m}'' := \mathfrak{m} \cap R''$. Since W is an $\mathfrak{m}$-adic p-basis of R/R', we have that $L = R/\mathfrak{m} = R''/\mathfrak{m}'' = L'[g(W)]$. It follows that there exists a subset W_2 of W such that $g(W_2)$ is a p-basis of L/L', where $g : W_2 \to g(W_2)$ is bijective. We set $W_1 := W - W_2$, and $W_1 := \{w_i \mid i \in I\}$. For each w_i, there exists $z_i \in R'[W_2]$ such that $g(w_i) = g(z_i)$, and hence $w_i - z_i := y_i \in \mathfrak{m}$. Therefore, we have that $W_1 = \{y_i + z_i \mid i \in I\}$. Since $R'' = R'[W_2][\{y_i\}]$, we see that $Y := \{y_i \mid i \in I\}$ is a p-basis of $R''/R'[W_2]$. Consequently, $Y \cup W_2$ is an $\mathfrak{m}$-adic p-basis of R/R'.

The canonical injection $R'' \to R$ induces an R''-module homomorphism $\mu : \Omega_{R'}(R'') \to \Omega_{R'}(R)$ such that $\mu \circ d_{R''/R'} = d_{R/R'}|_{R''}$. It follows that there exists an R-module homomorphism $\eta : R \otimes_{R''} \Omega_{R'}(R'') \to \Omega_{R'}(R)$ such that $\eta(a \otimes w) = a\mu(w)$ $(a \in R, w \in \Omega_{R'}(R''))$. Furthermore this induces an L-module homomorphism $\overline{\eta} : R \otimes_{R''} \Omega_{R'}(R'')/\mathfrak{m}(R \otimes_{R''} \Omega_{R'}(R'')) \to \Omega_{R'}(R)/\mathfrak{m}\Omega_{R'}(R)$. We claim that $\overline{\eta}$ is an isomorphism. Since $\Omega_{R'}(R/\mathfrak{m}^2) = \Omega_{R'}(R)/Rd_{R/R'}\mathfrak{m}^2$, we have $R/\mathfrak{m} \otimes_R \Omega_{R'}(R/\mathfrak{m}^2) \cong R/\mathfrak{m} \otimes_R \Omega_{R'}(R)$. Similarily, we have $R''/\mathfrak{m}'' \otimes_{R''} \Omega_{R'}(R''/\mathfrak{m}''^2) \cong R''/\mathfrak{m}'' \otimes_{R''} \Omega_{R'}(R'')$. It follows from $R''/\mathfrak{m}''^i \cong R/\mathfrak{m}^i$ $(i = 1, 2)$ that $R/\mathfrak{m} \otimes_R \Omega_{R'}(R/\mathfrak{m}^2) \cong R/\mathfrak{m} \otimes_{R''} \Omega_{R'}(R''/\mathfrak{m}''^2)$. Hence we get $R/\mathfrak{m} \otimes_{R''} \Omega_{R'}(R'') \cong R/\mathfrak{m} \otimes_R \Omega_{R'}(R)$ and so we have the following commutative diagram of isomorphisms:

$$\begin{tikzcd}
R/\mathfrak{m} \otimes_{R''} \Omega_{R'}(R'') \arrow[rr, "\overline{\eta}"] \arrow[dr] & & R/\mathfrak{m} \otimes_R \Omega_{R'}(R) \\
& R/\mathfrak{m} \otimes_R \Omega_{R'}(R/\mathfrak{m}^2) \arrow[ur] &
\end{tikzcd}$$

