

On Kenmotsu manifolds satisfying certain curvature conditions

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Abstract. The object of the present paper is to study some curvature conditions on Kenmotsu manifolds. Also, we classify Kenmotsu manifolds which satisfy $P \cdot \tilde{C} = 0$, $\tilde{C} \cdot \tilde{C} = 0$, $\tilde{Z} \cdot \tilde{C} = 0$, $\tilde{C} \cdot \tilde{Z} = 0$ and $C \cdot \tilde{C} = 0$, where P is the projective curvature tensor, \tilde{Z} is the concircular curvature tensor, \tilde{C} is the quasi-conformal curvature tensor and C is the conformal curvature tensor.

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§1. Introduction

The product of an almost contact manifold M and the real line \mathbb{R} carries a natural almost complex structure. However if one takes M to be an almost contact metric manifold and supposes that the product metric G on $M \times \mathbb{R}$ is Kaehlerian, then the structure on M is cosymplectic ([6]) and not Sasakian. On the other hand Oubina [9] pointed out that if the conformally related metric $e^{2t}G$, t being the coordinate on \mathbb{R} , is Kaehlerian, then M is Sasakian and conversely.

In [11], S. Tanno classified connected almost contact metric manifolds whose automorphism groups possess the maximum dimension. For such a manifold M , the sectional curvature of plane sections containing ξ is a constant, say c . If $c > 0$, M is a homogeneous Sasakian manifold of constant sectional curvature. If $c = 0$, M is the product of a line or a circle with a Kaehler manifold of constant holomorphic sectional curvature. If $c < 0$, M is a warped product space

$\mathbb{R} \times_f \mathbb{C}^n$. In 1971, Kenmotsu studied a class of contact Riemannian manifolds satisfying some special conditions ([8]). We call it Kenmotsu manifold.

The notion of the *quasi-conformal curvature tensor* was given by Yano and Sawaki [12]. According to them a quasi-conformal curvature tensor \tilde{C} is defined by

$$(1.1) \quad \begin{aligned} \tilde{C}(X, Y)Z &= aR(X, Y)Z + b[S(Y, Z)X - S(X, Z)Y \\ &\quad + g(Y, Z)QX - g(X, Z)QY] \\ &\quad - \frac{\tau}{n} \left[\frac{a}{n-1} + 2b \right] [g(Y, Z)X - g(X, Z)Y], \end{aligned}$$

where a and b are constants and R , S , Q and τ are the Riemannian curvature tensor type of $(1, 3)$, the Ricci tensor of type $(0, 2)$, the Ricci operator defined by $g(QX, Y) = S(X, Y)$ and scalar curvature of the manifold respectively. If $a = 1$ and $b = -\frac{1}{n-2}$ then (1.1) takes the form

$$(1.2) \quad \begin{aligned} \tilde{C}(X, Y)Z &= R(X, Y)Z \\ &\quad - \frac{1}{n-2} [S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY] \\ &\quad + \frac{\tau}{(n-1)(n-2)} [g(Y, Z)X - g(X, Z)Y] \\ &= C(X, Y)Z, \end{aligned}$$

where C is the conformal curvature tensor ([5]). Thus the conformal curvature tensor C is a particular case of the tensor \tilde{C} . For this reason \tilde{C} is called the quasi-conformal curvature tensor. A manifold (M^n, g) , $n > 1$, shall be called quasi-conformally flat if the quasi-conformal curvature tensor $\tilde{C} = 0$. It is known ([2]) that the quasi-conformally flat manifold is either conformally flat if $a \neq 0$ or, Einstein if $a = 0$ and $b \neq 0$. Since, they give no restrictions for manifolds if $a = 0$ and $b = 0$, it is essential for us to consider the case of $a \neq 0$ or $b \neq 0$.

We next define endomorphisms $R(X, Y)$ and $X \wedge_A Y$ of $\chi(M)$ by

$$R(X, Y)W = \nabla_X \nabla_Y W - \nabla_Y \nabla_X W - \nabla_{[X, Y]} W,$$

$$(X \wedge_A Y)W = A(Y, W)X - A(X, W)Y,$$

respectively, where $X, Y, W \in \chi(M)$ and A is the symmetric $(0, 2)$ -tensor.

