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# On $(2,3)$ torus decompositions of $QL$-configurations

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**Abstract.** Let $Q$ be an affine quartic which does not intersect transversely with the line at infinity $L_\infty$ such that there exists a $D_{2p}$-covering over $\mathbb{P}^2$ branched along $Q \cup L_\infty$. In this paper, we show the existence of a $(2,3)$ torus decomposition of the defining polynomial of $Q$ and its uniqueness except for one class.

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## Introduction

Let $C = \{f = 0\} \subset \mathbb{C}^2$ be an irreducible affine plane curve and let $a, b$ be coprime positive integers with $a, b \geq 2$. We say that $C$ is *a quasi torus curve of type* $(a, b)$ (c.f [2]) if there exist polynomials $f_r$, $f_p$ and $f_q$ such that they satisfy the following condition:

$$(*) \qquad f_r(x,y)^{ab} f(x,y) = f_p(x,y)^a + f_q(x,y)^b, \quad \deg f_j = j, \quad j = r, p, q$$

where $r \geq 0$ and $p$, $q > 0$. In the above affine equations, we also add conditions that any two polynomials of $f$, $f_r$, $f_p$ and $f_q$ are coprime. We say that such a decomposition $(*)$ is *a quasi torus decomposition of* $C$. A quasi torus curve $C$ is called *torus curve* if $f_r(x, y)$ is a non-zero constant.

In [8], H. Tokunaga studied $D_{2p}$ covers of $\mathbb{P}^2$ branched along a quintic $Q + L_\infty$ where $Q$ is a quartic and $L_\infty$ is the line at infinity and their relative positions are the following:

- $Q \cap L_\infty$ consists of two points.
  - (i) $L_\infty$ is bi-tangent to $Q$ at two distinct smooth points.
  - (ii) $L_\infty$ is tangent to a smooth point and passes through a singular point of $Q$.

(iii) $L_\infty$ passes through two distinct singular points of $Q$.

- $Q \cap L_\infty$ consists of a point.

  (iv) $L_\infty$ is tangent to $Q$ at a smooth point with intersection multiplicity 4.

  (v) $L_\infty$ intersects $Q$ at a singular point with intersection multiplicity 4.

Table 1 is the list of the possible configurations $Q + L_\infty$ which is given in [8].

| No. | Sing($Q$) | $Q \cap L_\infty$ | No. | Sing($Q$) | $Q \cap L_\infty$ |
|---|---|---|---|---|---|
| (1) | $2a_2$ | (i) | (11) | $e_6$ | (i) |
| (2) | $2a_2$ | (iv) | (12) | $e_6$ | (iv) |
| (3) | $2a_2 + a_1$ | (i) | (13) | $a_4 + a_2^\infty$ | (ii) |
| (4) | $2a_2 + a_1$ | (iv) | (14) | $a_3^\infty + a_2 + a_1$ | (ii) |
| (5) | $3a_2$ | (i) | (15) | $a_5 + a_1$ | (i) |
| (6) | $a_2 + a_3^\infty$ | (ii) | (16) | $a_5^\infty + a_2^\infty$ | (iii) |
| (7) | $a_5$ | (i) | (17) | $2a_3^\infty$ | (iii) |
| (8) | $a_5$ | (iv) | (18) | $a_7^\infty$ | (v) |
| (9) | $a_6^\infty$ | (ii) | (19) | $2a_3^\infty + a_1$ | (iii) |
| (10) | $a_4^\infty + a_2$ | (v) | | | |

Table 1.

Here the singularities of types $a_n$ and $e_6$ are defined by

$$a_n : x^2 + y^{n+1} = 0 \ (n \geq 1), \quad e_6 : x^3 + y^4 = 0$$

and the notation $*^\infty$ express singularities on the line $L_\infty$. We call such configurations *$QL$-configurations.* Note that $Q$ is irreducible for the cases $(1), \dots, (13)$ and $Q$ is not irreducible for the cases $(14), \dots, (19)$. We call configurations for the cases $(1), \dots, (13)$ (respectively for the cases $(14), \dots, (19)$) *irreducible $QL$-configurations* (resp. *non-irreducible $QL$-configurations*).

In [9], the second author studied two topological invariants of $QL$-configurations, the fundamental group $\pi_1(\mathbb{C}^2 \setminus Q)$ and the Alexander polynomial $\Delta_Q(t)$. In particular, he showed that

$$(\star) \qquad t^2 - t + 1 \text{ divides } \Delta_Q(t) \text{ except for the case (13) and (16).}$$

In [5], M. Oka studied a special type of degeneration family $\{C_\tau\}$ of irreducible torus sextics which degenerates into $C_0 := D + 2L_\infty$ where $D$ is a quartic and $L_\infty$ is a line. We call such a degeneration *a line degeneration of order* 2. (We will give the definition for general situation in §1). He showed the divisibility of the Alexander polynomials $\Delta_{C_\tau}(t) \mid \Delta_D(t)$ for $\tau \neq 0$. (Theorem 14 of [5])

In this paper, we study the possibilities of quasi torus decompositions of $QL$-configurations so that the above divisibility ($\star$) also follows from the results of the line degeneration by M. Oka.

This paper consists of 9 sections. In section 1, we recall the definition of line degenerated torus curves of type $(p,q)$. In section 2, we classify the singularities of line degenerated torus curves of type $(2,3)$. In section 3, we state our main theorem. In sections 4, 5, 6 and 7, we prove theorems which are stated in §3 using line degenerated torus curves of type $(2,3)$.

For the cases $(14),\dots,(19)$, $Q$ is not irreducible but we can also consider torus decomposition of non-irreducible $QL$-configurations without irreducibility and the condition $\gcd(a,b)=1$ in the definition of quasi torus curves. In section 8, we consider torus decomposition of above non-irreducible $QL$-configurations. In section 9, we will show that if a plane curve has a $(2,3)$ torus decomposition, then there exist infinite $(2,3)$ quasi torus decompositions.

## §1. Line degenerated torus curves

Let $U$ be an open neighborhood of 0 in $\mathbb{C}$ and let $\{C_s \mid s \in U\}$ be an analytic family of irreducible curves of degree $d$ which degenerates into $C_0 := D + j\,L_\infty$ $(1 \le j < d)$ where $D$ is an irreducible curve of degree $d-j$ and $L_\infty$ is a line. We denote this situation as $C_s \to C_0 = D + j\,L_\infty$. We assume that there is a point $B \in L_\infty \setminus L_\infty \cap D$ such that $B \in C_s$ and the multiplicity of $C_s$ at $P$ is $j$ for any non-zero $s \in U$. We call such a degeneration *a line degeneration of order* $j$ and we call $L_\infty$ *the limit line* of the degeneration. $B$ is called *the base point* of the degeneration. In [5], M. Oka showed that there exists a canonical surjection:

$$\varphi : \pi_1(\mathbb{C}^2 \setminus D) \to \pi_1(\mathbb{C}^2 \setminus C_s), \quad s : \text{sufficiently small},$$

where $\mathbb{C}^2 = \mathbb{P}^2 \setminus L_\infty$ and as a corollary he showed the divisibility among the Alexander polynomials of a line degeneration family:

$$\Delta_{C_s}(t) \mid \Delta_{D_0}(t).$$

### 1.1. Line degenerated torus curves

Let $C_{p,q} = \{F_{p,q} = 0\}$ be a $(p,q)$ torus curve $(p > q \ge 2)$ where $F_{p,q}$ is defined by

$$(1.1) \quad F_{p,q}(X,Y,Z) = F_p(X,Y,Z)^q + F_q(X,Y,Z)^p, \quad \deg F_k = k,\ k = p,q$$

where $(X, Y, Z)$ is a homogeneous coordinates system of $\mathbb{P}^2$. Suppose that $F_{p,q}$ is written by the following form:

$$F_{p,q}(X, Y, Z) = Z^j \, G(X, Y, Z) \tag{1.2}$$

where $G(X, Y, Z)$ is a homogeneous polynomial of degree $pq - j$. We call a curve $D = \{G = 0\}$ *a line degenerated torus curve of type* $(p, q)$ *of order* $j$ and the line $L_\infty = \{Z = 0\}$ *the limit line of the degeneration.* We divide the situations (1.2) into two cases.