Since $Y \cup W_2$ is a p-basis of R''/R', $\Omega_{R'}(R'')$ is a free R''-module with a basis $\{d_{R''/R'}(x) \mid x \in Y \cup W_2\}$. Thus $R \otimes_{R''} \Omega_{R'}(R'')/\mathfrak{m}(R \otimes_{R''} \Omega_{R'}(R'')) = (R/m) \otimes_{R''} \Omega_{R'}(R'')$ is a free L-module with a basis $\{1 \otimes d_{R''/R'}(x) \mid x \in Y \cup W_2\}$, and hence $\Omega_{R'}(R)/\mathfrak{m}\Omega_{R'}(R)$ is a free L-module with a basis $\{\overline{d_{R/R'}(x)} \mid x \in Y \cup W\}$, where $\overline{d_{R/R'}(x)} := d_{R/R'}(x) + \mathfrak{m}\Omega_{R'}(R)$. By Kunz ([4], (6.7)), there is a canonical exact sequence

$$0 \longrightarrow \mathfrak{m}/(\mathfrak{m}^2 + \mathfrak{m}'R) \xrightarrow{\alpha} \Omega_{R'}(R)/\mathfrak{m}\Omega_{R'}(R) \longrightarrow \Omega_{L'}(L) \longrightarrow 0.$$

Denote by N the L-module $\Omega_{R'}(R)/\mathfrak{m}\Omega_{R'}(R)$, and denote by N_1 and N_2 the submodules of N generated by $\overline{d_{R/R'}(Y)}$ and $\overline{d_{R/R'}(W_2)}$ respectively. Then we see that $N = N_1 \oplus N_2$. Let $\pi : N \to N/N_2$ be the canonical mapping, and put $\varphi := \pi \circ \alpha$. Then it is easy to check that φ is an isomorphism. It follows that $|Y| = \dim_L \mathfrak{m}/(\mathfrak{m}^2 + \mathfrak{m}'R) := r < \infty$, and we may put $Y = \{y_1, \ldots, y_r\}$. Thus we see that $\mathfrak{m}/(\mathfrak{m}^2 + \mathfrak{m}'R)$ is a vector space over L with a basis $\{\bar{y}_1, \ldots, \bar{y}_r\}$, where $\bar{y}_i := y_i + (\mathfrak{m}^2 + \mathfrak{m}'R)$. $\square$

Let $(R, \mathfrak{p}_1, \ldots, \mathfrak{p}_r)$ be a Noetherian semi-local ring with $\mathrm{char}(R) = p$, where $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$ are the maximal ideals of R. Let R' be a Noetherian subring of R such that $R^p \subset R'$, and let $f_i : R \to R_{\mathfrak{p}_i}$ $(i = 1, \ldots, r)$ be the canonical mappings. Put $\mathfrak{p}'_i := \mathfrak{p}_i \cap R'$ $(i = 1, \ldots, r)$ and denote by L'_i the residue field of the local ring $R'_{\mathfrak{p}'_i}$ for each $i (i = 1, \ldots, r)$. Let $g_i : R_{\mathfrak{p}_i} \to L_i$ $(i = 1, \ldots, r)$ be the canonical mappings and set $h_i := g_i \circ f_i$. Then the following statement is essentially the same as Lemma 2.1 of Ono [6], so we omit the proof.

Lemma 2.4. If W is a subset of R such that W is p-independent over R' and $h_i(W)$ is p-independent over L'_i for each $i (i = 1, \ldots, r)$, then $R'[W]$ is Noetherian.

Let R be a ring with $\mathrm{char}(R) = p$ and W a subset of R. From now on, denote by $m(W)$ the set of all monomials $w_1^{e_1} \cdots w_n^{e_n}$, where $w_1, \ldots, w_n$ are distinct elements of W and $0 \leq e_i \leq p - 1$ $(i = 1, \ldots, n)$.

Lemma 2.5. Let $(R, \mathfrak{m}, L)$ be a Noetherian local ring with $\mathrm{char}(R) = p$, R' a Noetherian subring of R such that $R^p \subset R'$, and let W be a subset of R. Put $\mathfrak{m}' := \mathfrak{m} \cap R'$. Then the following conditions are equivalent:

(1) W is an $\mathfrak{m}$-adic p-basis of R/R'.

(2) $m(W)$ is an $\mathfrak{m}'$-adic free basis of R as an R'-module.

