

On ϕ -Ricci symmetric $N(k)$ -contact metric manifolds

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Abstract. The object of the present paper is to study ϕ -Ricci symmetric $N(k)$ -contact metric manifolds and ϕ -Ricci symmetric $N(k)$ -contact metric manifolds of dimension three. It is proved that a ϕ -Ricci symmetric $N(k)$ -contact metric manifold M^{2n+1} ($n > 1$) is Sasakian. Next we prove that a three dimensional $N(k)$ -contact metric manifold is locally ϕ -Ricci symmetric if and only if the scalar curvature is constant. Finally we give examples of ϕ -Ricci symmetric $N(k)$ -contact metric manifolds.

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§1. Introduction

In modern mathematics geometry of contact manifolds have become a matter of growing interest. An important class of contact manifolds consists of Sasakian manifolds. Again, through the works of Ch. Baikoussis, D. E. Blair and Th. Koufogiorgos [2] a new class of non-Sasakian contact manifolds has evolved. Such manifolds are known as $N(k)$ -contact metric manifolds. However, among the geometric properties of contact metric manifold symmetry is an important one. Symmetry of a contact metric manifold has been studied in several ways by several authors. For instance, the notion of locally ϕ -symmetric Sasakian manifolds has been introduced by T. Takahashi [12]. He studied several interesting properties of such a manifold in the context of Sasakian geometry. Recently U. C. De, A. A. Shaikh and Sudipta Biswas [8] introduced the notion of ϕ -recurrent Sasakian manifolds which generalizes the notion of ϕ -symmetric Sasakian manifolds. Also in another paper U. C. De and Aboul Kalam Gazi [7] introduced the notion of ϕ -recurrent $N(k)$ -contact metric manifolds. In a recent paper U. C. De and Avijit Sarkar [9] introduced the

notion of ϕ -Ricci symmetric Sasakian manifolds. From the definitions given in Section 3, it follows that every locally ϕ -symmetric Sasakian manifold is locally ϕ -Ricci symmetric. But, the converse is not in general true. Also a $N(k)$ -contact metric manifold $M^{2n+1}(n > 1)$ is Sasakian manifold if $k = 1$. Considering the above facts we generalize the notion of ϕ -symmetric Sasakian manifolds and study ϕ -Ricci symmetric $N(k)$ -contact metric manifolds. In the present paper we study ϕ -Ricci symmetric $N(k)$ -contact metric manifolds. The paper is organized as follows: In section 2 we recall $N(k)$ -contact metric manifolds. ϕ -Ricci symmetric $N(k)$ -contact metric manifolds have been studied in section 3. We prove that a ϕ -Ricci symmetric $N(k)$ -contact metric manifold $M^{2n+1}(n > 1)$ is Sasakian. Also we prove that if a $N(k)$ -contact metric manifold $M^{2n+1}(n > 1)$ is an η -Einstein manifold with constant coefficients, then the manifold is locally ϕ -Ricci symmetric. In section 4, we prove that a three-dimensional $N(k)$ -contact metric manifold is locally ϕ -Ricci symmetric if and only if the scalar curvature is constant. Finally we give examples of ϕ -Ricci symmetric $N(k)$ -contact metric manifolds.

§2. $N(k)$ - contact metric manifolds

An $(2n + 1)$ -dimensional manifold M^{2n+1} is said to admit an almost contact structure if it admits a tensor field ϕ of type $(1, 1)$, a vector field ξ and a 1-form η satisfying

$$(2.1) \quad (a) \phi^2 = -I + \eta \otimes \xi, \quad (b) \eta(\xi) = 1, \quad (c) \phi\xi = 0, \quad (d) \eta \circ \phi = 0.$$