**First case.** Suppose that the defining polynomials of associated curves are written as follows:

$$F_p(X, Y, Z) = F'_{p-r}(X, Y, Z)Z^r, \quad F_q(X, Y, Z) = F'_{q-s}(X, Y, Z)Z^s$$

where $r$ and $s$ are positive integers such that $r < p$ and $s < q$. We assume that $sp \geq rq$. Factoring $F_{p,q}$ as $F_{p,q}(X, Y, Z) = Z^{rq}G(X, Y, Z)$, we can see that $G$ is defined as

$$G(X, Y, Z) = F'_{p-r}(X, Y, Z)^q + F'_{q-s}(X, Y, Z)^p Z^{sp-rq}. \tag{1.3}$$

We call such a factorization *visible factorization* and $D$ is called *a visible degeneration of torus curve of type* $(p, q)$. By the definition, $D \cap L_\infty = \{F'_{p-r}(X, Y, Z) = Z = 0\}$ and thus the limit line $L_\infty$ is singular with respect to the visible degeneration of torus curve $D$. In [5], M. Oka showed that a visible degeneration of torus curve of type $(p, q)$ can be expressed as a line degeneration of irreducible torus curves of degree $pq$.

**Second case.** Neither $F_p$ or $F_q$ factors through $Z$ but $F$ can be written as (1.2). Then $D$ is called *an invisible degeneration of torus curve of type* $(p, q)$.

## §2. Line degenerated $(2, 3)$ torus curves of degree 4

In this section, we consider a $(2, 3)$ sextic of torus type which is a visible factorization.

### 2.1. Visible factorization

Let $D = \{G = 0\}$ be a quartic associated with a visible factorization (1.3):

$$D: \quad G(X, Y, Z) = F'_2(X, Y, Z)^2 + F'_1(X, Y, Z)^3 Z = 0.$$

We consider the associated curves $C_2 := \{F'_2 = 0\}$, $L := \{F'_1 = 0\}$ and $L_\infty := \{Z = 0\}$. Let $P$ be an inner singularity of $D$, namely $P$ is on the intersection $C_2 \cap L$, $C_2 \cap L_\infty$ or $C_2 \cap L \cap L_\infty$. Then the topological type $(D, P)$ depends only on the intersection multiplicities of $C_2$, $L$ and $L_\infty$. To describe singularities of $D$, we put the intersection multiplicities $\iota_1 := I(C_2, L; P)$ and $\iota_2 := I(C_2, L_\infty; P)$. Note that $0 \le \iota_i \le 2$ for $i = 1, 2$ and $(\iota_1, \iota_2) \ne (2, 2)$ as $L \ne L_\infty$.

**Lemma 1.** *Suppose that $C_2$ is smooth at $P$. Then we have the following descriptions:*

(1) *If $P \in C_2 \cap L \setminus L_\infty$, then we have $(D, P) \sim a_{3\iota_1 - 1}$.*

(2) *If $P \in C_2 \cap L_\infty \setminus L$, then $D$ is smooth at $P$ and is tangent to $L_\infty$ with $I(D, L_\infty; P) = 2\iota_2$.*

(3) *If $P \in C_2 \cap L \cap L_\infty$, then we have $(D, P) \sim a_{3\iota_1 + \iota_2 - 1}$.*

*Proof.* The assertion (1) is shown in [7] and [1] for general cases. We consider the assertions (2) and (3). We use affine coordinates $(x, z) = (X/Y, Z/Y)$ on $\mathbb{C}^2 = \mathbb{P}^2 \setminus \{Y = 0\}$. Then the defining polynomial $g$ of $D$ in the affine coordinates is given as follows.

$$g(x, z) := G(x, 1, z) = f_2'(x, z)^2 + f_1'(x, z)^3 z, \quad f_j'(x, z) := F_j'(x, 1, z),\ j = 1, 2.$$

As $C_2$ is smooth at $P$, we can take local coordinates $(u, v)$ so that $L_\infty = \{v = 0\}$ and $f_2'(u, v) = c_1\, v - \varphi(u)$ where $c_1 \ne 0$. Then $\mathrm{ord}_u \varphi(u) = \iota_2$ and

$$\begin{aligned} g(u, v) &= (c_1 v - \varphi(u))^2 + f_1'(u, v)^3 v \\ &= c\, u^{2\iota_2} + f_1'(P)^3\, v + (\text{higher terms}),\ \ c \ne 0. \end{aligned}$$

Here "higher terms" are linear combinations of monomials $u^\alpha v^\beta$ such that $2\iota_2 \beta + \alpha > 2\iota_2$. They do not affect the topology of $D$ at $P$. As $P$ is not on $L$, $f_1'(P)$ is not 0. Hence we have the assertion (2).

Now we show the assertion (3). As $C_2$ is smooth at $P$, we can take local coordinates $(u, v)$ so that $f_2'(u, v) = v$, $f_1'(v, v) = c_1\, v - \varphi_1(u)$ and $L_\infty = \{c_2\, v - \varphi_2(u) = 0\}$ where $c_1$ and $c_2$ are non-zero constants. Then $\mathrm{ord}_u \varphi_1(u) = \iota_1$ and $\mathrm{ord}_u \varphi_2(u) = \iota_2$ and

$$\begin{aligned} g(u, v) &= v^2 + (c_1\, v - \varphi_1(u))^3 (c_2\, v - \varphi_2(u)) \\ &= v^2 - c\, u^{3\iota_1 + \iota_2} + (\text{higher terms}),\ \ c \ne 0. \end{aligned}$$

Here "higher terms" are linear combinations of monomials $u^\alpha v^\beta$ such that $2\alpha + (3\iota_1 + \iota_2)\beta > 2(3\iota_1 + \iota_2)$ if $\gcd(2, 3\iota_1 + \iota_2) = 1$ and $\alpha + (3\iota_1 + \iota_2)\beta/2 > (3\iota_1 + \iota_2)$ if $\gcd(2, 3\iota_1 + \iota_2) = 2$. In particular, $v\varphi_1(u)^3$ is in (higher terms). This shows the assertion (3). □

Next we consider the case that $C_2$ is singular at $P$. Then $C_2$ consists of two lines $\ell_1$ and $\ell_2$ such that $\ell_1 \cap \ell_2 = \{P\}$.

**Lemma 2.** *Suppose that $C_2$ is singular at $P$. Then singularities of $D$ at $P$ are described as follows:*

(1) *If $P \in C_2 \cap L \setminus L_\infty$, then we have $(D, P) \sim e_6$.*

(2) *If $P \in C_2 \cap L_\infty \setminus L$, then $D$ is smooth at $P$ and is tangent to $L_\infty$ with $I(D, L_\infty; P) = 4$.*

(3) *If $P \in C_2 \cap L \cap L_\infty$, then $D$ consists of four lines which intersect at $P$.*

*Proof.* The assertion (1) is shown in [7]. We show the assertion (2). We take a suitable local coordinates $(u, v)$ at $P$ so that $f_2(u, v) = u(b_1 u - b_2 v)$ and $L_\infty = \{v = 0\}$ where $b_i \neq 0$ for $i = 1, 2$. Then

$$\begin{aligned} g(u,v) &= u^2(b_1 u - b_2 v)^2 + b_3\, f_1'(P)^3 v \\ &= b_1^2 u^4 + b_3\, f_1'(P)^3 v + (\text{higher terms}). \end{aligned}$$

As $P$ is not on $L$, we have $f_1'(P) \neq 0$ and $I(D, L_\infty; P) = 4$. This shows the assertion (2). The assertion (3) is obvious from the defining polynomial of $D$. □

We say that $P$ is *an outer singularity of* $D$ if $P \in \mathrm{Sing}(D) \setminus C_2$. We consider possible outer singularities of $D$.

**Lemma 3.** *If $P \in D$ is an outer singularity, then $(D, P)$ is either $a_1$ or $a_2$.*

Our proof is computational and it is done in the same way as in [6].