If these conditions are satisfied, then $R'[W]$ is Noetherian and R is faithfully flat over $R'[W]$.

Proof. (1) $\Rightarrow$ (2). Suppose that W is an $\mathfrak{m}$-adic p-basis of R/R'. We use the same notations as those of Lemma 2.3. Then $R'[W_2]$ is Noetherian by Lemma 2.4. Since W_1 is a finite set, $R'' := R'[W] = R'[W_2][W_1]$ is Noetherian. Furthermore we have that $\mathfrak{m}/(\mathfrak{m}^2 + \mathfrak{m}'R)$ is a vector space over L with a basis $\{\bar{y}_1, \ldots, \bar{y}_r\}$, where $\bar{y}_i := y_i + (\mathfrak{m}^2 + \mathfrak{m}'R)$. Thus we see that $\mathfrak{m} = \mathfrak{m}''R$, where $\mathfrak{m}'' := \mathfrak{m} \cap R''$. Since $R/\mathfrak{m}^n R \cong R''/\mathfrak{m}''^n$ for every $n \geq 1$ by (2) and (3) of Definition 2.1, the $\mathfrak{m}$-adic completion $\varprojlim R/\mathfrak{m}^n$ is faithfully flat over both R and R'' by Theorem 8.14 of Matsumura [5]. Hence R is faithfully flat over R''. It follows that $\mathfrak{m}'^n R \cap R'' = \mathfrak{m}'^n R''$ for every $n \geq 1$. Since

$R'' = R'[W]$, R'' is a free R'-module with a basis $m(W)$. Furthermore we see that $R = \bigcap_{n=1}^{\infty}(R'' + \mathfrak{m}'^n R)$ because $\mathfrak{m}^n \subset \mathfrak{m}^{(p)}R \subset \mathfrak{m}'R$ for sufficiently large n. Consequently, $m(W)$ is an $\mathfrak{m}'$-adic free basis of R as an R'-module.

(2) $\Rightarrow$ (1). Suppose that $m(W)$ is an $\mathfrak{m}'$-adic free basis of R as an R'-module. Then $m(W)$ is linearly independent over R'. Hence we see immediately that W is p-independent over R'. Further we see that $R = R'' + \mathfrak{m}^n$ $(n = 1, 2, \dots)$ and $\mathfrak{m} = \mathfrak{m}''R$, where $R'' := R'[W]$ and $\mathfrak{m}'' := \mathfrak{m} \cap R''$. Since $R''/\mathfrak{m}'R'' \cong R/\mathfrak{m}'R$, $R''/\mathfrak{m}'R''$ is Noetherian. The canonical injection $R'' \to R$ induces an isomorphism $R''/\mathfrak{m}'' \to R/\mathfrak{m} = L$. Therefore $(R''/\mathfrak{m}'R'', \mathfrak{m}''/\mathfrak{m}'R'', L)$ is a Noetherian local ring. Thus we have that $\dim_L \mathfrak{m}''/(\mathfrak{m}''^2 + \mathfrak{m}'R'') < \infty$.

Let $g : R \to R/\mathfrak{m}$ be the canonical mapping. Then we have $g(R'') = L'[g(W)] = L$. Hence there exists a subset W_2 of W such that $g(W_2)$ is a p-basis of L/L'. We set $W_1 := W - W_2$. Again, by Kunz([4], (6.7)), there is a canonical exact sequence

$$0 \to \mathfrak{m}''/(\mathfrak{m}''^2 + \mathfrak{m}'R'') \to \Omega_{R'}(R'')/\mathfrak{m}''\Omega_{R'}(R'') \to \Omega_{L'}(L) \to 0.$$

