SUT Journal of Mathematics
Vol. 45, No. 1 (2009), 73–85

Pseudo-umbilical CR-submanifolds in a locally conformal Kaehler space form

Koji Matsumoto and Zerrin Şentürk

(Received November 28, 2008; Revised March 31, 2009)

Abstract. In this report, we consider pseudo-umbilical CR-submanifolds in a locally conformal Kaehler space form and we mainly get a relation of the scalar curvature and the coefficient functions of the shape operator of a pseudo-umbilical CR-submanifold in a locally conformal Kaehler space form.

AMS 2000 *Mathematics Subject Classification.* 53C40

Key words and phrases. Locally conformal Kaehler manifold, pseudo-umbilical CR-submanifold, Lee form, adapted frame.

§1. Introduction

As a special CR-submanifold of an almost Hermitian manifold, the notion of a pseudo-umbilical CR-submanifold was introduced by A. Bejancu and gave a lot of interesting properties of this submanifold in a Kaehler manifold ([1]).

We consider this submanifold in a locally conformal Kaehler space form which is a generalization of a complex space form and we prove some properties of this submanifold (See Theorems 5.1 and 6.3).

§2. Preliminaries

A Hermitian manifold $\tilde{M}$ with structure $(J, \tilde{g})$ is called a locally conformal Kaehler (an l.c.K.) manifold if each point $x \in \tilde{M}$ has an open neighbourhood U with differentiable function $\rho : U \to \mathcal{R}$ such that $\tilde{g}^* = e^{-2\rho}\tilde{g}_{|U}$ is a Kaehlerian metric on U, that is, $\nabla^* J = 0$, where J is the almost complex structure, $\tilde{g}$ is the Hermitian metric, ∇^* is the covariant differentiation with respect to $\tilde{g}^*$ and $\mathcal{R}$ is a real number space ([7]). Then we know

Proposition 2.1([5]). *A Hermitian manifold $\tilde{M}$ with structure $(J, \tilde{g})$ is l.c.K.- if and only if there exists a global 1-form α which is called the Lee form satisfying*

$$d\alpha = 0 \quad (\alpha : \textit{closed}), \tag{2.1}$$

$$(\tilde{\nabla}_X J)Y = -\tilde{g}(\alpha^\sharp, Y)JX + \tilde{g}(X, Y)\beta^\sharp + \tilde{g}(JX, Y)\alpha^\sharp - \tilde{g}(\beta^\sharp, Y)X \tag{2.2}$$

for any $X, Y \in \Gamma T\tilde{M}$, where $\tilde{\nabla}$ denotes the covariant differentiation with respect to $\tilde{g}$, $\alpha^\sharp$ is the dual vector field of α which is called the Lee vector field, the 1 form β is defined by $\beta(X) = -\alpha(JX)$, $\beta^\sharp$ is the dual vector field of β and $\Gamma T\tilde{M}$ means the set of all differentiable vector fields on $\tilde{M}$.

An l.c.K.-manifold $\tilde{M}(J, \tilde{g}, \alpha)$ is called an *l.c.K.-space form* if it has a constant holomorphic sectional curvature. We know that the Riemannian curvature tensor $\tilde{R}$ of an l.c.K.-space form with the constant holomorphic sectional curvature c is given by ([5])

$$\begin{aligned} 4\tilde{R}(X, Y, Z, W) \;=\; & c\{\tilde{g}(X, W)\tilde{g}(Y, Z) - \tilde{g}(X, Z)\tilde{g}(Y, W) \\ & +\tilde{g}(JX, W)\tilde{g}(JY, Z) - \tilde{g}(JX, Z)\tilde{g}(JY, W) \\ & -2\tilde{g}(JX, Y)\tilde{g}(JZ, W)\} + 3\{P(X, W)\tilde{g}(Y, Z) \\ & -P(X, Z)\tilde{g}(Y, W) + \tilde{g}(X, W)P(Y, Z) \\ & -\tilde{g}(X, Z)P(Y, W)\} - \tilde{P}(X, W)\tilde{g}(JY, Z) \\ & +\tilde{P}(X, Z)\tilde{g}(JY, W) - \tilde{g}(JX, W)\tilde{P}(Y, Z) \\ & +\tilde{g}(JX, Z)\tilde{P}(Y, W) + 2\{\tilde{P}(X, Y)\tilde{g}(JZ, W) \\ & +\tilde{g}(JX, Y)\tilde{P}(Z, W)\} \end{aligned} \tag{2.3}$$

for any $X, Y, Z, W \in \Gamma T\tilde{M}$, where P and $\tilde{P}$ are respectively defined by

$$\begin{cases} P(X, Y) = -(\tilde{\nabla}_X \alpha)Y - \alpha(X)\alpha(Y) + \frac{1}{2}\|\alpha\|^2 \tilde{g}(X, Y), \\ \tilde{P}(X, Y) = P(JX, Y) \end{cases} \tag{2.4}$$

for any $X, Y \in \Gamma T\tilde{M}$, where $\|\alpha\|$ is the length of the Lee form α.

