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On Trans-Sasakian manifolds

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Abstract. The notion of *generalized η-Einstein trans-Sasakian manifold* is introduced. Conformally flat trans-Sasakian manifolds are studied and introduced the idea of a manifold of *hyper generalized quasi-constant curvature* with various non-trivial examples.

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§1. Introduction

Recently, Oubina ([1]) introduced the notion of trans-Sasakian manifolds which contains both the class of Sasakian and cosymplectic structures and are closely related to the locally conformal Kähler manifolds. A trans-Sasakian manifold of type (0, 0), $(\alpha, 0)$ and $(0, \beta)$ are the cosymplectic, α-Sasakian and β-Kenmotsu manifold, respectively. The object of the present paper is to study conformally flat trans-Sasakian manifolds. Section 2 is concerned with some curvature identities of trans-Sasakian manifolds. In section 3, we introduce the notion of *generalized η-Einstein trans-Sasakian manifolds* and proved that in such a manifold the scalars $2n(\alpha^2-\beta^2-\xi\beta)$ and $\frac{r}{2n}-(\alpha^2-\beta^2-\xi\beta)$ are the Ricci curvatures in the direction of the vector fields associated with the 1-forms of the manifold and satisfies the inequality $\omega(\phi\ (\text{grad}\ \alpha)) < \frac{1}{\sqrt{2}}q+(2n-1)\omega(\text{grad}\ \beta)$ where q is the length of the Ricci tensor and ω is the associated non-zero 1-form. In 1972, Chen and Yano introduced the notion of a manifold of *quasi-constant curvature* ([3]). Generalizing this notion, M. C. Chaki ([4]) introduced the idea of a manifold of *generalized quasi-constant curvature* . It is shown that a 3-dimensional generalized η-Einstein trans-Sasakian manifold is a manifold of *generalized quasi-constant curvature.*

In 2000, M. C. Chaki and R. K. Ghosh ([4]) introduced the notion of quasi-Einstein manifold and then studied by various authors ([5], [14]). The same notion is also introduced and studied by R. Deszcz and his co-authors in several papers ([7], [8], [9], [10]). The existence and applications of quasi-Einstein manifolds have been studied by various authors. The notion of η-Einstein manifold for contact structures is an analogous situation as the quasi-Einstein manifold.

In 2001, M. C. Chaki ([5]) introduced the notion of generalized quasi-Einstein manifold and studied its geometrical significance as well as its applications to the general relativity and cosmology ([6]). Subsequently, the physical significance of the generalized quasi-Einstein manifold is interpreted in ([14]).

The notion of *generalized quasi-Einstein manifold* by Chaki stands an analogous situation to that of the *generalized η-Einstein trans-Sasakian manifold.* Thus the notion of *generalized η-Einstein manifold* is geometrically and physically important.

Section 4 deals with a conformally flat trans-Sasakian manifold. As an extension of *generalized η-Einstein trans-Sasakian manifold*, we introduce the notion of *hyper generalized η-Einstein trans-Sasakian manifold.* Especially, if the associated vector fields ρ and λ of the corresponding 1-forms ω and π of the *hyper generalized η-Einstein trans-Sasakian manifold* are linearly dependent, then it reduces to the notion of *generalized η-Einstein trans-Sasakian manifold.* The characteristic vector field ξ is always orthogonal to the associated vector field ρ but ξ is not necessarily orthogonal to the associated vector field λ, where $\omega(X) = g(X, \rho)$ and $\pi(X) = g(X, \lambda)$ for all X. In particular, if ρ and λ are linearly dependent, then ξ is orthogonal to both the vector fields ρ and λ in which case the notion reduces to the *generalized η-Einstein trans-Sasakian manifold.*

As in the case of *generalized η-Einstein trans-Sasakian manifold*, the notion of *hyper generalized η-Einstein trans-Sasakian manifold* is equally geometrically and physically importance. Not only that but also one can easily extend the notion of *generalized quasi-Einstein manifold* to the notion of *hyper generalized quasi-Einstein manifold* for the Riemannian case and study their geometrical significance as well as its applications to the general relativity and cosmology. It is proved that a conformally flat trans-Sasakian manifold is a *hyper generalized η-Einstein trans-Sasakian manifold.* It is shown that a conformally flat trans-Sasakian manifold is an η-Einstein if and only if ϕ (grad α) $= (2n-1)$ (grad β). Also it is proved that a conformally flat trans-Sasakian manifold is a *generalized η-Einstein manifold* if and only if the structure function β is a non-vanishing constant.

The notion of generalized quasi-constant curvature introduced by Chaki ([6]) is a geometrically important concept as its existence and physical interpretation is given by Chaki ([6]) and also by various authors ([14]). In this section we also introduce the notion of *hyper generalized quasi-constant curvature*.

Especially, if the associated vector fields ρ and λ of the corresponding 1-forms ω and π of the *hyper generalized quasi-constant curvature* are linearly dependent, then it reduces to the notion of *generalized quasi-constant curvature*. The characteristic vector field ξ is always orthogonal to the associated vector field ρ but ξ is not necessarily orthogonal to the associated vector field λ , where $\omega(X) = g(X, \rho)$ and $\pi(X) = g(X, \lambda)$ for all X. In particular, if ρ and λ are linearly dependent, then ξ is orthogonal to both the vector fields ρ and λ in which case the notion reduces to the *generalized quasi-constant curvature*.

It is proved that a conformally flat trans-Sasakian manifold of dimension greater than three is of quasi-constant curvature if and only if ϕ(grad $\alpha) = (2n-1)$ (grad β). Also it is shown that a conformally flat trans-Sasakian manifold is a manifold of *generalized quasi–constant curvature* if and only if the structure function β is a non-vanishing constant. Then we obtain some mutually equivalent conditions on a conformally flat trans-Sasakian manifold. The last section deals with several non-trivial examples of trans-Sasakian manifolds constructed with global vector fields.