Since $W = W_1 \cup W_2$ is a p-basis of R''/R', $\Omega_{R'}(R'')$ is a free R''-module with a basis $d_{R''/R'}(W)$. In the same way as the proof of Lemma 2.3, we see that $|W_1| = \dim_{L''} \mathfrak{m}''/(\mathfrak{m}''^2 + \mathfrak{m}'R'') < \infty$. By Lemma 2.4, $R'[W_2]$ is Noetherian, and $R'' = R'[W_2][W_1]$ is also Noetherian. By a similar argument as that in the proof of (1) $\Rightarrow$ (2) in Lemma 2.5, we can show that R is faithfully flat over R''. Therefore we have that $\mathfrak{m}^n \cap R'' = \mathfrak{m}''^n R \cap R'' = \mathfrak{m}''^n$ for every $n \geq 1$. Thus we see that W is an $\mathfrak{m}$-adic p-basis of R/R'. $\square$

§3. $\mathfrak{n}$-adic p-basis

In this section we use the following notations: Let $(R, \mathfrak{p}_1, \dots, \mathfrak{p}_r)$ be a Noetherian semi-local ring with $\mathrm{char}(R) = p$ and R' a Noetherian subring of R such that $R^p \subset R'$. Then R' is a Noetherian semi-local ring with the maximal ideals $\{\mathfrak{p}'_1, \dots, \mathfrak{p}'_r\}$, where $\mathfrak{p}'_i := \mathfrak{p}_i \cap R'$ $(i = 1, \dots, r)$. We set $R_i := R_{\mathfrak{p}_i}$, $R'_i := R'_{\mathfrak{p}'_i}$, $\mathfrak{m}_i := \mathfrak{p}_i R_i$, $\mathfrak{m}'_i := \mathfrak{p}'_i R'_i$, $L_i := R_i/\mathfrak{m}_i$ and $L'_i := R'_i/\mathfrak{m}'_i$. Then we may assume that $R_i \supset R'_i \supset R_i^p = (R^p)_{\mathfrak{p}_i^{(p)}}$ and $L_i \supset L'_i \supset L_i^p$ for every i $(i = 1, \dots, r)$. Furthermore, we set $\mathfrak{n} := \bigcap_{i=1}^{r} \mathfrak{p}_i$ (the Jacobson radical of R) and $\mathfrak{n}' := \bigcap_{i=1}^{r} \mathfrak{p}'_i$ (the Jacobson radical of R'). Let $f_i : R \to R_i$ and $g_i : R_i \to L_i$ $(i = 1, \dots, r)$ be the canonical mappings.

Definition 3.1. We say that R/R' has *locally adic p-bases* when R_i/R'_i has an $\mathfrak{m}_i$-adic p-basis $W^{(i)}$ for every i $(i = 1, \dots, r)$. Additionally when $W^{(1)}, \dots, W^{(r)}$ have the same cardinal number, we say that R/R' has locally adic p-bases which have the same cardinal number.

Definition 3.2. We say that R/R' satisfies property (P) when one of the following conditions is satisfied:

(1) p-$\deg(L_i/L_i') = \infty\,(i = 1,\dots,r)$ and L_i/L_i' has a p-basis $D_i\,(i = 1,\dots,r)$ such that $D_1,\dots,D_r$ have the same cardinal number.

(2) p-$\deg(L_i/L_i') < \infty\,(i = 1,\dots,r)$ and $\dim_{L_i} \mathfrak{m}_i/(\mathfrak{m}_i^2+\mathfrak{m}_i'R_i)+p$-$\deg(L_i/L_i')$ $(i = 1,\dots,r)$ have the same value.

With the notations stated above, now we can prove the following statements.

Proposition 3.3. The following conditions are equivalent:

(1) R/R' has locally adic p-bases which have the same cardinal number.

(2) R/R' has locally adic p-bases and R/R' satisfies property (P).