Remark. To get (2.3), we have to assume that the symmetric (0,2)-tensor P defined by (2.4) is hybrid or equivalently $\tilde{P}$ is skew-symmetric. This means the Ricci tensor $\tilde{R}_1$ is hybrid.

We write an l.c.K.-space form with the constant holomorphic sectional curvature c by $\tilde{M}(c)$

§3. CR-submanifolds in an l.c.K.-manifold

In generally, between a Riemannian manifold $(\tilde{M}, \tilde{g})$ and its submanifold, we know the Gauss and Weingarten formulas

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \tag{3.1}$$

$$\tilde{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi \tag{3.2}$$

for any $X, Y \in \Gamma TM$ and $\xi \in \Gamma T^\perp M$, where σ is the second fundamental form and A_ξ is the shape operator with respect to ξ. Moreover, we know the Gauss equation

$$\begin{aligned} R(X,Y,Z,W) &= \tilde{R}(X,Y,Z,W) + \tilde{g}(\sigma(X,W), \sigma(Y,Z)) \\ &\quad -\tilde{g}(\sigma(X,Z), \sigma(Y,W)) \end{aligned} \tag{3.3}$$

for any $X, Y, Z, W \in \Gamma TM$, where $\tilde{R}$ (resp. R) denotes the Riemannian curvature tensor with respect to $\tilde{g}$ (resp. the induced metric) ([3]).

A submanifold M in an l.c.K.-manifold $\tilde{M}$ is called a *CR-submanifold* if there exists a differentiable distribution $\mathcal{D} : x \to \mathcal{D}_x \subset T_x M$ on M satisfying the following conditions;

(i) $\mathcal{D}$ is holomorphic, i.e., $J\mathcal{D}_x = D_x$ for each $x \in M$ and

(ii) the complementary orthogonal distribution $\mathcal{D}^\perp : x \to \mathcal{D}^\perp_x \subset T_x M$ is totally real, i.e., $J\mathcal{D}^\perp_x \subset T^\perp_x M$ for each $x \in M$, where $T_x M$ (resp. $T^\perp_x M$) denotes the tangent (resp. normal) vector space at x of M ([1],[4], [6], etc.).

If $\dim \mathcal{D}^\perp_x = 0$ (resp. $\dim \mathcal{D}_x = 0$) for each $x \in M$, then the CR-submanifold is *holomorphic* (resp. *totally real*). A CR-submanifold M is said to be *anti-holomorphic* if $J\mathcal{D}^\perp_x = T^\perp_x M$ for any $x \in M$.

In [6], we proved that

Proposition 3.1([6]). *In a CR-submanifold M in an l.c.K.-manifold $\tilde{M}$, we have*

(i) *the distribution $\mathcal{D}^\perp$ is integrable,*

(ii) *the distribution $\mathcal{D}$ is integrable if and only if*

$$\tilde{g}(\sigma(X, JY) - \sigma(Y, JX) + 2\tilde{g}(JX, Y)\alpha^\sharp, JZ) = 0 \tag{3.4}$$

for any $X, Y \in \mathcal{D}$ and $Z \in \mathcal{D}^\perp$.

A CR-submanifold is said to be *proper* if it is neither holomorphic nor totally real.

In a CR-submanifold M in an l.c.K.-manifold $\tilde{M}$, we know the following formulas ([6]);

$$\begin{aligned} \tilde{g}(\nabla_U Z, X) &= \tilde{g}(JA_{JZ}U, X) + \tilde{g}(\alpha^\sharp, Z)\tilde{g}(U, X) \\ &\quad +\tilde{g}(U, Z)\tilde{g}(\alpha^\sharp, X) - \tilde{g}(\beta^\sharp, Z)\tilde{g}(JU, X), \end{aligned} \tag{3.5}$$

$$A_{JZ}W = A_{JW}Z + \tilde{g}(\beta^\sharp, Z)W - \tilde{g}(\beta^\sharp, W)Z \tag{3.6}$$

for any $U \in \Gamma TM$, $X \in \mathcal{D}$ and $Z, W \in \mathcal{D}^\perp$.

A CR-submanifold is said to be *mixed geodesic* if the second fundamental form σ satisfies $\sigma(\mathcal{D}, \mathcal{D}^\perp) = \{0\}$ and to be $\mathcal{D}$*-geodesic* if the second fundamental form σ satisfies $\sigma(\mathcal{D}, \mathcal{D}) = \{0\}$.