### 2.2. Invisible factorization

Let $D = \{G = 0\}$ be an invisible factorization of a $(2, 3)$ torus curve which satisfies the following equations.

$$F_{2,3}(X,Y,Z) = F_2(X,Y,Z)^3 - F_3(X,Y,Z)^2 = Z^2\, G(X,Y,Z) \tag{2.1}$$

We assume that $G$ is not divided by $Z$ and $G$ is reduced. We rewrite $F_2$ and $F_3$ as follows:

$$\begin{aligned} F_2(X,Y,Z) &= F_2^{(2)}(X,Y) + F_2^{(1)}(X,Y)Z + F_2^{(0)}(X,Y)Z^2, \\ F_3(X,Y,Z) &= F_3^{(3)}(X,Y) + F_3^{(2)}(X,Y)Z + F_3^{(1)}(X,Y)Z^2 + F_3^{(0)}(X,Y)Z^3 \end{aligned}$$

where $F_j^{(i)}$ is a homogeneous polynomial of degree $i$. By an easy calculation, we observe that there exists a linear form $\ell_1(X,Y)$ so that

$$\begin{cases} F_2^{(2)}(X,Y) = \ell_1(X,Y)^2, \\ F_3^{(3)}(X,Y) = \varepsilon\,\ell_1(X,Y)^3, \quad F_3^{(2)}(X,Y) = \dfrac{3\varepsilon}{2}\ell_1(X,Y)F_2^{(1)}(X,Y) \end{cases}$$

where $\varepsilon = 1$ or $-1$. We put $\ell_2(X,Y) := F_2^{(1)}(X,Y)$ and $\ell_3(X,Y) := F_3^{(1)}(X,Y)$. Then we may assume the defining polynomials of $C_2$ and $C_3$ as the following:

$$(\sharp) \qquad \begin{cases} F_2(X,Y,Z) = \ell_1(X,Y)^2 + \ell_2(X,Y)\,Z + a_{00}\,Z^2, \\ F_3(X,Y,Z) = \ell_1(X,Y)^3 + \dfrac{3}{2}\,\ell_1(X,Y)\,\ell_2(X,Y)\,Z + \ell_3(X,Y)\,Z^2 + b_{00}\,Z^3. \end{cases}$$

Then $F_{2,3}$ is factorized as

$$F_{2,3}(X,Y,Z) = F_2(X,Y,Z)^3 - F_3(X,Y,Z)^2 = Z^2 G(X,Y,Z).$$

To see the local geometry of $D$ at a intersection point $D$ and $L_\infty$, we may assume $\ell_1(X,Y) = X$ and we take the affine coordinates $(x,z) = (X/Y, Z/Y)$ at $O^* := [0:1:0]$. Let $g(x,z) = G(x,1,z)$, $f_2(x,z) = F_2(x,1,z)$ and $f_3(x,z) = F_3(x,1,z)$ be the local equations of $D$, $C_2$ and $C_3$ respectively. In the affine coordinates $(x,z)$, $f_2$ and $f_3$ are written as

$$\begin{aligned} f_2(x,z) &= x^2 + \ell_2(x,1)\,z + a_{00}z^2, \\ f_3(x,z) &= x^3 + \frac{3}{2}\ell_2(x,1)\,xz + \ell_3(x,1)z^2 + b_{00}z^3. \end{aligned}$$

We can see the local geometries of $C_2$ and $C_3$ at $O^*$. First we consider the case $\ell_2(0,1) \neq 0$. Then we have

(1) $C_2$ is smooth at $O^*$ and is tangent to the limit line $L_\infty$ at $O^*$.

(2) $C_3$ has an $a_1$ singularity at $O^*$.

(3) The intersection multiplicity $I(C_2, C_3; O^*)$ is 3.

Then, putting $c_1 = \ell_2(0,1)$, $g(x,z)$ is given as

$$g(x,z) = c_1^3\,z + \frac{3}{4}\,c_1^2\,x^2 + (\text{higher terms}).$$

Thus $D$ is simply tangent to $L_\infty$ at $O^*$. We write $g(x,0) = x^2\,(x-\alpha)(x-\beta)$ for some $\alpha, \beta$ such that $\alpha\beta = 3c_1^2/4 \neq 0$. Then if $\alpha \neq \beta$, then $L_\infty$ is tangent to $D$ at $O^*$ and intersects transversely with $D$ at other 2 points. If $\alpha = \beta$, then $L_\infty$ is a bi-tangent line of $D$.

**Lemma 4.** *If $c_1 \neq 0$, then the set of singularities* $\mathrm{Sing}(D)$ *is* $\{3a_2\}$ *or* $\{a_2 + a_5\}$.

*Proof.* As the intersection $C_2 \cap C_3 \cap L_\infty = \{O^*\}$ and $I(C_2, C_3; O^*) = 3$, the sum of the intersection numbers of $C_2 \cap C_3$ is 3 in the affine space $\mathbb{P}^2 \setminus L_\infty$. The possible configurations of $\mathrm{Sing}(D)$ are $\{3a_2\}$, $\{a_5 + a_2\}$ and $\{a_8\}$. The singularity $a_8$ is locally irreducible but the Milnor number of an irreducible quartics is less then or equal to 6. Hence the configuration $\{a_8\}$ does not occur. □

**Remark 1.** If $D$ is bi-tangent to $L_\infty$, then the configuration $\{a_5 + a_2\}$ does not exist. Indeed, if the configuration $\{a_5 + a_2\}$ exists, then $D$ can not be irreducible. If $D$ is a union of a line and a cubic, then a cubic can not have a bi-tangent line. If $D$ is a union of two conics which are tangent to $L_\infty$, then $D$ can not have any $a_2$ singularity.

Now we consider the case $c_1 = \ell_2(0,1) = 0$. Then putting $c_2 = \ell_3(0,1)$, their defining polynomials are given as

$$f_2(x,z) = a_{00}\, z^2 + \ell_2(1,0)\, xz + x^2,$$
$$f_3(x,z) = c_2\, z^2 + x^3 + \left(c_3\, x^2 z + c_4\, xz^2 + b_{00} z^3\right), \quad c_3, c_4 \in \mathbb{C}.$$

Thus $C_2$ consists of two lines $\ell_1$ and $\ell_2$ such that $\ell_1 \cap \ell_2 = \{O^*\}$ and $C_3$ has an $a_2$ singularity at $O^*$ and $I(C_2, C_3; O^*) = 4$. Then after an easy calculation, we have

$$g(x,z) = -c_2^2\, z^2 - 2\, c_2\, x^3.$$

Hence $D$ has an $a_2$ singularity at $O^*$. Thus we have:

**Lemma 5.** *If $c_1 = 0$, then $D$ has an $a_2$ singularity on $L_\infty$ and* $\mathrm{Sing}(D) = \{2a_2 + a_2^\infty\}$ *or* $\{a_5 + a_2^\infty\}$.

*Proof.* As $C_2 \cap C_3 \cap L_\infty = \{O^*\}$ and $I(C_2, C_3; O^*) = 4$, the intersection $C_2 \cap C_3$ generically consists of two points in the affine space $\mathbb{P}^2 \setminus L_\infty$. By a similar argument of Lemma 4, we have the assertion. □

## §3. Statement of the Theorem.

Let $Q$ be a quartic in $QL$-configurations. For a quartic $Q$ in one of the $(1), \dots, (13)$ of Table 1, $Q$ is irreducible and $Q$ is not irreducible for $(14), \dots, (19)$ of Table 1. Now our main results are the following:

**Theorem 1.** *Let $Q$ be an irreducible quartic in one of the $QL$-configurations.*

(1) *For $Q$ in the case* (13)*, there exists no* $(2,3)$ *torus decomposition.*

(2) *For $Q$ in the case* (5)*, there exist five torus decompositions of type* $(2,3)$ *whose three decompositions are visible decompositions and two are invisible decompositions.*

(3) *For $Q$ in the remaining cases, there exists a unique* $(2,3)$ *torus decomposition for each case.*

Note that a quartic $Q$ for the case (5) is a 3 cuspidal quartic.

**Remark 2.** In section 9, we will show that if a plane curve has a $(2,3)$ torus decomposition, then there exist infinite $(2,3)$ quasi torus decompositions.

**Theorem 2.** *For each quartic $Q$ of* $(1),\dots,(12)$*, there exists a line degeneration family of sextic $C(s) : H_3(X,Y,Z,s)^2 + H_2(X,Y,Z,s)^3 = 0$ which are* $(2,3)$ *torus curves such that $C(0) = Q + 2L_\infty$. In particular, we have the divisibility $\Delta_{C(s)}(t) \mid \Delta_Q(t)$.*

The divisibility $(\star)$ in Introduction also follows from Theorem 2 and Corollary 15 of [5].