§2. Trans-Sasakian manifolds

A $(2n+1)$-dimensional differentiable manifold M^{2n+1} is said to be an almost contact metric manifold ([12]) if it admits a (1, 1) tensor field ϕ, a contravariant vector field of ξ, a 1-form η and a Riemannian metric g which satisfy

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \quad \phi^2 X = -X + \eta(X)\xi, \tag{2.1}$$

$$g(\phi X, Y) = -g(X, \phi Y), \quad \eta(X) = g(X, \xi), \quad \eta(\xi) = 1, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.3}$$

for all vector fields X, Y on M^{2n+1}.

An almost contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$ is said to be trans-Sasakian manifold ([1]) if $(M \times R, J, G)$ belong to the class W_4 of the Hermitian manifolds where J is the almost complex structure on $M \times R$ defined by

$$J(Z, f\frac{d}{dt}) = (\phi Z - f\xi, \eta(Z)\frac{d}{dt})$$

for any vector field Z on M and smooth function f on $M \times R$ and G is the product metric on $M \times R$. This may be stated by the condition ([2])

$$(\nabla_X \phi)(Y) = \alpha\{g(X, Y)\xi - \eta(Y)X\} + \beta\{g(\phi X, Y)\xi - \eta(Y)\phi X\} \tag{2.4}$$

where α, β are smooth functions on M and we say such a structure the trans-Sasakian structure of type (α, β). From (2.4) it follows that

$$\nabla_X \xi = -\alpha\phi X + \beta\{X - \eta(X)\xi\}, \tag{2.5}$$

$$(\nabla_X \eta)(Y) = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y). \tag{2.6}$$

In a trans-Sasakian manifold $M^{2n+1}(\phi, \xi, \eta, g)$ the following relations hold ([11]):

$$\begin{aligned} R(X,Y)\xi &= (\alpha^2 - \beta^2)[\eta(Y)X - \eta(X)Y] - (X\alpha)\phi Y - (X\beta)\phi^2(Y) \\ &\quad +2\alpha\beta[\eta(Y)\phi X - \eta(X)\phi Y] + (Y\alpha)\phi X + (Y\beta)\phi^2(X), \end{aligned} \tag{2.7}$$

$$\begin{aligned} \eta(R(X,Y)Z) &= (\alpha^2 - \beta^2)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] \\ &\quad -2\alpha\beta[g(\phi X, Z)\eta(Y) - g(\phi Y, Z)\eta(X)] \\ &\quad -(Y\alpha)g(\phi X, Z) - (X\beta)\{g(Y,Z) - \eta(Y)\eta(Z)\} \\ &\quad +(X\alpha)g(\phi Y, Z) + (Y\beta)\{g(X,Z) - \eta(Z)\eta(X)\}, \end{aligned} \tag{2.8}$$

$$R(\xi, X)\xi = (\alpha^2 - \beta^2 - \xi\beta)[\eta(X)\xi - X], \tag{2.9}$$

$$S(X,\xi) = [2n(\alpha^2 - \beta^2) - (\xi\beta)]\eta(X) - ((\phi X)\alpha) - (2n-1)(X\beta), \tag{2.10}$$

$$S(\xi,\xi) = 2n(\alpha^2 - \beta^2 - \xi\beta), \tag{2.11}$$

$$(\xi\alpha) + 2\alpha\beta = 0, \tag{2.12}$$

$$Q\xi = [2n(\alpha^2 - \beta^2) - \xi\beta]\xi + \phi(\mathrm{grad}\alpha) - (2n-1)(\mathrm{grad}\beta). \tag{2.13}$$

for any vector fields X, Y on M.

§3. Generalized η-Einstein Trans-Sasakian manifolds

Definition 3.1. An almost contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$ is said to be *η-Einstein* if its Ricci tensor S of type (0, 2) is of the form

$$S = ag + b\eta \otimes \eta, \tag{3.1}$$

where a, b are smooth functions on M.

It is shown in ([11]) that the associated scalars a and b of the η-Einstein trans-Sasakian manifold are given by

$$a = \frac{r}{2n} - (\alpha^2 - \beta^2 - \xi\beta), \quad b = -\frac{r}{2n} + (2n+1)(\alpha^2 - \beta^2 - \xi\beta).$$

Definition 3.2. A trans-Sasakian manifold $M(\phi, \xi, \eta, g)$ is said to be *generalized η-Einstein* if its Ricci tensor S of type (0, 2) is of the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y) + c[\eta(X)\omega(Y) + \eta(Y)\omega(X)] \tag{3.2}$$

where a, b, c are non-zero scalars, ω is a non-zero 1-form such that $\omega(X) = g(X, \rho)$ for all X, and ξ and ρ are unit vector fields orthogonal to each other. The scalars a, b, c are called the associated scalars.

Proposition 1. *In a generalized η-Einstein trans-Sasakian manifold (M^{2n+1}, g), the associated scalars are given by*

$$a = \frac{r}{2n} - (\alpha^2 - \beta^2 - \xi\beta), \tag{3.3}$$

$$b = -\frac{r}{2n} + (2n+1)(\alpha^2 - \beta^2 - \xi\beta), \tag{3.4}$$

$$c = \omega(\phi \text{grad}\alpha) - (2n-1)\omega(\text{grad}\beta). \tag{3.5}$$

Proof. Setting $X = Y = \xi$ in (3.2) and then using (2.11), we get

$$S(\xi, \xi) = a + b = 2n(\alpha^2 - \beta^2 - \xi\beta). \tag{3.6}$$

Contracting (3.2) over X and Y, it yields

$$r = (2n+1)a + b, \tag{3.7}$$

where r is the scalar curvature of the manifold. From (3.6) and (3.7) we obtain (3.3) and (3.4).