For a CR-submanifold M of an almost Hermitian manifold $\tilde{M}$, we denote by ν the complementary orthogonal subbundle of $J\mathcal{D}^\perp$ in the normal bundle $T^\perp M$. Then we have the following direct sum decomposition

$$T^\perp M = J\mathcal{D}^\perp \oplus \nu, \qquad J\mathcal{D}^\perp \perp \nu. \tag{3.7}$$

Remark 3.1. By the definition of ν, a CR-submanifold is anti-holomorphic if $\nu_x = \{0\}$ for any $x \in M$.

Since the distribution $\mathcal{D}^\perp$ is integrable, we consider a maximal integral submanifold $M_\perp$ of the distribution. Let us cosider a necessary and sufficient condition that $M_\perp$ is totally geodesic in M, that is, $\nabla_Z W \in \mathcal{D}^\perp$ for any $Z, W \in \mathcal{D}^\perp$. This condition is equivalent to $\tilde{g}(J\nabla_Z W, \Gamma TM) = \{0\}$. The condition means (i) $\tilde{g}(J\nabla_Z W, X) = 0$ and (ii) $\tilde{g}(J\nabla_Z W, V) = 0$ for any $X \in \mathcal{D}$ and $Z, W, V \in \mathcal{D}^\perp$. But, the case (ii) is trivial. So, we only consider the case (i).

Using (2.2), we have

$$\begin{aligned} \tilde{g}(J\nabla_Z W, X) &= \tilde{g}(\nabla_Z JW, X) - \tilde{g}((\nabla_Z J)W, X) \\ &= \tilde{g}(\sigma(X, Z), JW) - \tilde{g}(Z, W)\tilde{\beta}^\sharp, X) \\ &= -\{\tilde{g}(\sigma(X, Z) - \tilde{g}(\alpha^\sharp, JX)JZ, JW)\} \end{aligned}$$

Thus we have

Proposition 3.2. *In a CR-submanifold M of an l.c.K.-manifold $\tilde{M}$, a maximal integral submanifold $M_\perp$ of the distribution $\mathcal{D}^\perp$ is totally geodesic in M if and only if*

$$\sigma(X, Z) - \tilde{g}(\alpha^\sharp, JX)JZ \in \nu \tag{3.8}$$

for any $X \in \mathcal{D}$ and $Z, W \in \mathcal{D}^\perp$.

Corollary 3.3. *Under the same assumption of the above proposition, if the Lee vector field $\alpha^\sharp$ is orthogonal to $\mathcal{D}$, then $M_\perp$ is totally geodesic in M if and only if $\sigma(\mathcal{D}, \mathcal{D}) \subset \nu$.*

Remark 3.2. The above corollary is the same with a Kaehlerian case ([2]).

§4. Pseudo-umbilical CR-submanifolds in an l.c.K.-manifold

Now, we put $\dim \tilde{M} = m$, $\dim M = n$, $\dim \mathcal{D} = 2p$, $\dim \mathcal{D}^{\perp} = q$ $(2p + q = n)$ and $\dim \nu = 2s$. Let $\{e_1, ..., e_p, e_1^*, ..., e_p^*\}$, $\{e_{2p+1}, ..., e_{2p+q}\}$, $\{e_{2p+1}^*, ..., e_{2p+q}^*\}$ and $\{e_{n+q+1}, ..., e_{n+q+2s}\}$ $(n + q + 2s = m)$ be a local orthonormal basis of $\mathcal{D}$, $\mathcal{D}^{\perp}$, $J\mathcal{D}^{\perp}$ and ν, respectively, where $e_i^* = Je_i$ for $i \in \{1, ..., p\}$ and $e_{2p+a}^* = Je_{2p+a}$ for $a \in \{1, ..., q\}$. We call such local basis an *adapted frame* of $\tilde{M}$.

Remark 4.1. It is known that the dimensions of the distributions $\mathcal{D}$ and ν are even and they have an almost complex tructure, respectively.

A CR-submanifold M in an l.c.K.-manifold $\tilde{M}$ is said to be *pseudo-umbilical* if the shape operator A satisfies, with respect to the adapted frame,

$$(4.1)\qquad \begin{cases} A_{e_{2p+a}^*} X = a_{2p+a} X + b_{2p+a} \tilde{g}(X, e_{2p+a}) e_{2p+a}, \\ A_{e_{n+q+\alpha}} X = a_{n+q+\alpha} X + \displaystyle\sum_{a=1}^{q} b_{n+q+\alpha}^{2p+a} \tilde{g}(X, e_{2p+a}) e_{2p+a}, \\ A_{e_{n+q+\alpha}^*} X = a_{(n+q+\alpha)^*} X + \displaystyle\sum_{a=1}^{q} b_{(n+q+\alpha)^*}^{2p+a} \tilde{g}(X, e_{2p+a}) e_{2p+a} \end{cases}$$

for any $X \in \Gamma TM$, where $a_{2p+a}, a_{n+q+\alpha}, a_{(n+q+\alpha)^*}, b_{2p+a}, b_{n+q+\alpha}^{2p+a}$ and $b_{(n+q+\alpha)^*}^{2p+a}$ are differentiable functions on M for any $a \in \{1, 2, ..., q\}$ and $\alpha \in \{1, 2..., s\}$ ([1]).