On the other hand, the *projective curvature tensor* P and the *concircular curvature tensor* \tilde{Z} in a Riemannian manifold (M^n, g) are defined by

$$(1.3) \quad P(X, Y)W = R(X, Y)W - \frac{1}{n-1} (X \wedge_S Y)W,$$

$$(1.4) \quad \tilde{Z}(X, Y)W = R(X, Y)W - \frac{\tau}{n(n-1)}(X \wedge_g Y)W,$$

respectively.

An almost contact metric manifold is said to be an η -Einstein manifold if the Ricci tensor S satisfies the condition

$$S(X, Y) = \lambda_1 g(X, Y) + \lambda_2 \eta(X)\eta(Y),$$

where λ_1, λ_2 are certain scalars. A Riemannian or a semi-Riemannian manifold is said to be semisymmetric if $R(X, Y) \cdot R = 0$, where $R(X, Y)$ is considered as a derivation of the tensor algebra at each point of the manifold for the tangent vectors X, Y .

Kenmotsu manifolds have been studied by many authors such as De and Pathak [3], Jun, De and Pathak [7], Ozgür and De [10] and many others.

In the present paper we have studied some curvature conditions on Kenmotsu manifolds. We have classified Kenmotsu manifolds which satisfy $P \cdot \tilde{C} = 0$, $\tilde{C} \cdot \tilde{C} = 0$, $\tilde{Z} \cdot \tilde{C} = 0$, $\tilde{C} \cdot \tilde{Z} = 0$ and $C \cdot \tilde{C} = 0$, where P is the projective curvature tensor, \tilde{Z} is the concircular curvature tensor, \tilde{C} is the quasi-conformal curvature tensor and C is the conformal curvature tensor.

§2. Preliminaries

Let $(M^n, \phi, \xi, \eta, g)$ be an n -dimensional (where $n = 2m + 1$) almost contact metric manifold, where ϕ is a $(1, 1)$ -tensor field, ξ is the structure vector field, η is a 1-form and g is the Riemannian metric. It is well known that the (ϕ, ξ, η, g) structure satisfies the conditions ([1])

$$(2.1) \quad \begin{aligned} \phi^2 X &= -X + \eta(X)\xi, \quad g(X, \xi) = \eta(X), \\ \phi\xi &= 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1, \\ g(\phi X, \phi Y) &= g(X, Y) - \eta(X)\eta(Y), \end{aligned}$$

for any vector fields X and Y on M^n .

If moreover

$$(\nabla_X \phi)Y = -g(X, \phi Y)\xi - \eta(Y)\phi X,$$

$$\nabla_X \xi = X - \eta(X)\xi,$$

where ∇ denotes the Riemannian connection of g hold, then $(M^n, \phi, \xi, \eta, g)$ is called a Kenmotsu manifold. In this case, it is well known ([8]) that

$$(2.2) \quad R(X, Y)\xi = \eta(X)Y - \eta(Y)X,$$

$$(2.3) \quad S(X, \xi) = -(n-1)\eta(X),$$

where S denotes the Ricci tensor. From (2.2), it easily follows that

$$(2.4) \quad R(X, \xi)Y = g(X, Y)\xi - \eta(Y)X,$$

$$(2.5) \quad R(X, \xi)\xi = \eta(X)\xi - X.$$

In a Kenmotsu manifold, using (2.3) and (2.4), equations (1.3), (1.4), (1.2), and (1.1) reduce to

$$(2.6) \quad P(\xi, X)Y = -g(X, Y)\xi - \frac{1}{n-1}S(X, Y)\xi,$$

$$(2.7) \quad \tilde{Z}(\xi, X)Y = (1 + \frac{\tau}{n(n-1)})(-g(X, Y)\xi + \eta(Y)X),$$

$$(2.8) \quad \begin{aligned} C(\xi, Y)W &= \frac{n-1+\tau}{(n-1)(n-2)}\{g(Y, W)\xi - \eta(W)Y\} \\ &\quad - \frac{1}{n-2}\{S(Y, W)\xi - \eta(W)QY\}, \end{aligned}$$

$$(2.9) \quad \begin{aligned} \tilde{C}(\xi, Y)W &= K\{\eta(W)Y - g(Y, W)\xi\} \\ &\quad + b\{S(Y, W)\xi - \eta(W)QY\}, \end{aligned}$$

respectively, where $K = a + (n-1)b + \frac{\tau}{n}(\frac{a}{n-1} + 2b)$.