Again replacing X by ρ and Y by ξ in (3.2), respectively, and keeping in mind the relation (2.10), we obtain (3.5). This proves the proposition.

Theorem 3.1. *In a generalized η-Einstein trans-Sasakian manifold (M^{2n+1}, g), the associated scalars $2n(\alpha^2 - \beta^2 - \xi\beta)$ and $\frac{r}{2n} - (\alpha^2 - \beta^2 - \xi\beta)$ are the Ricci curvatures in the direction of the vector fields ξ and ρ, respectively, and the inequality $\omega(\phi \text{grad}\alpha) < \frac{1}{\sqrt{2}}q + (2n-1)\omega(\text{grad}\beta)$ holds, where q is the length of the Ricci tensor S.*

Proof. Setting $X = Y = \rho$ in (3.2) we obtain by virtue of (3.3) that

$$S(\rho, \rho) = \frac{r}{2n} - (\alpha^2 - \beta^2 - \xi\beta). \tag{3.8}$$

From (3.6) and (3.8), it follows that $2n(\alpha^2 - \beta^2 - \xi\beta)$ and $\frac{r}{2n} - (\alpha^2 - \beta^2 - \xi\beta)$ are the Ricci curvatures in the direction of the vector fields ξ and ρ respectively. Let $g(QX, Y) = S(X, Y)$ and q^2 denote the square of the length of the Ricci tensor S, i.e.,

$$q^2 = \sum_{i=1}^{2n+1} S(Qe_i, e_i), \tag{3.9}$$

where $\{e_i \ : \ i = 1, 2, ..., 2n+1\}$ is an orthonormal basis of the tangent space at any point of the manifold. From (3.2) it follows that

$$\sum_{i=1}^{2n+1} S(Qe_i, e_i) = 2na^2 + (a+b)^2 + 2c^2$$

which implies that

$$q^2 - 2c^2 = 2na^2 + (a+b)^2.$$

Since $a \neq 0$ and $b \neq 0$, we obtain $q^2 - 2c^2 = 2na^2 + (a+b)^2 > 0$ and hence the equation

$$c < \frac{1}{\sqrt{2}} q.$$

Hence by virtue of (3.5) we have the required inequality. This proves the theorem.

Definition 3.3 ([3]). A Riemannian manifold (M^m, g) $(m \geq 3)$ is said to be of *quasi-constant curvature* if its curvature tensor $\tilde{R}$ of type (0, 4) satisfies the condition :

$$(3.10)\ \tilde{R}(X,Y,Z,W) = p_1[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ +p_2[g(X,W)A(Y)A(Z) - g(Y,W)A(X)A(Z) \\ +g(Y,Z)A(X)A(W) - g(X,Z)A(Y)A(W)]$$

where p_1, p_2 are non-zero scalars and A is a non-zero 1-form such that $g(X,U) = A(X)$ for all X, and U is a unit vector field. p_1, p_2 and A are called the associated scalars and associated 1-form of the manifold, respcetively.

The notion of a manifold of quasi-constant curvature is introduced by Chen and Yano ([3]). Generalizing this notion of quasi-constant curvature, Chaki ([4]) introduced the notion of generalized quasi-constant curvature as follows :

Definition 3.4. A Riemannian manifold $(M^m, g)(m \geq 3)$ is said to be of *generalized quasi-constant curvature* if its curvature tensor $\tilde{R}$ of type (0, 4) satisfies the condition

$$(3.11)\ \tilde{R}(X,Y,Z,W) = a[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ +b[g(X,W)A(Y)A(Z) - g(Y,W)A(X)A(Z) \\ +g(Y,Z)A(X)A(W) - g(X,Z)A(Y)A(W)]$$

$$
\begin{aligned}
&+c[g(X,W)\{A(Y)B(Z)+A(Z)B(Y)\}\\
&-g(X,Z)\{A(W)B(Y)+A(Y)B(W)\}\\
&+g(Y,Z)\{A(W)B(X)+A(X)B(W)\}\\
&-g(Y,W)\{A(Z)B(X)+A(X)B(Z)\}],
\end{aligned}
$$

where a, b and c are non-zero scalars, and A and B are non-zero 1-forms such that $A(X) = g(X,U)$ and $B(X) = g(X,V)$ for all X, and U and V are orthogonal vector fields.

Theorem 3.2. *A 3-dimensional generalized η-Einstein trans-Sasakian manifold is a manifold of generalized quasi-constant curvature.*

Proof. Since in a 3-dimensional Riemannian manifold the Weyl conformal curvature vanishes, its curvature tensor $\tilde{R}$ of type (0, 4) is given by

$$
\begin{aligned}
(3.12)\qquad \tilde{R}(X,Y,Z,W) &= g(Y,Z)S(X,W) - g(X,Z)S(Y,W)\\
&+S(Y,Z)g(X,W) - S(X,Z)g(Y,W)\\
&+\frac{r}{2}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)].
\end{aligned}
$$

By virtue of (3.2), (3.12) can be written as

$$
\begin{aligned}
(3.13)\quad \tilde{R}(X,Y,Z,W) &= a_1[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]\\
&+b_1[g(X,W)\eta(Y)\eta(Z) - g(Y,W)\eta(X)\eta(Z)\\
&+g(Y,Z)\eta(X)\eta(W) - g(X,Z)\eta(Y)\eta(W)]\\
&+c_1[g(X,W)\{\eta(Y)\omega(Z)+\eta(Z)\omega(Y)\}\\
&-g(X,Z)\{\eta(W)\omega(Y)+\eta(Y)\omega(W)\}\\
&+g(Y,Z)\{\eta(W)\omega(X)+\eta(X)\omega(W)\}\\
&-g(Y,W)\{\eta(Z)\omega(X)+\eta(X)\omega(Z)\}]
\end{aligned}
$$

where $a_1 = \frac{3r}{2} - 2(\alpha^2 - \beta^2 - \xi\beta)$, $b_1 = -\frac{r}{2} + 3(\alpha^2 - \beta^2 - \xi\beta)$ and $c_1 = \lambda(\phi\text{grad}\alpha) - \lambda(\text{grad}\beta)$ are three non-zero scalars. Comparing (3.11) with (3.13) , it follows that the manifold under consideration is of generalized quasi-constant curvature. This proves the theorem.