Now, we proved that

Proposition 4.1([6]). *Let M be a pseudo-umbilical CR-submanifold in an l.c.K.-manifold $\tilde{M}$. If $\dim \mathcal{D}_x > 1$ at each point $x \in M$, then the functions $a_{2p+a}, a_{n+q+\alpha}$ and $a_{(n+q+\alpha)^*}$ are vanish for each $a \in \{1, ..., q\}$ and $\alpha \in \{1, 2, ..., s\}$.*

By virtue of Proposition 4.1, the equation (4.1) can be written as

$$(4.2)\qquad \begin{cases} A_{e_{2p+a}^*} X = b_{2p+a} \tilde{g}(X, e_{2p+a}) e_{2p+a}, \\ A_{e_{n+q+\alpha}} X = \displaystyle\sum_{a=1}^{q} b_{n+q+\alpha}^{2p+a} \tilde{g}(X, e_{2p+a}) e_{2p+a}, \\ A_{e_{n+q+\alpha}^*} X = \displaystyle\sum_{a=1}^{q} b_{(n+q+\alpha)^*}^{2p+a} \tilde{g}(X, e_{2p+a}) e_{2p+a} \end{cases}$$

for any $X \in \Gamma TM$.

The equation (4.2) teaches us

Proposition 4.2. *A pseudo-umbilical CR-submanifold M in an l.c.K.-manifold $\tilde{M}$ is $\mathcal{D}$-geodesic, that is, $\sigma(\mathcal{D}, \mathcal{D}) = \{0\}$.*

Next, we prove

Proposition 4.3. *A pseudo-umbilical CR-submanifold M in an l.c.K.-manifold* $\tilde{M}$ *is a mixed geodesic, that is,* $\sigma(\mathcal{D}, \mathcal{D}^\perp) = \{0\}$.

Proof. It is enough to show $\tilde{g}(\sigma(X, Z), N) = 0$ for any $X \in \mathcal{D}$, $Z \in \mathcal{D}^\perp$ and $N \in \Gamma T^\perp M$.

We solve the above equation into three cases;

Case 1.

$$\begin{aligned} \tilde{g}(\sigma(e_i, e_{2p+a}), Je_{2p+b}) &= \tilde{g}(A_{e^*_{2p+b}} e_i, e_{2p+a}) \\ &= b_{2p+b} \tilde{g}(e_i, e_{2p+b}) \tilde{g}(e_{2p+b}, e_{2p+a}) = 0 \end{aligned}$$

for any $i \in \{1, 2, ..., 2p\}$ and $a, b \in \{1, 2, ..., q\}$.

Case 2.

$$\begin{aligned} \tilde{g}(\sigma(e_i, e_{2p+a}), e_{n+q+\alpha}) &= \tilde{g}(A_{e_{n+q+\alpha}} e_i, e_{2p+a}) \\ &= \sum_{b=1}^{q} b^{2p+b}_{n+q+\alpha} \tilde{g}(e_i, e_{2p+b}) \tilde{g}(e_{2p+b}, e_{2p+a}) = 0 \end{aligned}$$

for any $i \in \{1, 2, ..., 2p\}$, $a \in \{1, 2, ..., q\}$ and $\alpha \in \{1, 2, ..., s\}$.

Case 3.

$$\begin{aligned} \tilde{g}(\sigma(e_i, e_{2p+a}), e^*_{n+q+\alpha}) &= \tilde{g}(A_{e^*_{n+q+\alpha}} e_i, e_{2p+a}) \\ &= \sum_{b=1}^{q} b^{2p+b}_{(n+q+\alpha)^*} \tilde{g}(e_i, e_{2p+b}) \tilde{g}(e_{2p+b}, e_{2p+a}) = 0 \end{aligned}$$

for any $i \in \{1, 2, ..., 2p\}$, $a \in \{1, 2, ..., q\}$ and $\alpha \in \{1, 2, ..., s\}$.

The proof is complete. □

By virtue of Propositions 3.2 and 4.3, we have

Proposition 4.4. *In a pseudo-umbilical CR-submanifold M in an l.c.K.-manifold* $\tilde{M}$, *if the Lee vector field* $\alpha^\sharp$ *is not orthogonal to* $\mathcal{D}$, *the maximal integral submanifold* $M_\perp$ *of the distribution* $\mathcal{D}^\perp$ *is never totally geodesic in M.*

By virtue of Propositions 3.1 and 4.4, we have

Proposition 4.5. *In a pseudo-unbilical CR-submanifold M in an l.c.K.-manifold* $\tilde{M}$, *the distribution* $\mathcal{D}$ *is integrable if and only if* $\tilde{g}(\alpha^\sharp, JZ) = 0$ *for any* $Z \in \mathcal{D}^\perp$, *that is, the Lee vector field* $\alpha^\sharp$ *is orthogonal to* $J\mathcal{D}^\perp$, *or equivalently, the vector field* $\beta^\sharp$ *is orthogonal to* $\mathcal{D}^\perp$.