Let $\{e_i\}$ ($1 \leq i \leq n$) be an orthonormal basis of the tangent space at any point. Then the Ricci tensor and the scalar curvature of M are defined by

$$S(X, Y) = \sum_{i=1}^n g(R(e_i, X)Y, e_i),$$

and

$$\tau = \sum_{i=1}^n S(e_i, e_i),$$

respectively.

Since $S(X, Y) = g(QX, Y)$, we have

$$S(\phi X, \phi Y) = g(Q\phi X, \phi Y),$$

where Q is the Ricci operator. Using the properties $g(X, \phi Y) = -g(\phi X, Y)$, $Q\phi = \phi Q$, (2.1) and (2.3), we get

$$S(\phi X, \phi Y) = S(X, Y) + (n-1)\eta(X)\eta(Y).$$

Also we have ([1])

$$(\nabla_X \eta)Y = g(X, Y) - \eta(X)\eta(Y).$$

A Kenmotsu manifold M^n is said to be an η -Einstein manifold if its Ricci tensor S is of the form

$$S(X, Y) = \lambda_1 g(X, Y) + \lambda_2 \eta(X)\eta(Y),$$

for any vector fields X and Y , where $\lambda_1 = 1 + \frac{\tau}{n-1}$ and $\lambda_2 = -(n + \frac{\tau}{n-1})$.

Now, we define $P(X, Y) \cdot \tilde{C}$, $\tilde{Z}(X, Y) \cdot \tilde{C}$, $\tilde{C}(X, Y) \cdot \tilde{C}$, $\tilde{C}(X, Y) \cdot \tilde{Z}$ and $C(X, Y) \cdot \tilde{C}$ as

$$(2.10) \quad \begin{aligned} (P(X, Y) \cdot \tilde{C})(U, V)W &= P(X, Y)\tilde{C}(U, V)W - \tilde{C}(P(X, Y)U, V)W \\ &\quad - \tilde{C}(U, P(X, Y)V)W - \tilde{C}(U, V)P(X, Y)W, \end{aligned}$$

$$(2.11) \quad \begin{aligned} (\tilde{Z}(X, Y) \cdot \tilde{C})(U, V)W &= \tilde{Z}(X, Y)\tilde{C}(U, V)W - \tilde{C}(\tilde{Z}(X, Y)U, V)W \\ &\quad - \tilde{C}(U, \tilde{Z}(X, Y)V)W - \tilde{C}(U, V)\tilde{Z}(X, Y)W, \end{aligned}$$

$$(2.12) \quad \begin{aligned} (\tilde{C}(X, Y) \cdot \tilde{C})(U, V)W &= \tilde{C}(X, Y)\tilde{C}(U, V)W - \tilde{C}(\tilde{C}(X, Y)U, V)W \\ &\quad - \tilde{C}(U, \tilde{C}(X, Y)V)W - \tilde{C}(U, V)\tilde{C}(X, Y)W, \end{aligned}$$

$$(2.13) \quad \begin{aligned} (\tilde{C}(X, Y) \cdot \tilde{Z})(U, V)W &= \tilde{C}(X, Y)\tilde{Z}(U, V)W - \tilde{Z}(\tilde{C}(X, Y)U, V)W \\ &\quad - \tilde{Z}(U, \tilde{C}(X, Y)V)W - \tilde{Z}(U, V)\tilde{C}(X, Y)W, \end{aligned}$$

$$(2.14) \quad \begin{aligned} (C(X, Y) \cdot \tilde{C})(U, V)W &= C(X, Y)\tilde{C}(U, V)W - \tilde{C}(C(X, Y)U, V)W \\ &\quad - \tilde{C}(U, C(X, Y)V)W - \tilde{C}(U, V)C(X, Y)W, \end{aligned}$$

respectively, where $X, Y, U, V, W \in \chi(M)$.