Proof. (1) $\Rightarrow$ (2). Let $W^{(i)}$ be an $\mathfrak{m}_i$-adic p-basis of $R_i/R_i'\,(i = 1,\dots,r)$. Then $W^{(1)},\dots,W^{(r)}$ have the same cardinal number c. By Lemma 2.3, there are the two subsets $W_1^{(i)}$ and $W_2^{(i)}$ of $W^{(i)}$ such that $W_1^{(i)}$ is a finite set and $D_i := g_i(W_2^{(i)})$ is a p-basis of L_i/L_i' for every $i\,(i = 1,\dots,r)$. If c is infinite, then p-$\deg(L_i/L_i') = \infty$ and $|D_i| = |W_2^{(i)}| = |W^{(i)} - W_1^{(i)}| = |W^{(i)}| = c$, and hence $D_1,\dots,D_r$ have the same cardinal number. If c is finite, then p-$\deg(L_i/L_i') = |D_i| = |W_2^{(i)}|$ and $|W_1^{(i)}| = \dim_{L_i} \mathfrak{m}_i/(\mathfrak{m}_i^2 + \mathfrak{m}'{}_iR_i)(i = 1,\dots,r)$. Therefore it holds that p-$\deg(L_i/L_i')+\dim_{L_i} \mathfrak{m}_i/(\mathfrak{m}_i^2+\mathfrak{m}_i'R_i) = c\,(i = 1,\dots,r)$.

(2) $\Rightarrow$ (1). Let $W^{(i)}$ be an $\mathfrak{m}_i$-adic p-basis of $R_i/R_i'\,(i = 1,\dots,r)$. By Lemma 2.3, there are the two subsets $W_1^{(i)}$ and $W_2^{(i)}$ of $W^{(i)}$ such that $W^{(i)} = W_1^{(i)}\cup W_2^{(i)}$, $g_i(W_2^{(i)})$ is a p-basis of L_i/L_i' and $|W_1^{(i)}| = \dim_{L_i} \mathfrak{m}_i/(\mathfrak{m}_i^2+\mathfrak{m}_i'R_i) < \infty\,(i = 1,\dots,r)$. If p-$\deg(L_i/L_i') = \infty$ for any $i\,(i = 1,\dots,r)$, then property (P) implies that all $W_2^{(i)}$ have the same cardinal number. On the other hand, if p-$\deg(L_i/L_i') < \infty$ for any $i\,(i = 1,\dots,r)$, then $|W^{(i)}| = |W_1^{(i)}| + |W_2^{(i)}| = \dim_{L_i} \mathfrak{m}_i/(\mathfrak{m}_i^2+\mathfrak{m}_i'R_i)+p$-$\deg(L_i/L_i')$ so that $|W^{(1)}| = \cdots = |W^{(r)}|$ by property (P). Hence, all $W^{(i)}$ have the same cardinal number in either case. □

Theorem 3.4. If R/R' has an $\mathfrak{n}$-adic p-basis W, then $f_i(W)$ is an $\mathfrak{m}_i$-adic p-basis of $R_i/R_i'\,(i = 1,\dots,r)$ and W, $f_1(W),\dots,f_r(W)$ have the same cardinal number.

Proof. Putting $R'' := R'[W]$, then R'' is a semi-local ring with the maximal ideals $\{\mathfrak{p}_1'',\dots,\mathfrak{p}_r''\}$, where $\mathfrak{p}_i'' = \mathfrak{p}_i \cap R''\,(i = 1,\dots,r)$. We set $\mathfrak{n}'' := \mathfrak{p}_1'' \cap \cdots \cap \mathfrak{p}_r''$. Under the assumption, we see that $R''/\mathfrak{n}''^n \cong R/\mathfrak{n}^n$ for every $n \geq 1$. Putting $R_i'' := R''_{\mathfrak{p}_i''}$, $\mathfrak{m}_i'' := \mathfrak{p}_i''R_i''$ and $L_i'' := R_i''/\mathfrak{m}_i''$, then we may assume that $R_i \supset R_i'' \supset R_i' \supset R_i^p$ and $L_i \supset L_i'' \supset L_i' \supset L_i^p\,(i = 1,\dots,r)$. We put