**Proposition 6.** *For non-irreducible quartics* $(14),\dots,(19)$ *in Table* 1*, we have the following:*

(a) *There exist unique* $(2,3)$ *torus decompositions for quartics* (14) *and* (15) *and their decompositions are represented as visible decompositions.*

(b) *The quartic* (16) *does not admit any torus decompositions.*

(c) *There exist unique* $(2,4)$ *torus decompositions for the quartics* (17)*,* (18) *and* (19)*. Their decompositions are represented as invisible decompositions.*

**Remark 3.** For the quartics (13) and (16), there are not torus decomposition. By the classifications of singularities in §2, their singularities do not occur as the quartics with visible or invisible $(2,3)$ torus decompositions.

## §4. The proof of Theorem 1 (3)

### 4.1. Strategy

There are 13 configurations of singularities of the quintic $Q + L_\infty$ as in (1), ..., (13) in Table 1. We divide these quintic into 5 cases (i), ..., (v) as in Introduction. Note that the case (iii) does not appear when $Q$ is irreducible.

By the classification of the singularities for the visible and invisible factorizations in §2, for the quartics $(1),\dots,(12)$ except the case (5), the possible torus decomposition must be visible and unique if it exists. The quartic (5) has an exceptional property. It has both visible and invisible torus decompositions. Thus we treat this case in the next section.

First, we construct explicit quartics $Q := \{F = 0\}$ with the prescribed properties at infinity. By the action of $\mathrm{PSL}(3, \mathbb{C})$ of $\mathbb{P}^2$, we can put the singularities at fixed locations. Then we construct the respective torus decompositions in §2.

**Step 1. Construction of an explicit quartic $Q$.** By the classification of the singularities for invisible decomposition case (Lemma 4 and Lemma 5), the quartics in cases (1)$\sim$ (12) except the case (5) can not have invisible torus decomposition. So we only need the possible visible decomposition for these quartics. As the computations are boring and easy, we explain the quartic (1) in Table 1 in detail and for the other cases we simply give the result of the computations.

**The quartic (1) in Table 1.** In this case, $L_\infty$ is a bi-tangent line of $Q$ and the singularity is $\mathrm{Sing}(Q) = \{2a_2\}$. We construct a quartic $Q$ with $2a_2$ which $L_\infty$ is a bi-tangent line. Let $\Sigma(Q) := \{P_1, P_2\}$ be the singular locus of $Q$ and let $Q \cap L_\infty := \{R_1, R_2\}$ be the bi-tangent points. By the action of $\mathrm{PSL}(3, \mathbb{C})$ on $\mathbb{P}^2$, we can put the locations of points:

$$P_1 = [1 : 0 : 1], \quad P_2 = [-1 : 0 : 1], \quad R_1 = [1 : 1 : 0]$$

and we may assume that the tangent directions at $P_1$ and $P_2$ are given as

$$\{x - 1 = 0\}, \quad \{x + 1 = 0\}$$

respectively. We start from the generic quartic $F(X, Y, Z) = \sum_\nu c_\nu X^{\nu_1} Y^{\nu_2} Z^{\nu_3}$ with $\nu = (\nu_1, \nu_2, \nu_3)$ with $\nu_1 + \nu_2 + \nu_3 = 4$. The necessary conditions are

$$\begin{aligned}
&F(P_1) = \frac{\partial F}{\partial X}(P_1) = \frac{\partial F}{\partial Y}(P_1) = \frac{\partial^2 F}{\partial Y^2}(P_1) = \frac{\partial^2 F}{\partial X \partial Y}(P_1) = 0,\\
&F(P_2) = \frac{\partial F}{\partial X}(P_2) = \frac{\partial F}{\partial Y}(P_2) = \frac{\partial^2 F}{\partial Y^2}(P_2) = \frac{\partial^2 F}{\partial X \partial Y}(P_2) = 0,\\
&F(R_1) = \frac{\partial F}{\partial X}(R_1) = 0.
\end{aligned}$$

Under the above conditions, we have $F(X, Y, 0) = (X - Y)^2(X - \alpha Y)(X - \beta Y)$. As $L_\infty$ is bi-tangent to $Q$, we must have $\alpha = \beta$. Thus we have 13 equations of the coefficients of $F$. By solving these equations, $F$ has the following form:

$$Q: \quad F(X, Y, Z) = Z^4 + 2(Y^2 - X^2)Z^2 + t\,Y^3 Z + (Y^2 - X^2)^2 = 0, \quad t \neq 0.$$

Then another bi-tangent point $R_2$ is $[-1 : 1 : 0]$.

**Step 2. Torus decompositions.** Now we consider the possibilities of visible torus decompositions of $Q$. Thus we assume that $F$ is written as follows:

$$F(X,Y,Z) = F_2'(X,Y,Z)^2 + F_1'(X,Y,Z)^3 Z.$$

Two $a_2$ singularities are inner singularities of $Q$. Hence we assume that $C_2 \cap L = \{P_1, P_2\}$ and $C_2$ is smooth at $P_1$ and $P_2$ where $C_2 = \{F_2' = 0\}$ and $L = \{F_1' = 0\}$. Then we have

$$F_1'(P_1) = F_1'(P_2) = 0, \qquad F_2'(P_1) = F_2'(P_2) = 0.$$

Then $L$ is the line pass through $P_1$ and $P_2$. Hence we get $F_1'(X,Y,Z) = s_1 Y$ where $s_1 \in \mathbb{C}^*$. As $C_2$ is smooth at $P_i$, the tangent directions of $C_2$ at $P_i$ must coincide with that of $Q$ for $i = 1,2$. Hence $F_2'$ also satisfies the following conditions:

$$\frac{\partial F_2'}{\partial Y}(P_1) = \frac{\partial F_2'}{\partial Y}(P_2) = 0.$$

Then $\mathrm{Sing}(Q) = \{2a_2\}$ by Lemma 1. As $R_1 \in Q$, $C_2$ passes through $R_1$. Namely $F_2'$ satisfies the condition $F_2'(R_1) = 0$. Then $F_2'$ takes the following form:

$$F_2'(X,Y,Z) = s_2\,(X^2 - Y^2 - Z^2), \quad s_2 \in \mathbb{C}^*.$$

Note that $C_2$ also passes through another bi-tangent point $R_2$. Hence $Q$ satisfies the condition (i) by Lemma 1. Therefore we get the family of quartics with visible factorizations:

$$F(X,Y,Z) = s_2^2\,(X^2 - Y^2 - Z^2)^2 + s_1^3\,Y^3\,Z = 0.$$

Finally, we put $s_1^3 = t$ and $s_2^2 = 1$. Then we can see easily

$$\begin{aligned}(X^2 - Y^2 - Z^2)^2 + t\,Y^3 Z &= Z^4 - 2\left(X^2 + Y^2\right) Z^2 + t\,Y^3 Z + (X^2 + Y^2)^2 \\ &= F(X,Y,Z).\end{aligned}$$

**Step 3. Uniqueness.** By the classification of the singularities for the visible and invisible factorizations in §2, we can see easily that the possible torus decompositions are visible. Then two $a_2$ singularities must be inner singularities and the corresponding curves are uniquely determined by the above arguments.