§5. The length of the second fundamental form and the mean curvature

In this section, we consider the length of the second fundamental form and the mean curvature in a pseudo-umbilical CR-submanifold M in an l.c.K.-manifold $\tilde{M}$.

Let M be an n-dimensional pseudo-umbilical CR-submanifold in an m-dimensional l.c.K.-manifold $\tilde{M}$. The equation (4.2) implies

$$\begin{aligned} \sigma(U,V) &= \sum_{a=1}^{q} b_{2p+a}\tilde{g}(U,e_{2p+a})\tilde{g}(V,e_{2p+a})e^*_{2p+a} \\ &\quad + \sum_{a=1}^{q}\sum_{\alpha=1}^{s}\{b^{2p+a}_{n+q+\alpha}\tilde{g}(U,e_{2p+a})\tilde{g}(V,e_{2p+a})e_{n+q+\alpha} \\ &\quad + b^{2p+a}_{(n+q+\alpha)^*}\tilde{g}(U,e_{2p+a})\tilde{g}(V,e_{2p+a})e^*_{n+q+\alpha}\} \end{aligned} \tag{5.1}$$

for any $U, V \in \Gamma TM$.

Next, using (5.1), we calculate the length $\|\sigma\|$ of the second fundamental form σ and the length $\|H\|$ (the mean curvature) of the mean curvature vector field H, where the mean curvature vector field H is given by

$$H = \frac{1}{n}\sum_{\mu=1}^{n}\sigma(e_\mu, e_\mu) \tag{5.2}$$

for an adapted frame $\{e_1, e_2, ..., e_n\}$.

The length $\|\sigma\|$ of the second fundamental form σ is defined by

$$\|\sigma\|^2 = \sum_{\mu,\lambda=1}^{n}\tilde{g}(\sigma(e_\mu,e_\lambda),\sigma(e_\mu,e_\lambda)). \tag{5.3}$$

And it is separated to

$$\begin{aligned} \|\sigma\|^2 &= \sum_{\mu,\lambda=1}^{n}\{\sum_{a=1}^{q}\tilde{g}(\sigma(e_\mu,e_\lambda),e^*_{2p+a})^2 \\ &\quad + \sum_{\alpha=1}^{s}\tilde{g}(\sigma(e_\mu,e_\lambda),e_{n+q+\alpha})^2 + \sum_{\alpha=1}^{s}\tilde{g}(\sigma(e_\mu,e_\lambda),e^*_{n+q+\alpha})^2\}. \end{aligned} \tag{5.3$'$}$$

The mean curvature $\|H\|$ is defined

$$\|H\|^2 = \frac{1}{n^2}\sum_{\mu,\lambda=1}^{n}\tilde{g}(\sigma(e_\mu,e_\mu),\sigma(e_\lambda,e_\lambda)). \tag{5.4}$$

By virtue of Propositions 4.1, 4.2 and 4.3, the nontrivial components of σ are

$$
\begin{aligned}
(5.5)\quad \sigma(e_{2p+c}, e_{2p+b}) &= \sum_{a=1}^{q} b_{2p+a}\tilde{g}(e_{2p+c}, e_{2p+a})\tilde{g}(e_{2p+b}, e_{2p+a})e^*_{2p+a} \\
&\quad + \sum_{a=1}^{q}\sum_{\alpha=1}^{s}\{b^{2p+a}_{n+q+\alpha}\tilde{g}(e_{2p+c}, e_{2p+a})\tilde{g}(e_{2p+b}, e_{2p+a})e_{n+q+\alpha} \\
&\quad + b^{2p+a}_{(n+q+\alpha)^*}\tilde{g}(e_{2p+c}, e_{2p+a})\tilde{g}(e_{2p+b}, e_{2p+a})e^*_{n+q+\alpha}\} \\
&= \sum_{a=1}^{q} b_{2p+a}\delta_{ca}\delta_{ab}e^*_{2p+a} + \sum_{a=1}^{q}\sum_{\alpha=1}^{s}\{b^{2p+a}_{n+q+\alpha}\delta_{ca}\delta_{ba}e_{n+q+\alpha} \\
&\quad + b^{2p+a}_{(n+q+\alpha)^*}\delta_{ca}\delta_{ba}e^*_{n+q+\alpha}\}.
\end{aligned}
$$