§3. Kenmotsu manifolds satisfying $P(\xi, Y) \cdot \tilde{C} = 0$

In this section we consider a Kenmotsu manifold M^n satisfying the condition

$$(3.1) \quad P(\xi, Y) \cdot \tilde{C} = 0.$$

From (2.10), we have

$$(3.2) \quad \begin{aligned} (P(\xi, Y) \cdot \tilde{C})(Z, U)W &= P(\xi, Y)\tilde{C}(Z, U)W - \tilde{C}(P(\xi, Y)Z, U)W \\ &\quad - \tilde{C}(Z, P(\xi, Y)U)W - \tilde{C}(Z, U)P(\xi, Y)W = 0. \end{aligned}$$

Taking the inner product with X and using (2.6) in (3.2), we have

$$(3.3) \quad \begin{aligned} &g(Y, \tilde{C}(Z, U)W)\eta(X) - g(Y, Z)g(\tilde{C}(\xi, U)W, X) \\ &- g(Y, U)g(\tilde{C}(Z, \xi)W, X) - g(Y, W)g(\tilde{C}(Z, U)\xi, X) \\ &+ \frac{1}{n-1}\{S(Y, \tilde{C}(Z, U)W)\eta(X) - S(Y, Z)g(\tilde{C}(\xi, U)W, X) \\ &- S(Y, U)g(\tilde{C}(Z, \xi)W, X) - S(Y, W)g(\tilde{C}(Z, U)\xi, X)\} = 0. \end{aligned}$$

Taking $U = \xi$ in (3.3), we have

$$(3.4) \quad \begin{aligned} &g(Y, \tilde{C}(Z, \xi)W)\eta(X) - g(Y, W)g(\tilde{C}(Z, \xi)\xi, X) \\ &+ \frac{1}{n-1}\{S(Y, \tilde{C}(Z, \xi)W)\eta(X) - S(Y, W)g(\tilde{C}(Z, \xi)\xi, X)\} = 0. \end{aligned}$$

Using (2.9) in (3.4), we get

$$(3.5) \quad \begin{aligned} &K\{g(Y, Z)\eta(X)\eta(W) + \frac{1}{n-1}S(Y, Z)\eta(X)\eta(W) \\ &+ g(Y, W)\eta(X)\eta(Z) - g(Y, W)g(X, Z) \\ &+ \frac{1}{n-1}S(Y, W)\eta(X)\eta(Z) - \frac{1}{n-1}S(Y, W)g(X, Z)\} \\ &- b\{S(Y, Z)\eta(X)\eta(W) + \frac{1}{n-1}S(QY, Z)\eta(X)\eta(W) \\ &- S(X, Z)g(Y, W) - (n-1)g(Y, W)\eta(X)\eta(Z) \\ &- \frac{1}{n-1}S(Y, W)S(X, Z) - S(Y, W)\eta(X)\eta(Z)\} = 0, \end{aligned}$$

where $S(QY, Z) = S^2(Y, Z)$.

Let $\{e_i\}$ ($1 \leq i \leq n$) be an orthonormal basis of the tangent space at any point. Then the sum for $1 \leq i \leq n$ of the relation (3.5) for $Y = W = e_i$ gives

$$(3.6) \quad \{\tau + n(n-1)\}[bS(X, Z) - Kg(X, Z) + \{K + (n-1)b\}\eta(X)\eta(Z)] = 0.$$

Let U_1 and U_2 be a part of M satisfying $\tau + n(n-1) = 0$ and

$$(3.7) \quad bS(X, Z) - Kg(X, Z) + \{K + (n-1)b\}\eta(X)\eta(Z) = 0,$$

respectively. In the case of $\tau + n(n-1) \neq 0$, if $b = 0$, from (3.7) we get $a = 0$. This is the contradiction. Thus we find $b \neq 0$. By virtue of (3.7), we obtain $\frac{K}{b} = 1 + \frac{\tau}{n-1}$, which yields

$$S(X, Z) = (1 + \frac{\tau}{n-1})g(X, Z) - (n + \frac{\tau}{n-1})\eta(X)\eta(Z).$$

Hence we have the following:

Theorem 1. *Let M^n be an n -dimensional ($n > 1$) Kenmotsu manifold satisfying the condition $P(\xi, Y) \cdot \tilde{C} = 0$. Then M is a part of*

1. $\tau = -n(n-1)$, that is, the scalar curvature is the negative constant, or
2. an η -Einstein manifold.