An almost contact structure is said to be normal if the induced almost complex structure J on the product manifold $M^{2n+1} \times \mathbf{R}$ defined by $J(X, f \frac{d}{dt}) = (\phi X - f\xi, \eta(X) \frac{d}{dt})$ is integrable, where X is tangent to M , t is the coordinate of \mathbf{R} and f is a smooth function on $M \times \mathbf{R}$. Let g be a compatible Riemannian metric with almost contact structure (ϕ, ξ, η) , that is,

$$(2.2) \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).$$

Then M becomes an almost contact metric manifold equipped with an almost contact metric structure (ϕ, ξ, η, g) . From (2.2) it can be easily seen that

$$(2.3) \quad (a) g(X, \xi) = \eta(X), \quad (b) g(X, \phi Y) = -g(\phi X, Y),$$

for all vector fields X, Y . An almost contact metric structure becomes a contact metric structure if

$$(2.4) \quad g(X, \phi Y) = d\eta(X, Y),$$

for all vector fields X, Y . The 1-form η is then a contact form and ξ is its characteristic vector field. We define a $(1, 1)$ tensor field h by $h = \frac{1}{2}\mathcal{L}_\xi\phi$, where \mathcal{L} denotes the Lie-differentiation. Then h is symmetric and satisfies $h\phi = -\phi h$. We have $Tr.h = Tr.\phi h = 0$ and $h\xi = 0$. Also,

$$(2.5) \quad \nabla_X \xi = -\phi X - \phi h X,$$

holds in a contact metric manifold. A normal contact metric manifold is a Sasakian manifold. An almost contact metric manifold is Sasakian if and only if

$$(2.6) \quad (\nabla_X \phi)(Y) = g(X, Y)\xi - \eta(Y)X, \quad X, Y \in T_p M,$$

where ∇ is Levi-Civita connection of the Riemannian metric g . A contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$ for which ξ is a killing vector is said to be a K -contact manifold. A Sasakian manifold is K -contact but not conversely. However a 3-dimensional K -contact manifold is Sasakian ([10]). It is well known that the tangent sphere bundle of a flat Riemannian manifold admits a contact metric structure satisfying $R(X, Y)\xi = 0$ ([3]). On the other hand, on a Sasakian manifold the following holds:

$$(2.7) \quad R(X, Y)\xi = \eta(Y)X - \eta(X)Y.$$

The k -nullity distribution $N(k)$ of a Riemannian manifold M^{2n+1} ([13]) is defined by

$$N(k) : p \longrightarrow N_p(k) = \{Z \in T_p M : R(X, Y)Z = k[g(Y, Z)X - g(X, Z)Y]\},$$

k being a constant. If the characteristic vector field $\xi \in N(k)$, then we call a contact metric manifold an $N(k)$ -contact metric manifold ([5]). If $k = 1$, then $N(k)$ -contact metric manifold is Sasakian and if $k = 0$, then $N(k)$ -contact metric manifold is locally isometric to the product $E^{n+1} \times S^n(4)$ for $n > 1$ and flat for $n = 1$. If $k < 1$, the scalar curvature is $r = 2n(2n - 2 + k)$.

In [2], $N(k)$ -contact metric manifold were studied in some detail. For more details we refer to [6], [4].

In $N(k)$ -contact metric manifold the following relations hold:

$$(2.8) \quad h^2 = (k - 1)\phi^2, \quad k \leq 1,$$

$$(2.9) \quad (\nabla_X \phi)(Y) = g(X + hX, Y)\xi - \eta(Y)(X + hX),$$

$$(2.10) \quad R(\xi, X)Y = k[g(X, Y)\xi - \eta(Y)X],$$

$$(2.11) \quad S(X, \xi) = 2nk\eta(X),$$

$$(2.12) \quad S(X, Y) = 2(n-1)g(X, Y) + 2(n-1)g(hX, Y) \\ + [2(1-n) + 2nk]\eta(X)\eta(Y), \quad n \geq 1,$$

$$(2.13) \quad S(\phi X, \phi Y) = S(X, Y) - 2nk\eta(X)\eta(Y) - 4(n-1)g(hX, Y),$$

$$(2.14) \quad (\nabla_X \eta)(Y) = g(X + hX, \phi Y),$$

$$(2.15) \quad R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y],$$

$$(2.16) \quad \eta(R(X, Y)Z) = k[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)].$$

Lemma 2.1. ([12]) *Let M^{2n+1} be an η -Einstein manifold of dimension $(2n+1)$ ($n \geq 1$). If ξ belongs to the k -nullity distribution, then $k = 1$ and the structure is Sasakian.*

The above results will be used in the following sections.