Using (5.5), the equation (5.3) is written as

$$
\begin{aligned}
\|\sigma\|^2 &= \sum_{c,b,a=1}^{q} \tilde{g}(\sigma(e_{2p+c}, e_{2p+b}), e^*_{2p+a})^2 + \sum_{c,b=1}^{q}\sum_{\beta=1}^{s}\{\tilde{g}(\sigma(e_{2p+c}, e_{2p+b}), e_{n+q+\beta})^2 \\
&\quad + \tilde{g}(\sigma(e_{2p+c}, e_{2p+b}), e^*_{n+q+\beta})^2\} \\
&= \sum_{c,b,a=1}^{q} (b_{2p+b}\delta_{cb}\delta_{ba})^2 + \sum_{c,b=1}^{q}\sum_{\beta,\alpha=1}^{s}\{(b^{2p+b}_{n+q+\alpha}\delta_{cb}\delta_{\beta\alpha})^2 \\
&\quad + (b^{2p+b}_{(n+q+\alpha)^*}\delta_{cb}\delta_{\beta\alpha})^2\} \\
&= \sum_{a=1}^{q}(b_{2p+a})^2 + \sum_{b=1}^{q}\sum_{\alpha=1}^{s}\{(b^{2p+b}_{n+q+\alpha})^2 + (b^{2p+b}_{(n+q+\alpha)^*})^2\}.
\end{aligned}
$$

Hence, we get

$$
(5.6)\qquad \|\sigma\|^2 = \sum_{a=1}^{q}[(b_{2p+a})^2 + \sum_{\alpha=1}^{s}\{(b^{2p+a}_{n+q+\alpha})^2 + (b^{2p+a}_{(n+q+\alpha)^*})^2\}].
$$

Moreover, we have from (5.5)

$$
(5.7)\quad \sigma(e_{2p+b}, e_{2p+b}) = b_{2p+b}e^*_{2p+b} + \sum_{\alpha=1}^{s}\{b^{2p+b}_{n+q+\alpha}e_{n+q+\alpha} + b^{2p+b}_{(n+q+\alpha)^*}e^*_{n+q+\alpha}\}.
$$

By virtue of (5.4) and (5.7), we obtain

$$
\begin{aligned}
(5.8)\qquad n^2\|H\|^2 &= \sum_{b,a=1}^{q} \tilde{g}(\sigma(e_{2p+b}, e_{2p+b}), \sigma(e_{2p+a}, e_{2p+a})) \\
&= \sum_{a=1}^{q}(b_{2p+a})^2 + \sum_{a=1}^{q}\sum_{\alpha=1}^{s}\{(b^{2p+a}_{n+q+\alpha})^2 + (b^{2p+a}_{(n+q+\alpha)^*})^2\} \\
&\quad + \sum_{b\neq a=1}^{q}\sum_{\alpha=1}^{s}(b^{2p+b}_{n+q+\alpha}b^{2p+a}_{n+q+\alpha} + b^{2p+b}_{(n+q+\alpha)^*}b^{2p+a}_{(n+q+\alpha)^*}).
\end{aligned}
$$

Thus we have from (5.6) and (5.8)

$$(5.9)\qquad n^2\|H\|^2 = \|\sigma\|^2 + \sum_{b\neq a=1}^{q}\sum_{\alpha=1}^{s}(b^{2p+b}_{n+q+\alpha}b^{2p+a}_{n+q+\alpha} + b^{2p+b}_{(n+q+\alpha)^*}b^{2p+a}_{(n+q+\alpha)^*}).$$

The equation (5.9) means

Theorem 5.1. *If an n-dimensional pseudo-umbilical CR-submanifold M in an l.c.K.-manifold $\tilde{M}$ is anti-holomorhpic, then the submanifold M is totally geodesic or the length $\| \sigma \|$ of the second fundamental form σ and the mean curvature $\|H\|$ have the relation $\|\sigma\| = n\|H\|$.*

§6. Pseudo-umbilical CR-submanifolds in an l.c.K.-space form

Let $\tilde{M}(c)$ be an l.c.K-space form with the constant holomorphic sectional curvature c. Then, by virtue of (3.3), we have

$$(6.1)\qquad R_{\mu\lambda\lambda\mu} = \tilde{R}_{\mu\lambda\lambda\mu} + \tilde{g}(\sigma_{\mu\mu}, \sigma_{\lambda\lambda}) - \tilde{g}(\sigma_{\mu\lambda}, \sigma_{\mu\lambda}),$$

where $R_{\omega\nu\mu\lambda}$ and $\sigma_{\mu\lambda}$ are respectively the componernt of R and σ with respect to the adapted frame, that is,

$$(6.2)\qquad R_{\omega\nu\mu\lambda} = R(e_\omega, e_\nu, e_\mu, e_\lambda), \quad \sigma_{\mu\lambda} = \sigma(e_\mu, e_\lambda).$$

From (6.1), we have

$$(6.3)\qquad r = \sum_{\mu,\lambda=1}^{n}\tilde{R}_{\mu\lambda\lambda\mu} + n^2\|H\|^2 - \|\sigma\|^2,$$

where r is the scalar curvature with respect to the induced metric.