$W^{(i)} := f_i(W)$, then we have that $R''_i = R'_i[W^{(i)}]$ and $L''_i = L'_i[g_i(W^{(i)})]\,(i = 1,\dots,r)$. It is known that $W^{(i)}$ is p-independent over R'_i. Hence we see that $|W| = |W^{(i)}|\,(i = 1,\dots,r)$. Since $R_i = U_i^{-1}R\,(U_i := R^p - \mathfrak{p}_i^{(p)})$ and $R = R'' + \mathfrak{n}^n\,(n \geq 1)$, we see that $R_i = R''_i + \mathfrak{m}_i^n\,(n \geq 1)$. This means that R_i is the closure of R''_i in R_i with the $\mathfrak{m}_i$-adic topology for every $i(i = 1,\dots,r)$. It is easy to check that $L_i = L''_i$. Thus there are subsets $W_1^{(i)}$ and $W_2^{(i)}$ of $W^{(i)}$ such that $g_i(W_2^{(i)})$ is a p-basis of L_i/L'_i, $g_i : W_2^{(i)} \to g_i(W_2^{(i)})$ is bijective, $W^{(i)} = W_1^{(i)} \cup W_2^{(i)}$ and $W_1^{(i)} \cap W_2^{(i)} = \emptyset$. As the same way in the proof of Lemma 2.5, we see that $|W_1^{(i)}| = \dim_{L_i} \mathfrak{m}_i/(\mathfrak{m}_i^2 + \mathfrak{m}'_i R_i) < \infty\,(i = 1,\dots,r)$. We set $W_1^{(i)} := \{w_1^{(i)},\dots,w_{s_i}^{(i)}\}$. Lemma 2.3 says that there exists a subset $B^{(i)}$ of $\mathfrak{m}_i$ such that $R''_i = R'_i[B^{(i)}, W_2^{(i)}]$ and the image of $B^{(i)}$ in $\mathfrak{m}_i/(\mathfrak{m}_i^2 + \mathfrak{m}'_i R_i)$ forms a basis for this L_i-vector space. Hence $\mathfrak{m}_i = R_i\langle B^{(i)}\rangle + \mathfrak{m}'_i R_i = \mathfrak{m}''_i R_i$.

Now we shall show that $R'[W]$ is Noetherian. There are subsets W_{i1} of W such that $W_1^{(i)} = f_i(W_{i1})$ and $f_i : W_{i1} \to W_1^{(i)}$ is bijective $(i = 1,\dots,r)$. We set $W_1 := W_{11} \cup W_{21} \cup \cdots \cup W_{r1}$ and $W_2 := W - W_1$. By Lemma 2.4, we see that $R'[W_2]$ is Noetherian, and thus $R'[W] = R'[W_2][W_1]$ is Noetherian, since W_1 is a finite set.

By a similar argument as that in the proof of (1) $\Rightarrow$ (2) in Lemma 2.5, we can show that R is faithfully flat over R''.

Lastly we shall show that $\mathfrak{m}_i^n \cap R''_i = \mathfrak{m}''^n_i\,(n \geq 1)$. Since R is faithfully flat over R'', R_i is also faithfully flat over R''_i. It follows that $\mathfrak{m}_i^n \cap R''_i = \mathfrak{m}''^n_i R_i \cap R''_i = \mathfrak{m}''^n_i\,(n \geq 1)$. Thererfore $W^{(i)}$ is an $\mathfrak{m}_i$-adic p-basis of $R_i/R'_i\,(i = 1,\dots,r)$. □

Theorem 3.5. If R/R' has locally adic p-bases which have the same cardinal number, then R/R' has an $\mathfrak{n}$-adic p-basis.