§4. Kenmotsu manifolds satisfying $\tilde{C}(\xi, Y) \cdot \tilde{C} = 0$

In this section we consider a Kenmotsu manifold M^n satisfying the condition

$$\tilde{C}(\xi, Y) \cdot \tilde{C} = 0.$$

From (2.12), we have

$$(4.1) \quad \begin{aligned} (\tilde{C}(\xi, Y) \cdot \tilde{C})(U, V)W &= \tilde{C}(\xi, Y)\tilde{C}(U, V)W - \tilde{C}(\tilde{C}(\xi, Y)U, V)W \\ &\quad - \tilde{C}(U, \tilde{C}(\xi, Y)V)W - \tilde{C}(U, V)\tilde{C}(\xi, Y)W = 0. \end{aligned}$$

Taking the inner product with X and using $U = \xi$ in (4.1), we obtain

$$(4.2) \quad \begin{aligned} g(\tilde{C}(\xi, Y)\tilde{C}(\xi, V)W, X) - g(\tilde{C}(\tilde{C}(\xi, Y)\xi, V)W, X) \\ - g(\tilde{C}(\xi, \tilde{C}(\xi, Y)V)W, X) - g(\tilde{C}(\xi, V)\tilde{C}(\xi, Y)W, X) = 0. \end{aligned}$$

Let $\{e_i\}$ ($1 \leq i \leq n$) an orthonormal basis of the tangent space at any point. Now we put $X = W = e_i$ in (4.2). Straightforwardly we calculate the equation $\sum_{i=1}^n g((\tilde{C}(\xi, Y) \cdot \tilde{C})(\xi, e_i)W, e_i) = 0$. Then we obtain

$$(4.3) \quad \begin{aligned} g(\tilde{C}(\xi, Y)\tilde{C}(\xi, e_i)W, e_i) - g(\tilde{C}(\tilde{C}(\xi, Y)\xi, e_i)W, e_i) \\ - g(\tilde{C}(\xi, \tilde{C}(\xi, Y)e_i)W, e_i) - g(\tilde{C}(\xi, e_i)\tilde{C}(\xi, Y)W, e_i) = 0. \end{aligned}$$

Using (1.1) and (2.9) in (4.3), we get

$$\begin{aligned} \{a + (n-2)b\}[bS(QY, W) - \frac{1}{n(n-1)}\{a(\tau + n(n-1)) + 2(n-1)b\tau\}S(Y, W) \\ - (n-1)Kg(Y, W)] = 0. \end{aligned}$$

Thus we have $a + (n - 2)b = 0$, or

$$(4.4) \quad bS(QY, W) - \frac{1}{n(n-1)}\{a(\tau + n(n-1)) + 2(n-1)b\tau\}S(Y, W) \\ - (n-1)Kg(Y, W) = 0.$$

If $b = 0$, then we get

$$a\{\tau + n(n-1)\}\{S(Y, W) + (n-1)g(Y, W)\} = 0.$$

We can easily verify that

$$S(Y, W) = -(n-1)g(Y, W).$$

Therefore we have the following:

Theorem 2. *Let M^n be an n -dimensional ($n > 1$) Kenmotsu manifold satisfying the condition $\tilde{C}(\xi, Y) \cdot \tilde{C} = 0$. Then we get*

1. $a + (n - 2)b = 0$, or

2. we find

i) if $b = 0$, then M is an Einstein manifold,

ii) if $b \neq 0$, then we get

$$S(QY, W) = \left(\frac{K}{b} - n + 1\right)S(Y, W) + (n-1)\frac{K}{b}g(Y, W).$$

Now we need the following:

Lemma 1. ([4]) *Let A be a symmetric $(0, 2)$ -tensor at a point x of a semi-Riemannian manifold (M^n, g) , $n > 1$, and let $T = g \bar{\wedge} A$ be the Kulkarni-Nomizu product of g and A . Then, the relation*

$$T \cdot T = \alpha Q(g, T), \quad \alpha \in \mathbb{R}$$

is satisfied at x if and only if the condition

$$A^2 = \alpha A + \lambda g, \quad \lambda \in \mathbb{R}$$

holds at x .