Next, we calculate $\sum_{\mu\lambda=1}^{n}\tilde{R}_{\mu\lambda\lambda\mu}$ in an l.c.K.space form $\tilde{M}(c)$.

We can separate it as

$$\begin{aligned}\sum_{\mu,\lambda=1}^{n}\tilde{R}_{\mu\lambda\lambda\mu} &= \sum_{j,i=1}^{2p}\tilde{R}_{jiij} + 2\sum_{j=1}^{p}\sum_{a=1}^{q}\{\tilde{R}_{j(2p+a)(2p+a)j} \\ &\quad +\tilde{R}_{j^*(2p+a)(2p+a)j^*}\} + \sum_{b,a=1}^{q}\tilde{R}_{(2p+b)(2p+a)(2p+a)(2p+b)} \\ &= \sum_{j,i=1}^{p}\{\tilde{R}_{jiij} + 2\tilde{R}_{ji^*i^*j} + \tilde{R}_{j^*i^*i^*j^*}\} \\ &\quad +4\sum_{j=1}^{p}\sum_{a=1}^{q}\tilde{R}_{j(2p+a)(2p+a)j} + \sum_{b,a=1}^{q}\tilde{R}_{(2p+b)(2p+a)(2p+a)(2p+b)}.\end{aligned}$$

Since we know $\tilde{R}_{j^*i^*i^*j^*} = \tilde{R}_{jiij}$ and $\tilde{R}_{j^*(2p+a)(2p+a)j^*} = \tilde{R}_{j(2p+a)(2p+a)j}$, the above equation is

$$\begin{aligned}(6.4) \qquad \sum_{\mu,\lambda=1}^{n} \tilde{R}_{\mu\lambda\lambda\mu} &= 2\sum_{j,i=1}^{p}(\tilde{R}_{jiij} + \tilde{R}_{ji^*i^*j}) + 4\sum_{j=1}^{p}\sum_{a=1}^{q}\tilde{R}_{j(2p+a)(2p+a)j} \\ &\quad + \sum_{b,a=1}^{q}\tilde{R}_{(2p+b)(2p+a)(2p+a)(2p+b)}.\end{aligned}$$

Thus using (6.4), (6.3) is written as

$$\begin{aligned}(6.5) \qquad r &= 2\sum_{j,i=1}^{p}(\tilde{R}_{jiij} + \tilde{R}_{ji^*i^*j}) + 4\sum_{j=1}^{p}\sum_{a=1}^{q}\tilde{R}_{j(2p+a)(2p+a)j} \\ &\quad + \sum_{b,a=1}^{q}\tilde{R}_{(2p+b)(2p+a)(2p+a)(2p+b)} + n^2\|H\|^2 - \|\sigma\|^2.\end{aligned}$$

We have from (2.3)

$$4\tilde{R}_{jiij} = c(\delta_{jj}\delta_{ii} - \delta_{ji}\delta_{ji}) + 3(\delta_{ii}P_{jj} - \delta_{ji}P_{ji} + \delta_{jj}P_{ii} - \delta_{ji}P_{ij}).$$

So, we obtain

$$(6.6) \qquad 4\sum_{j,i=1}^{p}\tilde{R}_{jiij} = (p-1)(pc + 6\sum_{i=1}^{p}P_{ii}).$$

Similarly, we have from (2.3)

$$4\tilde{R}_{ji^*i^*j} = c(\delta_{jj}\delta_{ii} - \delta_{ji}\delta_{ji}) + 3(\delta_{ii}P_{jj} - \delta_{ji}P_{ji}).$$

So, we have

$$(6.7) \qquad 4\sum_{j,i=1}^{p}\tilde{R}_{ji^*i^*j} = (p-1)(pc + 3\sum_{i=1}^{p}P_{ii}).$$

Moreover, we have from (2.3)

$$4\tilde{R}_{j(2p+a)(2p+a)j} = c\delta_{jj}\delta_{aa} + 3(P_{jj}\delta_{aa} + \delta_{jj}P_{(2p+a)(2p+a)}).$$

Thus we get

$$(6.8) \qquad 4\sum_{j=1}^{p}\sum_{a=1}^{q}\tilde{R}_{j(2p+a)(2p+a)j} = pqc + 3\{q\sum_{j=1}^{p}P_{jj} + p\sum_{a=1}^{q}P_{(2p+a)(2p+a)}\}.$$