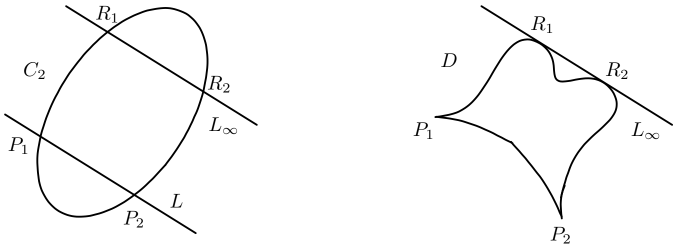


Figure 1: The quartic (1) in Table 1

For the other quartics, we only give the results of calculations.

| | |
|---|---|
| Quartic (1) | |
| Singularities | $2a_2$ at $[1:0:1]$, $[-1:0:1]$ |
| $Q \cap L_\infty$ | bi-tangent at $[1:1:0]$, $[-1:1:0]$ |
| Torus decomposition | $(X^2 - Y^2 - Z^2)^2 + t\,Y^3 Z = 0$ |
| Quartic (2) | |
| Singularities | $2a_2$ at $[1:0:1]$, $[-1:0:1]$ |
| $Q \cap L_\infty$ | tangent multiplicity 4 at $[0:1:0]$. |
| Torus decomposition | $(X^2 - Z^2)^2 + t\,Y^3 Z = 0.$ |
| Quartic (3) | |
| Singularities | $2a_2 + a_1$ at $[0:0:1]$, $[1:1:1]$, $[-1:1:0]$. |
| $Q \cap L_\infty$ | bi-tangent at $[\sqrt{3}:1:0]$, $[-\sqrt{3}:1:0]$. |
| Torus decomposition | $(4Z^2 - 6YZ - X^2 + 3Y^2)^2 + 16\,(Y - Z)Z = 0.$ |
| Quartic (4) | |
| Singularities | $2a_2 + a_1$ at $[0:0:1]$, $[1:1:1]$, $[-1:1:0]$. |
| $Q \cap L_\infty$ | tangent multiplicity 4 at $[0:1:0]$ |
| Torus decomposition | $(2Z^2 + X^2 - 3YZ)^2 + 4\,(Y - Z)Z = 0.$ |
| Quartic (6) | |
| Singularities | $a_2 + a_3^\infty$ at $[0:0:1]$, $[-1:1:0]$. |
| $Q \cap L_\infty$ | singular at $[-1:1:0]$ and tangent at $[1:1:0]$. |
| Torus decomposition | $(X^2 - 2\,XZ - Y^2)^2 + t\,(X + Y)Z = 0.$ |

| Quartic (7) | |
|---|---|
| Singularities | $a_5$ at $[0:0:1]$. |
| $Q \cap L_\infty$ | bi-tangent at $[1:1:0]$, $[-1:1:0]$. |
| Torus decomposition | $(XZ + s(X^2 - Y^2))^2 + s(ts-2)\,X^3Z = 0.$ |
| Quartic (8) | |
| Singularities | $a_5$ at $[0:0:1]$. |
| $Q \cap L_\infty$ | tangent multiplicity 4 at $[1:0:0]$. |
| Torus decomposition | $(XZ - s\,Y^2)^2 + t\,X^3Z = 0$ |
| Quartic (9) | |
| Singularities | $a_6^\infty$ at $[0:1:0]$. |
| $Q \cap L_\infty$ | singular at $[0:1:0]$ and tangent at $[-1:1:0]$. |
| Torus decomposition | $(t_2^2Z^2 + t_3XZ - t_2X(X+Y))^2$ $+t_2(2t_3 + t_1t_2)\,X^3Z = 0.$ |
| Quartic (10) | |
| Singularities | $a_4^\infty, a_2$ at $[0:0:1]$, $[0:1:0]$. |
| $Q \cap L_\infty$ | singular at $[0:1:0]$. |
| Torus decomposition | $(YZ - t_2X^2)^2 + t\,X^3Z = 0$ |
| Quartic (11) | |
| Singularities | $e_6$ at $[0:0:1]$. |
| $Q \cap L_\infty$ | bi-tangent at $[1:1:0]$, $[-1:1:0]$. |
| Torus decomposition | $(X^2 + Y^2)^2 + t\,X^3Z = 0$ |
| Quartic (12) | |
| Singularities | $e_6$ at $[0:0:1]$. |
| $Q \cap L_\infty$ | tangent multiplicity 4 at $[1:0:0]$. |
| Torus decomposition | $Y^4 + t\,X^3Z = 0.$ |

Table 2.

## §5. Proof of Theorem 1 (2).

In this section, we consider the exceptional case (5) in Table 1. This case (5) is exceptional as the classification of the singularities tells us that it may have a invisible case as well as visible decompositions. Let $Q = \{F = 0\}$ be a quartic with $\mathrm{Sing}(Q) = \{3a_2\}$. Then, by the class formula ([3]), $Q$ has a unique bi-tangent line and we take $L_\infty$ as the bi-tangent line of $Q$. We put the singular locus $\Sigma(Q) = \{P_1, P_2, P_3\}$ and the intersection $Q \cap L_\infty := \{R_1, R_2\}$. By the action of $\mathrm{PSL}(3, \mathbb{C})$ on $\mathbb{P}^2$, we can put the locations:

$$P_1 = [1 : 0 : 1], \quad P_2 = [-\frac{1}{2} : \frac{\sqrt{3}}{2} : 1], \quad P_3 = [-\frac{1}{2} : -\frac{\sqrt{3}}{2} : 1], \quad R_1 = [I, 1, 0]$$

where $I = \sqrt{-1}$. By direct computations, the defining polynomial $F$ of $Q$ is obtained by

$$F(X, Y, Z) = Z^4 - 6(X^2 + Y^2)Z^2 + 8(X^2 - 3Y^2)XZ - 3(X^2 + Y^2)^2.$$

Then another bi-tangent point $R_2$ is $[-I : 1 : 0]$. Note that there is no free parameters left.

Now we consider two transformations $\sigma$, $\tau : \mathbb{P}^2 \to \mathbb{P}^2$ which are defined as

$$\sigma(X : Y : Z) := (X : -Y : Z), \quad \tau(X : Y : Z) := (X : Y : Z)A,$$

where

$$A = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \theta = -\frac{2}{3}\pi.$$

Consider the subgroup $G$ of $\mathrm{PSL}(3; \mathbb{C})$ generated by $\sigma$ and $\tau$. Observe that $G \cong S_3$ and $\sigma^2 = \tau^3 = (\sigma\tau)^2 = e$. Then we can see that $L_\infty$ and $Q = \{F = 0\}$ are stable under the action of $G$:

$$F(X, Y, Z) = F(\sigma(X, Y, Z)) = F(\tau(X, Y, Z)).$$

We observe also the following.

$$\sigma(R_1) = R_2,\ \sigma(R_2) = R_1,\ \sigma(P_1) = P_1,\ \sigma(P_2) = P_3,\ \sigma(P_3) = P_2.$$

$$\tau(R_i) = R_i,\ i = 1, 2, \qquad \tau(P_i) = \begin{cases} P_{i+1} & \text{if } i = 1, 2 \\ P_1 & \text{if } i = 3. \end{cases}$$

**Visible factorization.** Now we consider the possibilities of $(2, 3)$ visible factorization of $Q = \{F = 0\}$. We assume that $F$ is written as follows:

$$F(X, Y, Z) = F_2'(X, Y, Z)^2 + F_1'(X, Y, Z)^3 Z.$$

In this case, two of $P_1$, $P_2$, $P_3$ must be inner singularities and the rest is an outer singularity. Thus we have three possible cases for these choices:

(1) $P_1$ is an outer singularity and $P_2$, $P_3$ are inner singularities.

(2) $P_2$ is an outer singularity and $P_1$, $P_3$ are inner singularities.

(3) $P_3$ is an outer singularity and $P_1$, $P_2$ are inner singularities.

First we assume the case (1). Then $L = \{F_1' = 0\}$ and $C_2 = \{F_2' = 0\}$ are satisfy the following.

- $P_1$ is an outer singularity.
- $L$ is the line passing $P_2$ and $P_3$.
- $C_2$ passes through $P_2$, $P_3$, $R_1$ and $R_2$.

Then the defining polynomials $F_1'$ and $F_2'$ are obtained by

$$F_1'(X,Y,Z) = -\frac{1}{3}t^2(Z+2X), \quad F_2'(X,Y,Z) = \frac{t^3}{6}(Z^2+4XZ+Y^2+X^2).$$

We take $t$ as one of the solutions $t^6 + 108 = 0$. Then $F$ is decomposed into

(V-1) $$F(X,Y,Z) = -3(Z^2+4XZ+Y^2+X^2)^2 + 4(Z+2X)^3 Z.$$

Note that $C_2$, $L$, $P_1$ and $\{P_2, P_3\}$ are stable by the action of $\sigma$.