From Theorem 2 and Lemma 1 we get the following:

Corollary 1. *Let M^n be an n -dimensional ($n > 1$) Kenmotsu manifold satisfying the condition $\tilde{C}(\xi, Y) \cdot \tilde{C} = 0$, then $T \cdot T = \alpha Q(g, T)$, where $T = g \bar{\wedge} A$ and $\alpha = \frac{K}{b} - n + 1$.*

§5. Kenmotsu manifolds satisfying $\tilde{Z}(\xi, Y) \cdot \tilde{C} = 0$

In this section we consider a Kenmotsu manifold M^n satisfying the condition

$$\tilde{Z}(\xi, Y) \cdot \tilde{C} = 0.$$

From (2.11), we have

$$(5.1) \quad \begin{aligned} (\tilde{Z}(\xi, Y) \cdot \tilde{C})(U, V)W &= \tilde{Z}(\xi, Y)\tilde{C}(U, V)W - \tilde{C}(\tilde{Z}(\xi, Y)U, V)W \\ &\quad - \tilde{C}(U, \tilde{Z}(\xi, Y)V)W - \tilde{C}(U, V)\tilde{Z}(\xi, Y)W = 0. \end{aligned}$$

Now using $U = \xi$ in (5.1), we have

$$(5.2) \quad \begin{aligned} \tilde{Z}(\xi, Y)\tilde{C}(\xi, V)W - \tilde{C}(\tilde{Z}(\xi, Y)\xi, V)W \\ - \tilde{C}(\xi, \tilde{Z}(\xi, Y)V)W - \tilde{C}(\xi, V)\tilde{Z}(\xi, Y)W = 0. \end{aligned}$$

Taking the inner product with X in (5.2) and using (2.7), we get

$$(5.3) \quad \begin{aligned} \{1 + \frac{\tau}{n(n-1)}\} \{g(X, Y)\eta(\tilde{C}(\xi, V)W) - g(Y, \tilde{C}(\xi, V)W)\eta(X) \\ - g(X, \tilde{C}(Y, V)W) + g(\tilde{C}(\xi, V)W, X)\eta(Y) \\ - g(\tilde{C}(\xi, Y)W, X)\eta(V) - g(\tilde{C}(\xi, V)Y, X)\eta(W) \\ + g(Y, W)g(\tilde{C}(\xi, V)\xi, X)\} = 0. \end{aligned}$$

Again from (2.7), we have $\tau \neq -n(n-1)$. Thus

$$(5.4) \quad \begin{aligned} g(X, Y)\eta(\tilde{C}(\xi, V)W) - g(Y, \tilde{C}(\xi, V)W)\eta(X) - g(X, \tilde{C}(Y, V)W) \\ + g(\tilde{C}(\xi, V)W, X)\eta(Y) - g(\tilde{C}(\xi, Y)W, X)\eta(V) - g(\tilde{C}(\xi, V)Y, X)\eta(W) \\ + g(Y, W)g(\tilde{C}(\xi, V)\xi, X) = 0. \end{aligned}$$

Using (2.9) in (5.4), we get

$$(5.5) \quad \begin{aligned} -a\{g(X, Y)g(V, W) + g(R(Y, V)W, X) - g(X, V)g(Y, W)\} \\ -b(n-1)\{g(X, Y)g(V, W) - g(X, Y)\eta(V)\eta(W) - g(X, V)g(Y, W) \\ + g(Y, W)\eta(X)\eta(V)\} + b\{S(Y, W)g(X, V) - S(X, Y)g(V, W) \\ - S(Y, W)\eta(X)\eta(V) + S(X, Y)\eta(V)\eta(W)\} = 0. \end{aligned}$$

Let $\{e_i\}$ ($1 \leq i \leq n$) be an orthonormal basis of the tangent space at any point. Then the sum for $1 \leq i \leq n$ of the relation (5.5) for $Y = W = e_i$ gives

$$(5.6) \quad \begin{aligned} (b-a)S(X, V) &= \{(n-1)a + (n-1)^2b + b\tau\}g(X, V) \\ &\quad - b\{\tau + n(n-1)\}\eta(X)\eta(V). \end{aligned}$$