Finally, since we get

$$\begin{aligned}4\tilde{R}_{(2p+b)(2p+a)(2p+a)(2p+b)} &= c(\delta_{bb}\delta_{aa} - \delta_{ba}\delta_{ba}) + 3(\delta_{aa}P_{(2p+b)(2p+b)} \\ &\quad -\delta_{ba}P_{(2p+b)(2p+a)} + \delta_{bb}P_{(2p+a)(2p+a)} \\ &\quad -\delta_{ba}P_{(2p+a)(2p+a)}),\end{aligned}$$

we obtain

$$4\sum_{b,a=1}^{q}\tilde{R}_{(2p+b)(2p+a)(2p+a)(2p+b)} = (q-1)(qc + 6\sum_{b=1}^{q}P_{(2p+b)(2p+b)}). \tag{6.9}$$

Substituting (6.6), (6.7), (6.8) and (6.9) into (6.5), we obtain

$$\begin{aligned}4r &= (n^2-n-2p)c + 6(2n-3-p)\sum_{j=1}^{p}P_{jj} \\ &\quad +6(n-1)\sum_{a=1}^{q}P_{(2p+a)(2p+a)} + 4n^2\|H\|^2 - 4\|\sigma\|^2.\end{aligned} \tag{6.10}$$

From (5.3), we have

Theorem 6.1. *In an n-dimensional pseudo-umbilical CR-submanifold M in an l.c.K.-space form $\tilde{M}(c)$, the mean curvature $\|H\|$ satisfies the following inequality.*

$$\begin{aligned}\|H\|^2 &\geq \frac{1}{4n^2}\{4r - (n^2-n-2p)c - 6(2n-3-p)\sum_{j=1}^{p}P_{jj} \\ &\quad -6(n-1)\sum_{a=1}^{q}P_{(2p+a)(2p+a)}\}.\end{aligned} \tag{6.11}$$

In particular, in the equality case of (6.11)*, we have from* (6.10) *and* (6.11)*, the submanifold M is totally geodesic and the scalar curvature r with respect to the induced metric satisfies*

$$4r = (n^2-n-2p)c + 6(2n-3-p)\sum_{j=1}^{p}P_{jj} + 6(n-1)\sum_{a=1}^{q}P_{(2p+a)(2p+a)}. \tag{6.12}$$

Corollary 6.2. *In an n-dimensional pseudo-umbilical CR-submanifold M in a complex space form $\tilde{M}(c)$, the mean curvature $\|H\|$ satisfies the following inequality.*

$$\|H\|^2 \geq \frac{1}{4n^2}\{4r - (n^2-n-2p)c\}. \tag{6.13}$$

In particular, in the equality case of (6.13)*, we have from* (6.10) *and* (6.11)*, the submanifold M is totally geodesic and the scalar curvature r with respect to the induced metric satisfies*

$$4r = (n^2 - n - 2p)c. \tag{6.14}$$

Substituting (5.9) into (6.10), we obtain

$$\begin{aligned} 4r = {} & (n^2 - n - 2p)c + 6(2n-3-p)\sum_{j=1}^{p} P_{jj} + 6(n-1)\sum_{a=1}^{q} P_{(2p+a)(2p+a)} \\ & +4\sum_{b\neq a=1}^{q}\sum_{\alpha=1}^{s}(b^{2p+b}_{n+q+\alpha}b^{2p+a}_{n+q+\alpha} + b^{2p+b}_{(n+q+\alpha)^*}b^{2p+a}_{(n+q+\alpha)^*}). \end{aligned} \tag{6.15}$$

Thus we have

Proposition 6.3. *In a pseudo-umbilical CR-submanifold M in an l.c.K.-space form $\tilde{M}(c)$, the scalar curvature r with respect to the induced metric is given by* (6.15)*.*

Corollary 6.4. *In a pseudo-umbilical CR-submanifold M in a complex space form $\tilde{M}(c)$, the scalar curvature r with respect to the induce metric is given by*

$$\begin{aligned} 4r = {} & (n^2 - n - 2p)c + 4\sum_{b\neq a=1}^{q}\sum_{\alpha=1}^{s}(b^{2p+b}_{n+q+\alpha}b^{2p+a}_{n+q+\alpha} \\ & + b^{2p+b}_{(n+q+\alpha)^*}b^{2p+a}_{(n+q+\alpha)^*}). \end{aligned} \tag{6.16}$$

Acknowlegements

The authors express their hearty thanks to the referee who give them very kind and important suggestions.

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Koji Matsumoto
2-3-65, Nishi-Odor,
Yonezawa, Yamagata,992-0059, Japan
E-mail: tokiko_ matsumoto@yahoo.com

Zerrin Şentürk
Department of Mathematics,
Fen-Edebiyat Faculty,
Istanbul Technical University,Maslak,
Istanbul, Turkey
E-mail: senturk@itu.edu.tr