Next we consider the case (2). The singular locus $\Sigma(Q)$ is stable by the action of $\tau$ and $\tau(C_2 \cap L) = \{P_3, P_1\}$. Hence $P_2$ is the outer singularity. Thus we have

(V-2)
$$\begin{aligned} F(X,Y,Z) &= F(\tau(X,Y,Z)) = F_2(\tau(X,Y,Z))^2 + F_1(\tau(X,Y,Z))^3 Z \\ &= -3(Z^2 - 2XZ - 2\sqrt{3}YZ + X^2 + Y^2)^2 + 4(Z - X - \sqrt{3}Y)^3. \end{aligned}$$

By the same argument, we have one more different torus decomposition:

(V-3)
$$\begin{aligned} F(X,Y,Z) &= F(\tau^2(X,Y,Z)) = F_2'(\tau^2(X,Y,Z))^2 + F_1'(\tau^2(X,Y,Z))^3 Z \\ &= -3(Z^2 - 2XZ + 2\sqrt{3}YZ + X^2 + Y^2)^2 + 4(Z - X + \sqrt{3}Y)^3. \end{aligned}$$

Thus we have three different torus decompositions (V-1), (V-2) and (V-3).

**Invisible factorization.** Next we consider $(2,3)$ invisible factorization (§2):

$$Z^2 F(X,Y,Z) = F_2(X,Y,Z)^3 - F_3(X,Y,Z)^2$$

where $F_2$ and $F_3$ are defined by

$$(\sharp)\quad \begin{aligned} F_2(X,Y,Z) &= \ell_1(X,Y)^2 + \ell_2(X,Y)\,Z + a_{00}\,Z^2,\\ F_3(X,Y,Z) &= \ell_1(X,Y)^3 + \frac{3}{2}\,\ell_1(X,Y)\,\ell_2(X,Y)\,Z + \ell_3(X,Y)\,Z^2 + b_{00}\,Z^3 \end{aligned}$$

where $\ell_i$ is a linear form for $i = 1, 2, 3$. By the argument in §2.2, the singularity locus $P_1$, $P_2$ and $P_3$ are inner singularities. Hence we have the conditions:

$$(*_1) \quad F_2(P_i) = F_3(P_i) = 0, \quad i = 1, 2, 3.$$

Moreover one of the bi-tangent points is obtained by the intersection point $\{\ell_1 = 0\} \cap L_\infty$.

First we assume that $\{\ell_1 = 0\} \cap L_\infty = \{R_1\}$. By solving conditions $(*_1)$ and $\ell_1(R_1) = 0$, we have $a_{00} = 0$, $b_{00} = t^3/2$ and

$$\ell_1(X,Y) = t\,(X - IY), \quad \ell_2(X,Y) = -t^2\,(X + IY), \quad \ell_3(X,Y) = 0.$$

Thus $F_2$ and $F_3$ are given by

$$\begin{aligned} F_2(X,Y,Z) &= t^2(X - IY)^2 - t^2(X + IY)Z,\\ F_3(X,Y,Z) &= \frac{t^3}{2}\left(Z^3 - 3(X^2+Y^2)Z + 2(X - IY)^3\right). \end{aligned}$$

Note that $C_2 = \{F_2 = 0\}$ and $C_3 = \{F_3 = 0\}$ are stable by the action $\sigma$. Then we have

$$\frac{t^6}{4} Z^2 F(X,Y,Z) = F_2(X,Y,Z)^3 - F_3(X,Y,Z)^2.$$

Hence taking $t$ as one of the solutions $t^6 = 4$, we have an invisible torus decomposition:

$$(\text{In-1})\quad \begin{aligned} Z^2F(X,Y,Z) = 4((X+IY)Z &- (X - IY)^2)^3\\ &+ (Z^3 - 3(X^2+Y^2)Z + 2(X-IY)^3)^2. \end{aligned}$$

Next we consider the case $\{\ell_1 = 0\} \cap L_\infty = \{R_2\}$. As the singular locus $\Sigma(Q)$ is stable by the action of $\sigma$ and $\sigma(R_1) = R_2$, we have another invisible torus decomposition from (In-1):

$$(\text{In-2})\quad \begin{aligned} Z^2F(\sigma(X,Y,Z)) = 4((X-IY)Z &- (X + IY)^2)^3\\ &+ (Z^3 - 3(X^2+Y^2)Z + 2(X+IY)^3)^2. \end{aligned}$$

Thus we have two different invisible torus decompositions (In-1) and (In-2).

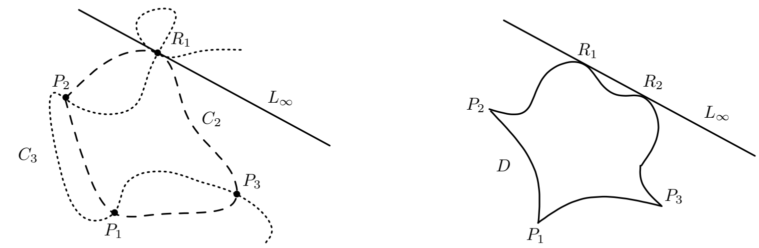


Figure 2: Invisible factorization (In-1) of the quartic (5)

We have shown in the above argument that the three visible decompositions move each other by the action of $\sigma$:

$$\text{(V-1)} \quad \overset{\sigma}{\to} \quad \text{(V-2)} \quad \overset{\sigma}{\to} \quad \text{(V-3)}.$$

We also showed that the two invisible decompositions move each other by the action of $\tau$:

$$\text{(In-1)} \quad \overset{\tau}{\to} \quad \text{(In-2)}.$$

In this case, there exist other quasi torus decompositions. We discuss in §9.

## §6. Proof of Theorem 2

Let $U$ be an open neighborhood of 0 and let $C(t) = \{F_{2,3}(X,Y,Z,t) = 0\}$ $(t \in U)$ be a $(2,3)$ torus curve which is defined by

$$F_{2,3}(X,Y,Z,t) = H_3(X,Y,Z,t)^2 + H_2(X,Y,Z,t)^3.$$

Fix $B = [0 : 1 : 0]$ in $\mathbb{P}^2$. We consider the family $\{C(t)\}_{t \in U}$ which has $B$ as the base point with multiplicity 2. We assume that the defining polynomials have the following form:

$$(*_2) \qquad \begin{cases} H_2(X,Y,Z,t) = F_1'(X,Y,Z)Z + t\,X K_1(X,Y), \\ H_3(X,Y,Z,t) = F_2'(X,Y,Z)Z + t\,X K_2(X,Y) \end{cases}$$

where $F_i'(X,Y,Z)$ and $K_i(X,Y)$ are homogeneous polynomials of degree $i$ such that $K_i(0,1) \neq 0$ for $i = 1,2$. Then $B$ is in $C(t)$ with multiplicity 2 for $t \neq 0$ and $\{C(t)\}_{t \in U}$ degenerates into $C(0) = D + 2L_\infty$ where $D = \{G = 0\}$ is a visible factorization of torus curve which is defined as

$$G(X,Y,Z) = F_2'(X,Y,Z)^2 + F_1'(X,Y,Z)^3 Z.$$

Thus we can construct a line degeneration of $(2,3)$ torus sextic. For example, we give line degeneration for the case (1). Let $Q = \{F = 0\}$ be the quartic which is satisfies the condition of (1). In §4, we obtained the defining polynomials $F$, $F_1$ and $F_2$ as

$$\begin{aligned} Q:&\quad F(X,Y,Z) = Z^4 + 2(Y^2 - X^2)Z^2 + t\,Y^3 Z + (Y^2 - X^2)^2 = 0, \\ L:&\quad F_1'(X,Y,Z) = s_1\,Y = 0, \\ C_2:&\quad F_2'(X,Y,Z) = s_2\,(X^2 + Y^2 - Z^2) = 0 \end{aligned}$$

where $s_1$ and $s_2$ are one of the solutions $s_1^3 = t$ and $s_2^2 = 1$ respectively. Let $K_i(X,Y)$ be any homogeneous polynomial of degree $i$ such that $K_i(0,1) \neq 0$ for $i = 1,\ 2$. We take $H_2$ and $H_3$ as $(*_2)$ and then the family $\{C(t)\}_{t\in U}$ degenerates into $C(0) = Q + 2L_\infty$.