If $a = b(\neq 0)$, then we have $a\{\tau + n(n-1)\}\{g(X, V) - \eta(X)\eta(V)\} = 0$. Because of (2.7), we find $\tau + n(n-1) \neq 0$. Thus $a \neq b$ holds. We obtain $\{a + (n-2)b\}\{\tau + n(n-1)\} = 0$ from (5.6), which means that $a + (n-2)b = 0$. Thus equation (5.6) can be rewritten as follows:

$$S(X, V) = (1 + \frac{\tau}{n-1})g(X, Y) - (n + \frac{\tau}{n-1})\eta(X)\eta(Y).$$

Hence we have the following:

Theorem 3. *An n -dimensional ($n > 1$) Kenmotsu manifold M^n satisfying the condition $\tilde{Z}(\xi, Y) \cdot \tilde{C} = 0$ is an η -Einstein manifold.*

§6. Kenmotsu manifolds satisfying $\tilde{C}(\xi, Y) \cdot \tilde{Z} = 0$

In this section we consider a Kenmotsu manifold M^n satisfying the condition

$$\tilde{C}(\xi, Y) \cdot \tilde{Z} = 0.$$

From (2.13), we have

$$\begin{aligned} (\tilde{C}(\xi, Y) \cdot \tilde{Z})(U, V)W &= \tilde{C}(\xi, Y)\tilde{Z}(U, V)W - \tilde{Z}(\tilde{C}(\xi, Y)U, V)W \\ &\quad - \tilde{Z}(U, \tilde{C}(\xi, Y)V)W - \tilde{Z}(U, V)\tilde{C}(\xi, Y)W = 0. \end{aligned} \quad (6.1)$$

Putting $U = \xi$ in (6.1), we have

$$\begin{aligned} &\tilde{C}(\xi, Y)\tilde{Z}(\xi, V)W - \tilde{Z}(\tilde{C}(\xi, Y)\xi, V)W \\ &- \tilde{Z}(\xi, \tilde{C}(\xi, Y)V)W - \tilde{Z}(\xi, V)\tilde{C}(\xi, Y)W = 0. \end{aligned} \quad (6.2)$$

Taking the inner product with $X \in \chi(M)$ in (6.2) and using (2.9), we get

$$\begin{aligned} &K\{g(Y, \tilde{Z}(\xi, V)W)\eta(X) - \eta(\tilde{Z}(\xi, V)W)g(Y, X) - g(\tilde{Z}(\xi, V)W, X)\eta(Y) \\ &\quad + g(\tilde{Z}(Y, V)W, X) + g(\tilde{Z}(\xi, Y)W, X)\eta(V) - g(Y, W)g(\tilde{Z}(\xi, V)\xi, X) \\ &\quad + g(\tilde{Z}(\xi, V)Y, X)\eta(W)\} - b\{S(Y, \tilde{Z}(\xi, V)W)\eta(X) - \eta(\tilde{Z}(\xi, V)W)S(Y, X) \\ &\quad + (n-1)g(\tilde{Z}(\xi, V)W, X)\eta(Y) + g(\tilde{Z}(QY, V)W, X) + g(\tilde{Z}(\xi, QY)W, X)\eta(V) \\ &\quad - S(Y, W)g(\tilde{Z}(\xi, V)\xi, X) + g(\tilde{Z}(\xi, V)QY, X)\eta(W)\} = 0. \end{aligned}$$

Using (1.4) and (2.7) in the above equation, we obtain

$$\begin{aligned} &K\{g(R(Y, V)W, X) + g(Y, X)g(V, W) - g(X, V)g(Y, W)\} \\ &- b\{g(R(QY, V)W, X) + S(Y, X)g(V, W) - S(Y, W)g(X, V)\} = 0. \end{aligned} \quad (6.3)$$