§4. Conformally flat Trans-Sasakian manifolds

Let (M^{2n+1}, g) $(n > 1)$ be a conformally flat trans-Sasakian manifold. Then its curvature tensor is given by

$$
\begin{aligned}
(4.1)\ R(X,Y)Z &= \frac{1}{2n-1}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX\\
&-g(X,Z)QY] - \frac{r}{2n(2n-1)}[g(Y,Z)X - g(X,Z)Y]
\end{aligned}
$$

for any vector fields X, Y and Z on M. Setting $Z = \xi$ in (4.1) and using (2.7) and (2.10), we obtain

$$\begin{aligned}(4.2)\qquad & [(\alpha^2-\beta^2)-\frac{2n(\alpha^2-\beta^2)-\xi\beta}{2n-1}+\frac{r}{2n(2n-1)}][\eta(Y)X-\eta(X)Y] \\ & +2\alpha\beta[\eta(Y)\phi X-\eta(X)\phi Y] \\ & -(X\alpha)\phi Y-(X\beta)\phi^2(Y)+(Y\alpha)\phi X+(Y\beta)\phi^2(X) \\ = & \frac{1}{2n-1}[\{\eta(Y)QX-\eta(X)QY\}-(2n-1)\{(Y\beta)X-(X\beta)Y\} \\ & -\{((\phi Y)\alpha)X-((\phi X)\alpha)Y\}].\end{aligned}$$

Again replacing Y by ξ in (4.2), we obtain by virtue of (2.12) that

$$\begin{aligned}(4.3)\qquad QX = & [\frac{r}{2n}-(\alpha^2-\beta^2-\xi\beta)]X \\ & +[-\frac{r}{2n}+(2n+1)(\alpha^2-\beta^2)+(2n-3)(\xi\beta)]\eta(X)\xi \\ & -(2n-1)\{(X\beta)\xi+\eta(X)\mathrm{grad}\beta\}-((\phi X)\alpha)\xi \\ & +\eta(X)\phi(\mathrm{grad}\alpha)+(2n-1)(\xi\alpha)\phi X,\end{aligned}$$

which can also be written as

$$\begin{aligned}(4.4)\ S(X,Y) = & ag(X,Y)+b\eta(X)\eta(Y) \\ & -(2n-1)\{(X\beta)\eta(Y)+(Y\beta)\eta(X)\}-[((\phi X)\alpha)\eta(Y) \\ & +((\phi Y)\alpha)\eta(X)]+(2n-1)(\xi\alpha)g(\phi X,Y)\end{aligned}$$

where $a = \frac{r}{2n}-(\alpha^2-\beta^2-\xi\beta)$ and $b = -\frac{r}{2n}+(2n+1)(\alpha^2-\beta^2)-(2n-3)(\xi\beta)$.

The symmetry property of the Ricci tensor yields from (4.4) that

$$(4.5)\qquad (\xi\alpha) = 0.$$

Extending the notion of *generalized η-Einstein manifold* we introduce the notion of *hyper generalized η-Einstein manifold* as follows :

Definition 4.1. A trans-Sasakian manifold (M^{2n+1}, g) is said to be *hyper generalized η-Einstein manifold* if its Ricci tensor S of type (0, 2) is of the form

$$\begin{aligned}(4.6)\ S(X,Y) = & ag(X,Y)+b\eta(X)\eta(Y)+c[\eta(X)\omega(Y)+\eta(Y)\omega(X)] \\ & +d[\eta(X)\pi(Y)+\eta(Y)\pi(X)]\end{aligned}$$

where a, b, c and d are non-zero scalars which are called the associated scalars, ω and π are non-zero 1-forms such that $\omega(X) = g(X,\rho)$, $\pi(X) = g(X,\lambda)$ for all

X ; ρ and λ being associated vector fields of the 1-forms ω and π respectively such that ξ is orthogonal to ρ.

The name 'hyper' is used as in the case of hyper real numbers. Especially, if $\lambda = \delta\rho$, δ being a scalar, then the notion of *hyper generalized η-Einstein manifold* reduces to the notion of *generalized η-Einstein manifold.* This implies that ρ and λ are not necessarily mutually orthogonal whereas ξ is always orthogonal to ρ.

Theorem 4.1. *A conformally flat trans-Sasakian manifold* (M^{2n+1}, g) $(n > 1)$ *is a hyper generalized η-Einstein manifold.*

Proof. If a trans-Sasakian manifold (M^{2n+1}, g) $(n > 1)$ is conformally flat, then we have the relation (4.4). By virtue of (4.5), (4.4) yields,

$$\begin{aligned} S(X,Y) &= ag(X,Y) + b\eta(X)\eta(Y) - (2n-1)\{(X\beta)\eta(Y) \\ &\quad +(Y\beta)\eta(X)\} - [((\phi X)\alpha)\eta(Y) + ((\phi Y)\alpha)\eta(X)], \end{aligned} \tag{4.7}$$

which can also be written as

$$\begin{aligned} S(X,Y) &= ag(X,Y) + b\eta(X)\eta(Y) + c[\eta(X)\omega(Y) + \eta(Y)\omega(X)] \\ &\quad +d[\eta(X)\pi(Y) + \eta(Y)\pi(X)] \end{aligned} \tag{4.8}$$

where a, b, c and d are non-zero scalars given by where $a = \frac{r}{2n} - (\alpha^2 - \beta^2 - \xi\beta)$, $b = -\frac{r}{2n} + (2n+1)(\alpha^2 - \beta^2) - (2n-3)(\xi\beta)]$, $c = 1$ and $d = -(2n-1)$; ω and π are non-zero 1-forms such that $\omega(X) = g(X, \rho) = g(X, \phi(\text{grad}\alpha)) = -((\phi X)\alpha)$, $\pi(X) = g(X, \lambda) = g(X, \text{grad}\beta) = (X\beta)$ for all X. This proves the theorem.