§3. ϕ -Ricci symmetric $N(k)$ -contact metric manifolds

Definition 3.1. *A $N(k)$ -contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$ is said to be ϕ -Ricci symmetric if the Ricci operator satisfies*

$$\phi^2(\nabla_X Q)(Y) = 0,$$

for all vector fields $X, Y \in \chi(M)$ and $S(X, Y) = g(QX, Y)$. In particular, if X, Y are orthogonal to ξ then the manifold is said to be locally ϕ -Ricci symmetric.

Definition 3.2. *If the Ricci tensor S of the manifold M^{2n+1} is of the form $S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y)$, where a and b are smooth functions on M^{2n+1} , and $X, Y \in \chi(M)$, then the manifold is called an η -Einstein manifold.*

Let us suppose that the manifold is ϕ -Ricci symmetric. Then by definition

$$\phi^2(\nabla_X Q)(Y) = 0.$$

Using (2.1) we have from above

$$(3.1) \quad -(\nabla_X Q)(Y) + \eta((\nabla_X Q)(Y))\xi = 0.$$

or,

$$-g((\nabla_X Q)(Y), Z) + \eta((\nabla_X Q)(Y))\eta(Z) = 0.$$

Then we obtain

$$-g(\nabla_X Q(Y) - Q(\nabla_X Y), Z) + \eta((\nabla_X Q)(Y))\eta(Z) = 0,$$

which induces

$$(3.2) \quad -g(\nabla_X Q(Y), Z) + S(\nabla_X Y, Z) + \eta((\nabla_X Q)(Y))\eta(Z) = 0.$$

Putting $Y = \xi$ we get from (3.2)

$$(3.3) \quad -g(\nabla_X Q(\xi), Z) + S(\nabla_X \xi, Z) + \eta((\nabla_X Q)(\xi))\eta(Z) = 0.$$

In view of (2.11) and (2.5) it follows from (3.3)

$$(3.4) \quad \begin{aligned} 2nkg(\phi X, Z) + 2nkg(\phi hX, Z) - S(\phi X, Z) \\ - S(\phi hX, Z) + \eta((\nabla_X Q)(\xi))\eta(Z) = 0. \end{aligned}$$

Replacement of Z by ϕZ in (3.4) yields

$$(3.5) \quad 2nkg(\phi X, \phi Z) + 2nkg(\phi hX, \phi Z) - S(\phi X, \phi Z) - S(\phi hX, \phi Z) = 0.$$

In a $N(k)$ -contact metric manifold we have,

$$(3.6) \quad g(\phi X, \phi Z) = g(X, Z) - \eta(X)\eta(Z).$$

Replacing X by hX and multiplying both sides of (3.6) with $2nk$ we get

$$(3.7) \quad 2nkg(\phi hX, \phi Z) = 2nkg(hX, Z) - 2nk\eta(hX)\eta(Z).$$

Again in a contact metric manifold we have

$$(3.8) \quad \eta(hX) = g(hX, \xi) = g(X, h\xi) = 0.$$

Using (3.8) in (3.7) we obtain

$$(3.9) \quad 2nkg(\phi hX, \phi Z) = 2nkg(hX, Z).$$

Replacing X by hX and using (3.8) we obtain from (2.13)

$$(3.10) \quad S(\phi hX, \phi Y) = S(hX, Y) - 4(n-1)g(h^2 X, Y).$$

Using (2.1) and (2.8) in (3.10) we obtain,

$$(3.11) \quad \begin{aligned} S(\phi hX, \phi Y) &= S(hX, Y) + 4(n-1)(k-1)g(X, Y) \\ &\quad - 4(n-1)(k-1)\eta(X)\eta(Y). \end{aligned}$$