Let $\{e_i\}$ ($1 \leq i \leq n$) be an orthonormal basis of the tangent space at any point. Then the sum for $1 \leq i \leq n$ of the relation (6.3) for $X = V = e_i$ gives

$$K\{S(Y, W) + (n-1)g(Y, W)\} - b\{S^2(Y, W) + (n-1)S(Y, W)\} = 0.$$

When $b = 0$, the above equation can be rewritten as follows:

$$K\{S(Y, W) + (n-1)g(Y, W)\} = 0,$$

which means that $K\{\tau + n(n-1)\} = 0$. From (2.7), we find $\tau + n(n-1) \neq 0$. Thus we get $K = 0$, namely, $a = 0$. Therefore we get $b \neq 0$ and

$$(6.4) \quad S(QY, W) = \left(\frac{K}{b} - n + 1\right)S(Y, W) + (n-1)\frac{K}{b}g(Y, W).$$

This leads to the following:

Theorem 4. *In an n -dimensional ($n > 1$) Kenmotsu manifold M if the condition $\tilde{C}(\xi, Y) \cdot \tilde{Z} = 0$ holds on M , then the equation (6.4) is satisfied on M .*

From Theorem 4 and Lemma 1 we get the following:

Corollary 2. *Let M be an n -dimensional ($n > 1$) Kenmotsu manifold satisfying the condition $\tilde{C}(\xi, Y) \cdot \tilde{Z} = 0$, then $T \cdot T = \alpha Q(g, T)$, where $T = g \bar{\wedge} A$ and $\alpha = \frac{K}{b} - n + 1$.*

§7. Kenmotsu manifolds satisfying $C(\xi, Y) \cdot \tilde{C} = 0$

In this section we consider a Kenmotsu manifold M^n satisfying the condition

$$C(\xi, Y) \cdot \tilde{C} = 0.$$

From (2.14), we have

$$(7.1) \quad \begin{aligned} (C(\xi, Y) \cdot \tilde{C})(U, V)W &= C(\xi, Y)\tilde{C}(U, V)W - \tilde{C}(C(\xi, Y)U, V)W \\ &\quad - \tilde{C}(U, C(\xi, Y)V)W - \tilde{C}(U, V)C(\xi, Y)W = 0. \end{aligned}$$

Taking the inner product with X and using $U = \xi$ in (7.1), we obtain

$$(7.2) \quad \begin{aligned} g(C(\xi, Y)\tilde{C}(\xi, V)W, X) - g(\tilde{C}(C(\xi, Y)\xi, V)W, X) \\ - g(\tilde{C}(\xi, C(\xi, Y)V)W, X) - g(\tilde{C}(\xi, V)C(\xi, Y)W, X) = 0. \end{aligned}$$

Let $\{e_i\}$ ($1 \leq i \leq n$) be an orthonormal basis of the tangent space at any point. Now we put $X = V = e_i$ in (7.2). Straightforwardly we calculate

the equation $\sum_{i=1}^n g((C(\xi, Y) \cdot \tilde{C})(\xi, e_i)W, e_i) = 0$. Then from (1.1), (2.8) and (2.9), we obtain

$$\left\{ \frac{a}{n-2} + b \right\} \{ S(QY, W) - [1 + \frac{\tau}{n-1} - (n-1)]S(Y, W) - [\tau + n - 1]g(Y, W) \} = 0.$$

Let U_1 and U_2 be a part of M satisfying $a + b(n-2) = 0$ and

$$(7.3) \quad S(QY, W) = [\frac{\tau}{n-1} + 2 - n]S(Y, W) + [\tau + n - 1]g(Y, W).$$

This leads to the following:

Theorem 5. *In n -dimensional ($n > 1$) Kenmotsu manifold M^n satisfying the condition $C(\xi, Y) \cdot \tilde{C} = 0$. Then we get*

1. $a + (n-2)b = 0$, or
2. $a + b(n-2) \neq 0$, then the equation (7.3) holds on M .

From Theorem 5 and Lemma 1 we get the following:

Corollary 3. *Let M be an n -dimensional ($n > 1$) Kenmotsu manifold satisfying the condition $C(\xi, Y) \cdot \tilde{C} = 0$, then $T \cdot T = \alpha Q(g, T)$, where $T = g \bar{\wedge} A$ and $\alpha = \frac{\tau}{n-1} + 2 - n$.*

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