Theorem 4.2. *A conformally flat trans-Sasakian manifold* (M^{2n+1}, g) $(n > 1)$ *is an η-Einstein manifold if and only if*

$$\phi(\text{grad}\alpha) = (2n-1)(\text{grad}\beta). \tag{4.9}$$

Proof. For a conformally flat trans-Sasakian manifold we have the relation (4.8). We first suppose that the conformally flat trans-Sasakian manifold is η-Einstein. Then (4.8) yields

$$[\eta(X)\omega(Y) + \eta(Y)\omega(X)] - (2n-1)[\eta(X)\pi(Y) + \eta(Y)\pi(X)] = 0 \tag{4.10}$$

where $\omega(X) = g(X, \phi\text{grad}\alpha)$ and $\pi(X) = g(X, \text{grad}\beta)$. Setting $X = \xi$ in (4.10) we get

$$\omega(Y) - (2n-1)[\pi(Y) + (\xi\beta)\eta(Y)] = 0. \tag{4.11}$$

Again replacing $Y = \xi$ in (4.11), we have

$$(\xi\beta) = 0. \tag{4.12}$$

In view of (4.12) and (4.11) we obtain (4.9).

Conversely, if (4.9) holds, then $\pi(X) = \frac{1}{(2n-1)}\omega(X)$ and hence $(\xi\beta) = g(\xi, \text{grad}\beta) = \frac{1}{2n-1}g(\xi, \phi\text{grad}\alpha) = 0$ and hence (4.8) reduces to

$$S(X,Y) = \tilde{a}g(X,Y) + \tilde{b}\eta(X)\eta(Y), \tag{4.13}$$

where $\tilde{a}$ and $\tilde{b}$ are non-zero scalars given by

$$\tilde{a} = \frac{r}{2n} - (\alpha^2 - \beta^2), \quad \tilde{b} = -\frac{r}{2n} + (2n+1)(\alpha^2 - \beta^2).$$

The relation (4.13) implies that the manifold under consideration (4.9) is an η-Einstein manifold. This proves the theorem.

Corollary 4.1. *A conformally flat trans-Sasakian manifold* (M^{2n+1}, g) $(n > 1)$ *is a generalized* η*-Einstein manifold if and only if the structure function* β *is a non-vanishing constant.*

Proof. If β is a non-vanishing constant, then $(X\beta) = 0$ for all X and hence (4.8) reduces to

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y) + c[\eta(X)\omega(Y) + \eta(Y)\omega(X)], \tag{4.14}$$

where a, b and c are non-zero scalars. The relation (4.14) is of the form (3.2) and hence the manifold is generalized η-Einstein. Conversely, if a conformally flat trans-Sasakian manifold (M^{2n+1}, g) $(n > 1)$ is a generalized η-Einstein manifold, then we have the relation (4.14). From (4.8) and (4.14), we have

$$d[\eta(X)\pi(Y) + \eta(Y)\pi(X)] = 0,$$

which yields for $Y = \xi$

$$(X\beta) + (\xi\beta)\eta(X) = 0, \tag{4.15}$$

since $d \neq 0$. Again, setting $X = \xi$ in (4.15), we have $(\xi\beta) = 0$. Therefore, (4.15) takes the form

$$(X\beta) = 0,$$

for all X and hence β is a constant. This proves the corollary.

Extending the notion of generalized quasi-constant curvature of M. C. Chaki ([4]), we introduce the notion of *hyper generalized quasi-constant curvature* as follows:

Definition 4.2. A Riemannian manifold $(M^m, g)(m \geq 3)$ is said to be of *hyper generalized quasi-constant curvature* if its curvature tensor $\tilde{R}$ of type (0, 4) is of the form

$$(4.16)\ \tilde{R}(X,Y,Z,W) = \delta_1[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ +\delta_2[g(X,W)A(Y)A(Z) - g(Y,W)A(X)A(Z) \\ +g(Y,Z)A(X)A(W) - g(X,Z)A(Y)A(W)] \\ +\delta_3[g(X,W)\{A(Y)B(Z) + A(Z)B(Y)\} \\ -g(X,Z)\{A(Y)B(W) + A(W)B(Y)\} \\ +g(Y,Z)\{A(X)B(W) + A(W)B(X)\} \\ -g(Y,W)\{A(X)B(Z) + A(Z)B(X)\}] \\ +\delta_4[g(X,W)\{A(Y)D(Z) + A(Z)D(Y)\} \\ -g(X,Z)\{A(Y)D(W) + A(W)D(Y)\} \\ +g(Y,Z)\{A(X)D(W) + A(W)D(X)\} \\ -g(Y,W)\{A(X)D(Z) + A(Z)D(X)\}],$$

where δ_i ($i = 1, 2, 3, 4$) are non-vanishing scalars and A, B and D are non-zero 1-forms given by $A(X) = g(X,\xi)$, $B(X) = g(X,\rho)$, $D(X) = g(X,\lambda)$ such that ξ is orthogonal to ρ.