Using (3.6), (3.9), (2.13) and (3.11) in (3.5) we get

$$(3.12) \quad \begin{aligned} S(X, Y) + S(hX, Y) &= [2nk - 4(n-1)(k-1)]g(X, Y) \\ &\quad + [2nk + 4(n-1)]g(hX, Y) \\ &\quad + 4(n-1)(k-1)\eta(X)\eta(Y). \end{aligned}$$

Again from (2.12) we get

$$(3.13) \quad \begin{aligned} S(hX, Y) &= 2(n-1)g(hX, Y) - 2(n-1)(k-1)g(X, Y) \\ &\quad + 2(n-1)(k-1)\eta(X)\eta(Y). \end{aligned}$$

Using (3.13) in (3.12) we obtain

$$(3.14) \quad \begin{aligned} S(X, Y) &= 2(n+k-1)g(X, Y) + 2(nk+n-1)g(hX, Y) \\ &\quad + 2(n-1)(k-1)\eta(X)\eta(Y). \end{aligned}$$

Now comparing the value of $S(X, hY)$ from (2.12) and (3.14) we get

$$(3.15) \quad \begin{aligned} 2(n-1)g(X, hY) + 2(n-1)g(X, h^2Y) &= 2(n+k-1)g(X, hY) \\ &\quad + 2(nk+n-1)g(X, h^2Y) \\ &\quad + 2(n-1)(k-1)\eta(X)\eta(hY). \end{aligned}$$

Using (2.1), (2.8) and (3.8) in (3.16) we obtain

$$(3.16) \quad g(X, hY) = g(hX, Y) = n(k-1)g(X, Y) - n(k-1)\eta(X)\eta(Y).$$

Using (3.16) in (3.15) we get

$$(3.17) \quad S(X, Y) = Ag(X, Y) + B\eta(X)\eta(Y),$$

where $A = 2[(n+k-1) + n(k-1)(nk+n-1)]$ and $B = 2(k-1)[(n-1) - n(nk+n-1)]$.

Hence the manifold is an η -Einstein manifold. Thus we can state the following:

Proposition 3.1. *A $(2n+1)$ -dimensional ϕ -Ricci symmetric $N(k)$ -contact metric manifold is an η -Einstein manifold.*

If $k = 1$, then the manifold reduces to a Sasakian manifold. Therefore from Proposition 3.1 for $k = 1$ we can state the following:

Proposition 3.2. *A $(2n+1)$ -dimensional ϕ -Ricci symmetric Sasakian manifold is an Einstein manifold.*

The above Proposition 3.2 have been proved by De and Sarkar [9].

Now from Lemma 1 and Proposition 3.1 we get the following:

Theorem 3.1. *A ϕ -Ricci symmetric $N(k)$ -contact metric manifold M^{2n+1} ($n > 1$) is Sasakian.*

Definition 3.3. ([12]) *A Sasakian manifold is said to be a locally ϕ -symmetric manifold if*

$$\phi^2((\nabla_W R)(X, Y)Z) = 0,$$

for all vector fields X, Y, Z, W orthogonal to ξ . If X, Y, Z, W are not orthogonal to ξ , then the manifold is called ϕ -symmetric.

Remark: *Since ϕ -symmetry implies ϕ -Ricci symmetry, therefore the above Propositions and Theorem 3.1 also hold for ϕ -symmetric $N(k)$ -contact metric manifold M^{2n+1} ($n > 1$).*

Next suppose that the $N(k)$ -contact metric manifold M^{2n+1} ($n > 1$) is an η -Einstein manifold with constant coefficients. Then

$$(3.18) \quad S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y),$$

where $S(X, Y) = g(QX, Y)$ and a and b are constants M^{2n+1} . Hence

$$(3.19) \quad QX = aX + b\eta(X)\xi.$$

Using (3.19) we get

$$(3.20) \quad (\nabla_X Q)(Y) = b(\nabla_X \eta)(Y)\xi + b\eta(Y)(\nabla_X \xi).$$