Proof. Let $W^{(i)}$ be an $\mathfrak{m}_i$-adic p-basis of $R_i/R'_i\,(i = 1,\dots,r)$ and let $W^{(i)}\,(i = 1,\dots,r)$ have the same cardinal number. By the assumption, $W^{(i)}$ is expressed as $\{w_j^{(i)} \mid j \in J\}$ with J a set of suffixes for every $i\,(i = 1,\dots,r)$. Since $R_i = R_{\mathfrak{p}_i^{(p)}}$, we may assume that there is a set $A_i := \{a_{ij} \in R \mid j \in J\}$ such that $f_i(a_{ij}) = w_j^{(i)}$. Since $\mathfrak{p}_i^{(p)}\,(i = 1,\dots,r)$ are the distinct maximal ideals of R^p, there exist elements $b_i\,(i = 1,\dots,r)$ of R^p such that $b_i \notin \mathfrak{p}_i^{(p)}$ and $b_i \in \cap_{j \neq i}\mathfrak{p}_j^{(p)}$. Put $x_j := b_1 a_{1j} + \cdots + b_r a_{rj}$, $X := \{x_j \mid j \in J\}$, $x_j^{(i)} := f_i(x_j)$ and $X^{(i)} := \{x_j^{(i)} \mid j \in J\}$. By Lemma 2.5, $R''_i := R'_i\langle m(W^{(i)})\rangle$ is a free R'_i-module with a basis $m(W^{(i)})$. Then $R''_i/\mathfrak{m}'_i R''_i = L'_i\langle \overline{m(W^{(i)})}\rangle$ is a free L'_i-module with a basis $\overline{m(W^{(i)})}$, where $\overline{m(W^{(i)})} := \{x + m'_i R''_i \mid x \in m(W^{(i)})\}$. We remark that $R''_i = R'_i[W^{(i)}]$, and so R''_i is a subring of R_i. Since $R''_i/\mathfrak{m}'_i R''_i \cong R_i/\mathfrak{m}'_i R_i$, we see that $R_i/\mathfrak{m}'_iR_i$ is a free L'_i-module with a basis $\{x+\mathfrak{m}'_iR_i \mid x \in m(W^{(i)})\}$. It is easy to check that $x^{(i)}_j+\mathfrak{m}'_iR_i = u_iw^{(i)}_j+\mathfrak{m}'_iR_i$ in $R_i/\mathfrak{m}'_iR_i$, where $u_i := \underline{f_i(b_i)}$ is an unit in $\underline{(R_i)^p}$. Hence we have that $R_i/\mathfrak{m}'_iR_i = L'_i\langle\overline{m(W^{(i)})}\rangle = L'_i\langle\overline{m(X^{(i)})}\rangle$, where $\overline{m(X^{(i)})} := \{x+\mathfrak{m}'_iR_i \mid x \in m(X^{(i)})\}$. Thus $R_i/\mathfrak{m}'_iR_i$ is a free L'_i-module with a basis $\overline{m(X^{(i)})}$. It follows from Suzuki ([7], Theorem 1) that $m(X^{(i)})$ is an $\mathfrak{m}'_i$-adic free basis of the R'_i-module R_i. Therefore $X^{(i)}$ is an $\mathfrak{m}_i$-adic p-basis of R_i/R'_i by Lemma 2.5. We set $T := R'[X]$, $\mathfrak{q}_i := \mathfrak{p}_i \cap T$, and $T_i := T_{\mathfrak{q}_i}$. Then $(T, \mathfrak{q}_1, \dots, \mathfrak{q}_r)$ is a semi-local ring. We may assume that $R_i \supset T_i \supset R'_i$. Furthermore, we see that $T_i = R'_i[X^{(i)}]$. Since $X^{(i)}$ is p-independent over R'_i for every $i\,(i = 1, \dots, r)$, X is p-independent over R'. Furthermore, since $X^{(i)}$ is an $\mathfrak{m}_i$-adic p-basis of R_i/R'_i, then R_i is faithfully flat over T_i for every $i\,(i = 1, \dots, r)$ by Lemma 2.5, and hence R is faithfully flat over T. For any $a \in R$, since $R_i = T_i+\mathfrak{m}'_iR_i$, there are elements $t_i \in T$, $s_i \in \mathfrak{p}'_iR$ and $u_i \in R^p-\mathfrak{p}^{(p)}_i$ such that $u_ia = s_i+t_i\,(i = 1, \dots, r)$. Since $\mathfrak{p}'_i\,(i = 1, \dots, r)$ are the distinct maximal ideals of R', there exist elements $c_i\,(i = 1, \dots, r)$ of R' such that $c_i \notin \mathfrak{p}'_i$ and $c_i \in \cap_{j\neq i}\mathfrak{p}'_j$. Putting $g := c_1u_1+\dots+c_ru_r$, then we see that g is a unit of R'. For any $a \in R$, $a = g^{-1}ga = g^{-1}(c_1s_1+\dots+c_rs_r)+g^{-1}(c_1t_1+\ldots+c_rt_r) \in T+\mathfrak{n}'R$. This implies that $R = T + \mathfrak{n}'R$. Therefore we have that $R = T + \mathfrak{n}^n\,(n \geq 1)$. We set $\mathfrak{a} := \mathfrak{q}_1\cap\dots\cap\mathfrak{q}_r$ (the Jacobson radical of T). Then we see that $\mathfrak{n} = \mathfrak{a}R$. Furthermore, since R is faithfully flat over T, it holds that $\mathfrak{n}^n\cap T = \mathfrak{a}^n\,(n \geq 1)$. Consequently, X is an $\mathfrak{n}$-adic p-basis of R/R'. □