Especially, if $\lambda = \delta\rho$, δ being a scalar, then the notion of a manifold of *hyper generalized quasi-constant curvature* reduces to the notion of *generalized quasi-constant curvature*. This implies that ρ and λ are not necessarily mutually orthogonal whereas ξ is always orthogonal to ρ. We have used the term *"hyper"* , since if B and D are linearly dependent, then (4.16) reduces to the form of (3.11).

Theorem 4.3. *A conformally flat trans-Sasakian manifold* (M^{2n+1}, g) $(n > 1)$ *is a manifold of hyper generalized quasi-constant curvature.*

Proof. In a conformally flat trans-Sasakian manifold (M^{2n+1}, g) $(n > 1)$ we have the relations (4.1) and (4.8). By virtue of (4.8) the relation (4.1) can be written as

$$(4.17)\ \tilde{R}(X,Y,Z,W) = \gamma_1[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ +\gamma_2[g(X,W)\eta(Y)\eta(Z) - g(Y,W)\eta(X)\eta(Z) \\ +g(Y,Z)\eta(X)\eta(W) - g(X,Z)\eta(Y)\eta(W)] \\ +\gamma_3[g(X,W)\{\eta(Y)\omega(Z) + \eta(Z)\omega(Y)\} \\ -g(X,Z)\{\eta(W)\omega(Y) + \eta(Y)\omega(W)\} \\ +g(Y,Z)\{\eta(W)\omega(X) + \eta(X)\omega(W)\} \\ -g(Y,W)\{\eta(Z)\omega(X) + \eta(X)\omega(Z)\}]$$

$$+\gamma_4[g(X,W)\{\eta(Y)\pi(Z)+\eta(Z)\pi(Y)\}$$
$$-g(X,Z)\{\eta(W)\pi(Y)+\eta(Y)\pi(W)\}$$
$$+g(Y,Z)\{\eta(W)\pi(X)+\eta(X)\pi(W)\}$$
$$-g(Y,W)\{\eta(Z)\pi(X)+\eta(X)\pi(Z)\}]$$

where γ_i, $i = 1, 2, 3, 4$ are non-zero scalars given by $\gamma_1 = \frac{1}{2n-1}[\frac{r}{2n} - 2(\alpha^2 - \beta^2 - \xi\beta)]$, $\gamma_2 = \frac{1}{2n-1}[-\frac{r}{2n} + (2n+1)(\alpha^2 - \beta^2) - (2n-3)(\xi\beta)]$, $\gamma_3 = \frac{1}{2n-1}$ and $\gamma_4 = -1$, $\omega(X) = g(X, \phi\text{grad}\alpha)$, and $\pi(X) = g(X, \text{grad}\beta)$ for all X. From (4.16) and (4.17) , it follows that the manifold under consideration is *hyper generalized quasi-constant curvature.*

Theorem 4.4. *A conformally flat trans-Sasakian manifold* (M^{2n+1}, g) $(n > 1)$ *is a manifold of quasi-constant curvature if and only if*

$$\phi(\text{grad}\alpha) = (2n-1)(\text{grad}\beta).$$

Proof. We first suppose that in a conformally flat trans-Sasakian manifold (M^{2n+1}, g) $(n > 1)$, the relation $\phi(\text{grad}\alpha) = (2n-1)(\text{grad}\beta)$ holds. Then we have the relation (4.13). By virtue of (4.13) the relation (4.1) can be written as

$$\tag{4.18} \begin{aligned}\tilde{R}(X,Y,Z,W) = {} & \tilde{\gamma}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ & +\tilde{\delta}[g(X,W)\eta(Y)\eta(Z) - g(Y,W)\eta(X)\eta(Z) \\ & +g(Y,Z)\eta(X)\eta(W) - g(X,Z)\eta(Y)\eta(W)]\end{aligned}$$

where $\tilde{\gamma}$ and $\tilde{\delta}$ are non-zero scalars given by

$$\begin{aligned}\tilde{\gamma} &= \frac{1}{2n-1}[\frac{r}{2n} - 2(\alpha^2 - \beta^2 - \xi\beta)], \\ \tilde{\delta} &= \frac{1}{2n-1}[-\frac{r}{2n} + (2n+1)(\alpha^2 - \beta^2) - (2n-3)(\xi\beta)].\end{aligned}$$

From (4.18) it follows by virtue of Definition 3.3 that the manifold is of quasi-constant curvature.

Conversely, if the manifold is of quasi-constant curvature, then (4.17) yields

$$\tag{4.19} \begin{aligned}& \gamma_3[g(X,W)\{\eta(Y)\omega(Z) + \eta(Z)\omega(Y)\} - g(X,Z)\{\eta(W)\omega(Y) \\ & +\eta(Y)\omega(W)\} + g(Y,Z)\{\eta(W)\omega(X) + \eta(X)\omega(W)\} \\ & -g(Y,W)\{\eta(Z)\omega(X) + \eta(X)\omega(Z)] + \gamma_4[g(X,W)\{\eta(Y)\pi(Z) \\ & +\eta(Z)\pi(Y)\} - g(X,Z)\{\eta(W)\pi(Y) + \eta(Y)\pi(W)\} \\ & +g(Y,Z)\{\eta(W)\pi(X) + \eta(X)\pi(W)\} - g(Y,W)\{\eta(Z)\pi(X) \\ & +\eta(X)\pi(Z)\}] = 0.\end{aligned}$$

Let $\{e_i\}$, $i = 1, 2, \ldots, 2n+1$ be an orthonormal basis of the tangent space at any point of the manifold. Setting $X = W = e_i$ in (4.19) and taking summation over i, $1 \leq i \leq 2n+1$, we get

$$\begin{aligned}(4.20)\qquad &\gamma_3(2n-1)[\eta(Y)\omega(Z)+\eta(Z)\omega(Y)]\\ &+\gamma_4[(2n-1)\{\eta(Y)\pi(Z)+\eta(Z)\pi(Y)\}+2g(Y,Z)(\xi\beta)] \;=\; 0.\end{aligned}$$