## §7. Degenerate families of $QL$-configurations.

Let $\mathcal{QL}$ be the set of quartics of $QL$-configurations and let $L_\infty$ be the fixed line at infinity. We consider the subset $\mathcal{QL}(n)$ of $\mathcal{QL}$ which is the set of a quartic $Q$ whose configuration $Q \cup L_\infty$ is of the type $(n)$ in Table 1 for $n = 1, \dots, 12$. It is easy to see that $\mathcal{QL}(n)$ is a connected subspace of the space of quartics under the canonical topology. This implies that for any $Q$, $Q' \in \mathcal{QL}(n)$, the topology of $\mathbb{C}^2 \setminus Q$ and $\mathbb{C}^2 \setminus Q'$ are homeomorphic.

For the comparison of the topology of $\mathbb{C}^2 \setminus Q$ and $\mathbb{C}^2 \setminus Q'$ where $Q \in \mathcal{QL}(n)$ and $Q' \in \mathcal{QL}(m)$ $(n \neq m)$, we consider the degeneration problem among quartics in these subsets. Suppose that there exists an analytic family $\{Q(s)\}_{s\in U}$ of quartics such that $Q(s) \in \mathcal{QL}(n)$ for $s \in U \setminus \{0\}$ and $Q(0) \in \mathcal{QL}(m)$ where $U$ is an open neighborhood of 0 in $\mathbb{C}$. In particular, $Q(s) \cup L_\infty \to Q(0) \cup L_\infty$. We denote this situation as $\mathcal{QL}(n) \succ \mathcal{QL}(m)$ and say that $\mathcal{QL}(m)$ *is a t-boundary of* $\mathcal{QL}(n)$. If $\mathcal{QL}(n) \succ \mathcal{QL}(m)$, then, by degenerate properties ([4]), we have the surjectivity of the homomorphism: $\pi_1(\mathbb{C}^2 \setminus Q(0)) \to \pi_1(\mathbb{C}^2 \setminus Q(s))$ $(s \neq 0)$ and the divisibility of the Alexander polynomials: $\Delta_{Q(s)}(t) \mid \Delta_{Q(0)}(t)$.

**Proposition 7.** *The following diagram denotes the various t-boundary relations among the set* $\mathcal{QL}(n)$, $n = 1, \dots, 12$.

$$\begin{array}{ccccccccc}
 & & & & \mathcal{QL}(10) & & & & \\
 & & & & \curlywedge & & & & \\
 & & \mathcal{QL}(4) & \prec & \mathcal{QL}(2) & \succ & \mathcal{QL}(8) & \succ & \mathcal{QL}(12) \\
 & & \curlywedge & & \curlywedge & & \curlywedge & & \curlywedge \\
\mathcal{QL}(5) & \prec & \mathcal{QL}(3) & \prec & \mathcal{QL}(1) & \succ & \mathcal{QL}(7) & \succ & \mathcal{QL}(11) \\
 & & & & \curlyvee & & & & \\
 & & & & \mathcal{QL}(6) & \succ & \mathcal{QL}(9) & &
\end{array}$$

For a proof, we give explicit defining equations and degenerations of quartics for each case.

(1) Let $\{C_{s,t,u}\} = \{F_{s,t,u} = 0\} \subset \mathcal{QL}(1)$ be a family of quartics with 3 parameters where

$$\begin{aligned} F_{s,t,u}(X,Y,Z) = s^4Z^4 - 2s^2uYZ^3 + (2s^2(t^2Y^2 - X^2) + u^2Y^2)Z^2 \\ + (Y^2 + 2u(X^2 - t^2Y^2))YZ + (X^2 - t^2Y^2)^2 \end{aligned}$$

such that

$$\begin{aligned} &C_{0,t,u} \in \mathcal{QL}(7), \quad C_{0,t,0} \in \mathcal{QL}(11) \ \text{ for } \ t \neq 0, \\ &C_{s,0,u} \in \mathcal{QL}(2), \quad C_{0,0,u} \in \mathcal{QL}(8), \quad C_{0,0,0} \in \mathcal{QL}(12). \end{aligned}$$

(2) Let $\{C_{s,t,u}\} = \{F_{s,t,u} = 0\} \subset \mathcal{QL}(1)$ be a family of quartics with 3 parameters where

$$\begin{aligned} F_{s,t,u}(X,Y,Z) = s(2+s)Z^4 - 3sXZ^3 + \frac{1}{4}(x^2(3s + 8u + 8su) \\ - y^2(s+1)(8u+3))Z^2 - \frac{1}{8}(u(x^2 - t^2y^2) \\ + (x^2 - 9t^2y^2))XZ + \frac{1}{64}(8u+3)^2(X^2 - t^2Y^2)^2 \end{aligned}$$

such that

$$\begin{aligned} &C_{0,t,u} \in \mathcal{QL}(3), \quad C_{0,t,0} \in \mathcal{QL}(5) \ \text{ for } \ t \neq 0, \\ &C_{s,0,u} \in \mathcal{QL}(2), \quad C_{0,0,u} \in \mathcal{QL}(4) \ \text{ for } \ u \neq 0. \end{aligned}$$

(3) Let $\{C_{s,t}\} = \{F_{s,t} = 0\} \subset \mathcal{QL}(1)$ be a family of quartics with 2 parameters where

$$\begin{aligned} F_{s,t}(X,Y,Z) = (t^3+1)Z^4 + 3t^2\ell_1(X,Y,s)Z^3 + (3\ell_1(X,Y,s)^2t \\ - 2(X^2 - Y^2))Z^2 + \ell_1(X,Y,s)^3Z + (X^2 - Y^2)^2 \\ \text{where } \ell_1(X,Y,s) = X - sY - Y \end{aligned}$$

such that

$$C_{0,t} \in \mathcal{QL}(6), \quad C_{0,0} \in \mathcal{QL}(9).$$

(4) Let $\{C_{s,t}\} = \{F_{s,t} = 0\} \subset \mathcal{QL}(1)$ be a family of quartics with 2 parameters where

$$F_{s,t}(X,Y,Z) = (X+Y)Z^2 + ((X-sY)^3 - 2(X+Y)(t^2Y^2 - X^2))Z + (t^2Y^2 - X^2)^2$$

such that

$$C_{0,t} \in \mathcal{QL}(2), \quad C_{0,0} \in \mathcal{QL}(10).$$

## §8. Non-irreducible $QL$-configurations

In this section, we consider torus decompositions of non-irreducible $QL$-configurations for the quartics $(14), \dots, (19)$. Then we will get defining polynomials and torus decompositions by the same argument of irreducible case. Recall that the situations of each case:

| No. | Sing($Q$) | $Q \cap L_\infty$ | irreducible components |
|---|---|---|---|
| (14) | $a_3^\infty + a_2 + a_1$ | (ii) | a cuspidal cubic and a line |
| (15) | $a_5 + a_1$ | (i) | two conics |
| (16) | $a_5^\infty + a_2^\infty$ | (iii) | a cuspidal cubic and a line |
| (17) | $2a_3^\infty$ | (iii) | two conics |
| (18) | $a_7^\infty$ | (v) | two conics |
| (19) | $2a_3^\infty + a_1$ | (iii) | a conic and two lines |

Table 3.

### 8.1. Invisible factorization of (2,4) torus curves

Let $D$ be an invisible factorization of $(2,4)$ torus curve which satisfies the following.

$$\begin{aligned} F_{2,4}(X,Y,Z) &= F_2(X,Y,Z)^4 - F_4(X,Y,Z)^2 \\ &= (F_2(X,Y,Z)^2 - F_4(X,Y,Z))(F_2(X,Y,Z)^2 + F_4(X,Y,Z)) \\ &= Z^4 G(X,Y,Z). \end{aligned}$$

By the same argument in §2.2, we can assume that the forms of $F_2$ and $F_4$ are

$$\begin{aligned} &F_2(X,Y,Z) = F_2^{(2)}(X,Y) + F_2^{(1)}(X,Y)Z + a_{00}Z^2, \quad \deg F_2^{(i)} = i, \\ &F_4(X,Y,Z) = F_2(X,Y,Z)^2 - c\,Z^4, \quad c = b_{00} - a_{00}^2 \neq 0. \end{aligned}$$

Then $D = \{G = 0\}$ is defined by

$$D: \quad G(X,Y,Z) = F_2(X,Y,Z)^2 - c'Z^4 = 0, \quad c' \neq 0.$$

By the form of the defining polynomial of $D$, the inner singularities of $D$ is on $L_\infty$. Singularities of $D$ are described as follows.