Corollary 3.6. The following two conditions are equivalent:

(1) R/R' has an $\mathfrak{n}$-adic p-basis.

(2) R/R' has locally adic p-bases which have the same cardinal number.

Now we shall prove the generalization of Theorem 3.4 of Furuya and Niitsuma [1]. Before the proof, we notice that a regular semi-local ring is Noetherian.

Theorem 3.7. Let R be a regular semi-local ring with $\mathrm{char}(R) = p$, R' a Noetherian subring of R such that $R^p \subset R'$ and let $\mathfrak{n}$ be the Jacobson radical of R. Then the following two conditions are equivalent:

(1) R/R' has an $\mathfrak{n}$-adic p-basis.

(2) R' is regular and R/R' satisfies property (P).

Proof. (1) $\Rightarrow$(2). Since R/R' has an $\mathfrak{n}$-adic p-basis, R_i/R'_i has an $\mathfrak{m}_i$-adic p-basis by Theorem 3.4. It follows from Furuya and Niitsuma ([1], Theorem 3.4) that R'_i is regular. Therefore R' is regular.

(2) $\Rightarrow$ (1). Since R' is regular, R'_i is a regular local ring. Hence R_i/R'_i has an $\mathfrak{m}_i$-adic p-basis by Theorem 3.4 of [1]. Thus R/R' has locally adic p-bases

which have the same cardinal number by Proposition 3.3. Therefore R/R' has an $\mathfrak{n}$-adic p-basis by Theorem 3.5. $\square$

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References

[1] M. Furuya, H. Niitsuma, $\mathfrak{m}$-adic p-basis and regular local ring, *Proc. Amer. Math. Soc.,* **132** (11) (2004), 3189–3193.

[2] M. Furuya, H. Niitsuma, Regularity of Noetherian local rings, *J. Algebra,* **279** (2004), 214–225.

[3] T. Kimura and H. Niitsuma, Regular local ring of prime characteristic p and p-basis, *J. Math. Soc. Japan*, **32** (1980), 363–371.

[4] E. Kunz, *Kähler Differentials*, Vieweg Advanced lectures notes in Mathematics, Braunschweig/Wiesbaden: Vieweg & Sohn, 1986.

[5] H. Matsumura, *Commutative Ring Theory*, Cambridge Studies in Advanced Mathematics, Vol. 8. Cambridge: Cambridge University Press, 1986.

[6] T. Ono, On the existence of p-bases and $\mathfrak{a}$-adic p-bases of a semi-local ring extension, *Comm. Alg.*, **34** (2006), 1897–1907.

[7] S. Suzuki, Some results on Hausdorff $\mathfrak{m}$-adic modules and $\mathfrak{m}$-adic differentials, *J. Math. Kyoto Univ.*, **2** (1962), 157–182.

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