Since $\gamma_3 = \frac{1}{2n-1}$ and $\gamma_4 = -1$, (4.20) implies that

$$\begin{aligned}(4.21)\qquad &\eta(Y)\omega(Z)+\eta(Z)\omega(Y)-2g(Y,Z)(\xi\beta)\\ &-(2n-1)\{\eta(Y)\pi(Z)+\eta(Z)\pi(Y)\}=0.\end{aligned}$$

Replacing Y by ξ in (4.21), we get

$$(4.22)\qquad \omega(Z)-(2n-1)\pi(Z) \;=\; 0,$$

which implies $\phi(\text{grad}\alpha) = (2n-1)(\text{grad}\beta)$. This proves the theorem.

Corollary 4.2. *A conformally flat trans-Sasakian manifold* (M^{2n+1}, g) $(n > 1)$ *is a manifold of generalized quasi-constant curvature if and only if the structure function* β *is a non-vanishing constant.*

Proof. If β is constant, then $(Y\beta) = 0$ for all Y and hence (4.17) reduces to the form of generalized quasi-constant curvature.

Conversely, if the manifold is of generalized quasi-constant curvature, then, from the relation (4.17), it follows that

$$\begin{aligned}(4.23)\qquad &\gamma_4[g(X,W)\{\eta(Y)\pi(Z)+\eta(Z)\pi(Y)\}\\ &-g(X,Z)\{\eta(W)\pi(Y)+\eta(Y)\pi(W)\}\\ &+g(Y,Z)\{\eta(W)\pi(X)+\eta(X)\pi(W)\}\\ &-g(Y,W)\{\eta(Z)\pi(X)+\eta(X)\pi(Z)\}]=0.\end{aligned}$$

Contracting (4.23) over X and W, we get

$$(4.24)\qquad \gamma_4[(2n-1)\{\eta(Y)\pi(Z)+\eta(Z)\pi(Y)\}-2g(Y,Z)(\xi\beta)] \;=\; 0,$$

which yields for $Y = \xi$

$$(4.25)\qquad (2n-1)\pi(Z)-(2n+1)(\xi\beta)\eta(Z) \;=\; 0.$$

Now, setting $Z = \xi$ in the above relation, we have $(\xi\beta) = 0$. Hence, (4.25) takes the form $(Z\beta) = 0$ for all Z, which implies that β is a constant. This proves the corollary.

Theorem 4.5. *Let (M^{2n+1}, g) $(n > 1)$ be a conformally flat trans-Sasakian manifold. Then the following conditions are mutually equivalent:*
(1) M is η-Einstein.
(2) M is a manifold of quasi-constant curvature.
(3) ξ is the eigenvector field of the Ricci operator Q.
(4) M satisfies $\phi(\text{grad}\alpha) = (2n-1)(\text{grad}\beta)$.

Proof. Let (M^{2n+1}, g) $(n > 1)$ be a conformally flat trans-Sasakian manifold. We first suppose that M is η-Einstein. Then (4.1) and (3.1) hold good. In view of (4.1) and (3.1) we have

$$
\begin{aligned}
(4.26)\quad \tilde{R}(X,Y,Z,W) &= \frac{1}{2n-1}(2a - \frac{r}{2n})[g(Y,Z)g(X,W) \\
&\quad -g(X,Z)g(Y,W)] + \frac{b}{2n-1}[g(X,W)\eta(Y)\eta(Z) \\
&\quad -g(Y,W)\eta(X)\eta(Z) + g(Y,Z)\eta(X)\eta(W) \\
&\quad -g(X,Z)\eta(Y)\eta(W)],
\end{aligned}
$$

where a and b are non-zero scalars given by

$$
a = \frac{r}{2n} - (\alpha^2 - \beta^2 - \xi\beta), \quad b = -\frac{r}{2n} + (2n+1)(\alpha^2 - \beta^2 - \xi\beta).
$$

The relation (4.26) implies that the manifold under consideration is a manifold of quasi-constant curvature. Hence (1) $\Rightarrow$ (2).

Next, let M^{2n+1} $(n > 1)$ be a conformally flat trans-Sasakian manifold which is of quasi-constant curvature. Then (3.10) holds good. For $U = \xi$, (3.10) can be written as

$$
\begin{aligned}
\tilde{R}(X,Y,Z,W) &= p_1[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\
&\quad +p_2[g(X,W)\eta(Y)\eta(Z) - g(Y,W)\eta(X)\eta(Z) \\
&\quad +g(Y,Z)\eta(X)\eta(W) - g(X,Z)\eta(Y)(W)],
\end{aligned}
$$

which yields

$$
(4.27)\qquad S(Y,Z) = (2np_1 + p_2)g(Y,Z) + (2n-1)p_2\eta(Y)\eta(Z).
$$

From (4.27) it follows that $Q\xi = 2n(p_1 + p_2)\xi$ which yields ξ is the eigenvector of the Ricci operator Q. Hence (2) $\Rightarrow$ (3).

Again, let in a conformally flat trans-Sasakian manifold M^{2n+1} $(n > 1)$ ξ is the eigenvector of the Ricci operator Q. Then from (4.3) it follows by virtue of (4.5) that $\phi(\text{grad}\alpha) = (2n-1)(\text{grad}\beta)$. Thus (3) $\Rightarrow$ (4).

Finally, let in a conformally flat trans-Sasakian manifold M^{2n+1} $(n > 1)$ the condition $\phi(\text{grad}\alpha) = (2n-1)(\text{grad}\beta)$ holds. Using this condition in (4.4) we obtain by virtue of (4.5) that the manifold is η-Einstein. Hence (4) $\Rightarrow$ (1). This completes the proof.