Applying ϕ^2 on both sides of (3.20) and using (2.1)(c) we get

$$(3.21) \quad \phi^2((\nabla_X Q)(Y)) = b\eta(Y)\phi^2(\nabla_X \xi).$$

If Y is orthogonal to ξ , then (3.21) yields

$$\phi^2((\nabla_X Q)(Y)) = 0.$$

Hence the manifold is locally ϕ -Ricci symmetric. This helps us to conclude the following:

Theorem 3.2. *If a $N(k)$ -contact metric manifold M^{2n+1} ($n > 1$) is an η -Einstein manifold with constant coefficients, then the manifold is locally ϕ -Ricci symmetric.*

§4. Three-dimensional ϕ -Ricci symmetric $N(k)$ -contact metric manifolds

In a 3-dimensional Riemannian manifold we have ([14])

$$(4.1) \quad R(X, Y)Z = g(Y, Z)QX - g(X, Z)QY + S(Y, Z)X - S(X, Z)Y \\ + \frac{r}{2}[g(X, Z)Y - g(Y, Z)X],$$

where Q is the Ricci-operator, that is, $g(QX, Y) = S(X, Y)$ and r is the scalar curvature of the manifold. Now putting $Z = \xi$ in (4.1) and using (2.11), we get

$$(4.2) \quad R(X, Y)\xi = \eta(Y)QX - \eta(X)QY + 2k[\eta(Y)X - \eta(X)Y] \\ + \frac{r}{2}[\eta(X)Y - \eta(Y)X].$$

Using (2.15) in (4.2), we have

$$(4.3) \quad (k - \frac{r}{2})[\eta(Y)X - \eta(X)Y] = \eta(X)QY - \eta(Y)QX.$$

Putting $Y = \xi$ in (4.3) and using (2.1)(b), we obtain

$$(4.4) \quad QX = (\frac{r}{2} - k)X + (3k - \frac{r}{2})\eta(X)\xi.$$

Differentiating (4.4) covariantly with respect to W we obtain

$$(4.5) \quad (\nabla_W Q)(X) = \frac{1}{2}\{dr(W)X + (6k - r)(\nabla_W \eta)(X)\xi \\ + (6k - r)\eta(X)(\nabla_W \xi) - dr(W)\eta(X)\xi\}.$$

Applying ϕ^2 on both sides of above and using (2.1) we have

$$(4.6) \quad \phi^2((\nabla_W Q)(X)) = \frac{1}{2}\{dr(W)(-X + \eta(X)\xi) + (6k - r)\eta(X)\phi^2(\nabla_W \xi)\}.$$

Hence (4.5) yields the following:

Proposition 4.1. *If the scalar curvature r of a three-dimensional $N(k)$ -contact metric manifold equal to $6k$ then the manifold is ϕ -Ricci symmetric.*

If we take the vector field X orthogonal to ξ , then (4.6) gives

$$\phi^2((\nabla_W Q)(X)) = -\frac{1}{2}dr(W)X.$$

Thus we are in a position to conclude the following:

Theorem 4.1. *A three-dimensional $N(k)$ -contact metric manifold is locally ϕ -Ricci symmetric if and only if the scalar curvature is constant.*

§5. Examples

Example 5.1. ([1]): Let $M^{2n+1}(\varphi, \xi, \eta, g)$ be a contact metric manifold with ξ belonging to the k -nullity distribution. If the curvature tensor is harmonic, then for $k \neq 0$, the manifold M^{2n+1} is an Einstein Sasakian manifold. Hence a $N(k)$ -contact metric manifold with $k \neq 0$ whose curvature tensor is harmonic is ϕ -Ricci symmetric which is not ϕ -symmetric.

Example 5.2. ([11]): Ricci symmetric ($\nabla S = 0$) $N(k)$ -contact metric manifold with $k \neq 0$ is an Einstein manifold. Hence a Ricci symmetric $N(k)$ -contact metric manifold with $k \neq 0$ is ϕ -Ricci symmetric which is not ϕ -symmetric.

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