**Lemma 8.** *Under the above notations, $D$ has the following singularities.*

(1) *If $C_2$ is smooth at $P \in C_2 \cap L_\infty$, then $(D, P) \sim a_{3\iota-1}$ where $\iota = I(C_2, L_\infty; P)$.*

(2) *If $C_2$ is singular at $P \in C_2 \cap L_\infty$, then $D$ consists of four lines.*

(3) *If $P \in D$ is an outer singularity, then $(D, P) \sim a_1$.*

Our proof is done in the same way as Lemma 1 and [6].

### 8.2. Torus decompositions of non-irreducible $QL$-configurations

In this section, we show the possibilities of torus decompositions for non-irreducible $QL$-configurations. Our proof is similar to the cases of irreducible $QL$-configurations. For the quartics (14) and (15), we use $(2,3)$ visible factorizations. For the quartics (17), (18) and (19), we use $(2,4)$ invisible factorizations.

| | |
|---|---|
| Quartic (14) | |
| Singularities | $a_3^\infty + a_2 + a_1$ at $[1:1:0]$, $[0:0:1]$, $[-1:0:1]$. |
| $Q \cap L_\infty$ | singular at $[1:1:0]$ and tangent at $[-1:1:0]$. |
| Torus decomposition | $(X^2 - Y^2 - XZ + 3YZ)^2 + 4\,(X-Y)^3 Z = 0.$ |
| Quartic (15) | |
| Singularities | $a_5 + a_1$ at $[0:0:1]$, $[-1:0:1]$. |
| $Q \cap L_\infty$ | bi-tangent at $[1:1:0]$ and $[-1:1:0]$. |
| Torus decomposition | $(X^2 - Y^2 + XZ)^2 + 4\,X^3 Z = 0.$ |
| Quartic (17) | |
| Singularities | $2a_3$ at $[1:1:0]$ and $[-1:0:0]$. |
| $Q \cap L_\infty$ | singular at $[1:1:0]$ and $[-1:0:0]$. |
| Torus decomposition | $\frac{1}{64}(2X^2 - 2Y^2 - t_2 Z^2)^4$ $-(\frac{1}{8}(2X^2 - 2Y^2 - t_2 Z^2)^2 + \frac{1}{4}(4t_1 - t_2^2)Z^4)^2.$ |

| Quartic (18) | |
|---|---|
| Singularities | $a_7$ at $[0:1:0]$. |
| $Q \cap L_\infty$ | singular at $[0:1:0]$. |
| Torus decomposition | $\left(Z^2 - \frac{2}{a_{01}c_2}(c_3X - c_2Y)Z - 2\frac{c_2}{a_{01}}X^2\right)^4$ $-\left(\left(Z^2 - \frac{2}{a_{01}c_2}(c_3X - c_2Y)Z - 2\frac{c_2}{a_{01}}X^2\right)^2 + 2\frac{4a_{00}-a_{01}^2}{a_{01}^2}Z^4\right)^2.$ |
| Quartic (19) | |
| Singularities | $2a_3 + a_1$ at $[1:1:0]$, $[-1:1:0]$, $[0:0:1]$. |
| $Q \cap L_\infty$ | singular at $[1:1:0]$ and $[-1:1:0]$. |
| Torus decomposition | $\frac{1}{t^2}\Big((-X^2+Y^2-\frac{1}{2}tZ^2)^4$ $-\left((-X^2+Y^2-\frac{1}{2}tZ^2)^2 - \frac{1}{2}t^2Z^4\right)^2\Big).$ |

Table 4.

## §9. Infiniteness of $(2,3)$ quasi torus decompositions

In this section, we consider the possibilities of $(2,3)$ quasi torus decompositions of a plane curve which admits a $(2,3)$ torus decomposition. We assert:

**Proposition 9.** *Let $C = \{f = 0\} \subset \mathbb{C}^2$ be a $(2,3)$ torus curve of any degree. Then $C$ has infinitely many $(2,3)$ quasi torus decompositions.*

*Proof.* Suppose that $f(x,y)$ can be written as $f(x,y) = h_0(x,y)^2 - g_0(x,y)^3$. We put inductively

$$g_{i+1}(x,y) = -\frac{4}{3}\,h_i(x,y)^2 + g_i(x,y)^3,$$
$$h_{i+1}(x,y) = \frac{\sqrt{-3}}{9}h_i(x,y)(-8\,h_i(x,y)^2 + 9\,g_i(x,y)^3)$$

for $i \geq 0$. Then we claim that they satisfy the following equality:

$$(*_3) \qquad \left(\prod_{k=0}^{i} g_k(x,y)\right)^6 f(x,y) = h_{i+1}(x,y)^2 - g_{i+1}(x,y)^3, \quad i \geq 0.$$

Indeed, by a simple calculation, we have

$$h_{i+1}(x,y)^2 - g_{i+1}(x,y)^3 = g_i(x,y)^6\left(h_i(x,y)^2 - g_i(x,y)^3\right).$$

The assertion follows immediately from this equality. □

**Remark 4.** Let $C(t)$ be a family of curves given by

$$C(t) : t\, h_{i+1}(x,y)^2 - g_{i+1}(x,y)^3 = 0, \quad t \in \mathbb{C}.$$

We thank Professor J. I. Cogolludo for informing us the generic fiber of this pencil is not irreducible.

Now we study the location of singularities of a family of $(2,3)$ quasi torus decompositions which has the form $(*_3)$.
We put $r_i(x,y) := \prod_{k=0}^{i} g_k(x,y)$ and $\Sigma_i := \{h_i = 0\} \cap \{g_i = 0\}$ for $i \geq 0$. Then, by the definitions, we have the followings:

(1) $\Sigma_0$ is the set of inner singularities of $\{f = 0\}$.

(2) $\Sigma_i \subset \{r_i = 0\}$ for all $i \geq 0$.

(3) $\Sigma_0 \subset \Sigma_1 \subset \cdots \subset \Sigma_i \subset \cdots$.

In particular, $\{r_i = 0\}$ contains the inner singularities of $\{f = 0\}$ for all $i \geq 0$.

By Proposition 9 and above observations, it is important to study the existence of $(2,3)$ torus decompositions which is obtained by visible or invisible degenerations. We are also interested in quasi torus decompositions which does not come from a torus decomposition as in $(*_3)$.

We will give such an example $(2,3)$ quasi torus decomposition. Let $Q = \{f = 0\}$ be the three cuspidal quartic which has three $(2,3)$ torus decompositions (V-1), (V-2) and (V-3) as in §5. Recall that the $(2,3)$ torus decomposition (V-1) and locations of singularities:

$$f(x,y) = -3(x^2 + y^2 + 4x + 1)^2 + 4(2x+1)^3,$$
$$P_1 = (1,0), \quad P_2 = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \quad P_3 = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$$

where $P_2, P_3$ are the inner singularities of torus decomposition (V-1). Now we take following three polynomials $s_1$ $s_3$ and $s_5$ of degree 1, 3 and 5 respectively:

$$\begin{aligned}
s_1(x,y) &= x - Iy - 1,\\
s_3(x,y) &= \sqrt[3]{4}(3Iy^3 - (5x+7)y^2 - I(x-1)^2 y - (x-1)^3),\\
s_5(x,y) &= \sqrt{3}(y^5 + 3I(x+5)y^4 - 2(x^2+13x+10)y^3 + 2I(x-4)(x-1)^2y^2\\
&\qquad - 3(x+1)(x-1)^3 y - I(x-1)^5).
\end{aligned}$$

Then they satisfy the following equality:

$$s_1(x,y)^6 f(x,y) = s_5(x,y)^2 + s_3(x,y)^3.$$

Note that $\{s_1 = 0\}$ does not pass through the inner singularities of $Q$. Thus this decomposition is an example of $(2,3)$ quasi torus decomposition which does not come from as in $(*_3)$.

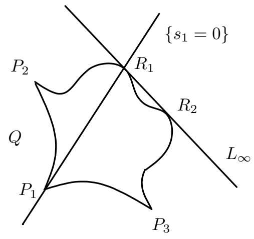


Figure 3:

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