§5. Examples of trans-Sasakian manifolds

Example 1 We consider the 3-dimensional manifold $M = \{(x, y, z) \in R^3 : z \neq 0\}$, where (x, y, z) are the standard coordinates in R^3. Let $\{E_1, E_2, E_3\}$ be linearly independent global frame on M given by

$$E_1 = e^{-z}\frac{\partial}{\partial y}, \quad E_2 = e^{-z}(\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}), \quad E_3 = \frac{\partial}{\partial z}.$$

Let g be the Riemannian metric defined by $g(E_1, E_3) = g(E_2, E_3) = g(E_1, E_2) = 0, \quad g(E_1, E_1) = g(E_2, E_2) = g(E_3, E_3) = 1$. Let η be the 1-form defined by $\eta(U) = g(U, E_3)$ for any $U \in \chi(M)$. Let ϕ be the (1, 1) tensor field defined by $\phi E_1 = E_2$, $\phi E_2 = -E_1$, $\phi E_3 = 0$. Then using the linearity of ϕ and g, we have $\eta(E_3) = 1$, $\phi^2 U = -U + \eta(U)E_3$ and $g(\phi U, \phi W) = g(U, W) - \eta(U)\eta(W)$ for any $U, W \in \chi(M)$. Thus for $E_3 = \xi$, (ϕ, ξ, η, g) defines an almost contact metric structure on M.

Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g and R the curvature tensor of g. Then we have

$$[E_1, E_2] = ye^{-z}E_1 + e^{-2z}E_3, \quad [E_1, E_3] = E_1, \quad [E_2, E_3] = E_2.$$

Taking $E_3 = \xi$ and using Koszul formula for the Riemannian metric g, we can easily calculate

$$\nabla_{E_1}E_3 = E_1 - \frac{1}{2}e^{-2z}E_2, \quad \nabla_{E_3}E_3 = 0, \quad \nabla_{E_2}E_3 = E_2 + \frac{1}{2}e^{-2z}E_1,$$
$$\nabla_{E_2}E_2 = -E_3, \quad \nabla_{E_2}E_1 = -\frac{1}{2}e^{-2z}E_3, \quad \nabla_{E_1}E_2 = \frac{1}{2}e^{-2z}E_3 + ye^{-z}E_1,$$
$$\nabla_{E_1}E_1 = -E_3 - ye^{-z}E_2, \quad \nabla_{E_3}E_2 = \frac{1}{2}e^{-2z}E_1, \quad \nabla_{E_3}E_1 = -\frac{1}{2}e^{-2z}E_2.$$

From the above it can be easily seen that (ϕ, ξ, η, g) is an trans-Sasakian structure on M. Consequently, $M^3(\phi, \xi, \eta, g)$ is a trans-Sasakian manifold with $\alpha = -\frac{1}{2}e^{-2z} \neq 0$ and $\beta = 1$.

Example 2. We consider the 3-dimensional manifold $M = \{(x, y, z) \in R^3 : z \neq 0\}$, where (x, y, z) are the standard coordinates in R^3. Let $\{E_1, E_2, E_3\}$ be linearly independent global frame on M given by

$$E_1 = -z(\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}), \quad E_2 = -z\frac{\partial}{\partial y}, \quad E_3 = \frac{\partial}{\partial z}.$$

Let g be the Riemannian metric defined by $g(E_1, E_3) = g(E_2, E_3) = g(E_1, E_2) = 0, \quad g(E_1, E_1) = g(E_2, E_2) = g(E_3, E_3) = 1$. Let η be the 1-form defined by $\eta(U) = g(U, E_3)$ for any $U \in \chi(M)$. Let ϕ be the (1, 1) tensor field defined by $\phi E_1 = E_2$, $\phi E_2 = -E_1$, $\phi E_3 = 0$. Then using the linearity of ϕ and g we have

$\eta(E_3) = 1$, $\phi^2 U = -U + \eta(U)E_3$ and $g(\phi U, \phi W) = g(U, W) - \eta(U)\eta(W)$ for any $U, W \in \chi(M)$. Thus, for $E_3 = \xi$, (ϕ, ξ, η, g) defines an almost contact metric structure on M.

Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g and R the curvature tensor of g. Then we have

$$[E_1, E_2] = -yE_2 - z^2 E_3, \quad [E_1, E_3] = \frac{1}{z}E_1, \quad [E_2, E_3] = \frac{1}{z}E_2.$$

Taking $E_3 = \xi$ and using Koszul formula for the Riemannian metric g, we can easily calculate

$$\begin{aligned}
\nabla_{E_1} E_3 &= \frac{1}{z}E_1 + \frac{1}{2}z^2 E_2, \quad \nabla_{E_3} E_3 = 0, \quad \nabla_{E_2} E_3 = \frac{1}{z}E_2 - \frac{1}{2}z^2 E_1, \\
\nabla_{E_2} E_2 &= -yE_1 - \frac{1}{z}E_3, \quad \nabla_{E_1} E_2 = -\frac{1}{2}z^2 E_3, \quad \nabla_{E_2} E_1 = \frac{1}{2}z^2 E_3 + yE_2, \\
\nabla_{E_1} E_1 &= -\frac{1}{z}E_3, \quad \nabla_{E_3} E_2 = -\frac{1}{2}z^2 E_1, \quad \nabla_{E_3} E_1 = \frac{1}{2}z^2 E_2.
\end{aligned}$$

From the above it can be easily seen that (ϕ, ξ, η, g) is an trans-Sasakian structure on M. Consequently, $M^3(\phi, \xi, \eta, g)$ is a trans-Sasakian manifold with $\alpha = -\frac{1}{2}z^2 \neq 0$ and $\beta = \frac{1}{z}$.

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