SUT Journal of Mathematics
Vol. 47, No. 1 (2011), 15–23

Generalized finite operators and orthogonality

Smail Bouzenada

(Received November 7, 2009; Revised April 9, 2011)

Abstract. In this paper we prove that a spectraloid operator is finite, we present some generalized finite operators and we give a new class of finite operators. Also, the orthogonality of some operators is studied.

AMS 2010 *Mathematics Subject Classification.* 47B47, 47A30, 47A12

Key words and phrases. Finite operator, orthogonality, numerical range.

§1. Introduction

Let H be a separable infinite dimensional complex Hilbert space, and let $\mathcal{L}(H)$ denote the algebra of all bounded linear operators on H. For $A, B \in \mathcal{L}(H)$, the generalized derivation $\delta_{A,B} : \mathcal{L}(H) \to \mathcal{L}(H)$ is defined by

$$\delta_{A,B}(X) = AX - XB.$$

We denote $\delta_{A,A}$ by δ_A. Let E be a complex Banach space. We say [1] that $b \in E$ is orthogonal to $a \in E$ if for all complex λ there holds $\|a + \lambda b\| \geq \|a\|$. An operator $A \in \mathcal{L}(H)$ is called *finite* by J. P. Williams [12] if $\|AX - XA - I\| \geq 1$ for all $X \in \mathcal{L}(H)$, i.e. the range of δ_A is orthogonal to the identity operator. The pair $(A, B) \in \mathcal{L}(H) \times \mathcal{L}(H)$ is said to be generalized finite operators [7] if $\|AX - XB - I\| \geq 1$ for all $X \in \mathcal{L}(H)$. $\mathcal{F}(H)$ and $\mathcal{GF}(H)$ denote the class of finite operators and the class of generalized finite operators respectively.

For $A \in \mathcal{L}(H)$ the set $W(A) = \{(Ax, x) : x \in H \text{ and } \|x\| = 1\}$ is called the numerical range of A.

In the following we will denote the spectrum, the point spectrum, the approximate spectrum and the approximate reducing spectrum of $A \in \mathcal{L}(H)$ by $\sigma(A)$, $\sigma_p(A)$, $\sigma_a(A)$ and $\sigma_{ar}(A)$ respectively.

An operator $A \in \mathcal{L}(H)$ is said to be spectraloid if $\omega(A) = r(A)$, where $r(A)$ (resp. $\omega(A)$) is the spectral radius (resp. numerical radius) of A, convexoid if $\overline{W(A)} = co\sigma(A)$, where $co\sigma(A)$ is the convex hull of $\sigma(A)$, and transaloid if $r\left((A-\lambda I)^{-1}\right) = \left\|(A-\lambda I)^{-1}\right\|$ for all $\lambda \notin \sigma(A)$. We have the following inclusions:

$$\begin{array}{ccccccc} & & \text{paranormal} & \longrightarrow & \text{normaloid} & & \\ & \nearrow & & & & \searrow & \\ \text{hyponormal} & & & \nearrow & & & \text{spectraloid} \\ & \searrow & & & & \nearrow & \\ & & \text{transaloid} & \longrightarrow & \text{convexoid} & & \end{array}$$

A bounded linear operator A is in the class $\mathcal{Y}_\alpha$ for certain $\alpha \geq 1$ if there exists a positive number k_α such that

$$|AA^* - A^*A|^\alpha \leq k_\alpha^2\left((A-\lambda I)^*(A-\lambda I)\right), \text{ for all } \lambda \in \mathbb{C}.$$

It is known that $\mathcal{Y}_\alpha \subseteq \mathcal{Y}_\beta$ for each α, β such as $1 \leq \alpha \leq \beta$ [11], where $\mathcal{Y} = \cup_{\alpha \geq 1}\mathcal{Y}_\alpha$.

In this paper we prove that a spectraloid operator is finite and that the operator of the form $A + K$ is also finite, where A is convexoid and K is compact. We present some generalized finite operators and we give a new class of finite operators. Also we study the orthogonality of certain operators.

§2. Preliminaries

Lemma 1. *Let $A \in \mathcal{L}(H)$. If $\sigma_{ar}(A)$ is not empty, then A is finite.*

Proof. Let $\lambda \in \sigma_{ar}(A)$ and $\{x_n\}$ be a normalized sequence such that $(A-\lambda I)x_n \longrightarrow 0$ and $(A-\lambda I)^* x_n \longrightarrow 0$. If $X \in \mathcal{L}(H)$, then we have

$$\begin{aligned} \|AX - XA - I\| &= \|(A-\lambda I)X - X(A-\lambda I) - I\| \\ &\geq |\langle (A-\lambda I)Xx_n, x_n\rangle - \langle X(A-\lambda I)x_n, x_n\rangle - 1|. \end{aligned}$$

Letting $n \longrightarrow \infty$, we obtain $\|AX - XA - I\| \geq 1$. □

Lemma 2. *Let $A \in \mathcal{L}(H)$. If $\mathcal{R}eA \geq 0$, then $\sigma_a(A) \subset \sigma_{ar}(A)$.*

Proof. For $\lambda \in \sigma_a(A)$, there exists a sequence $\{x_n\}$ such that $(A-\lambda I)x_n \longrightarrow 0$, and then

$$B = \mathcal{R}e(A-\lambda I) = \frac{1}{2}\left[(A-\lambda I) + (A-\lambda I)^*\right]$$

satisfies $\langle Bx_n, x_n\rangle \longrightarrow 0$. Since $B \geq 0$, it results that $Bx_n \longrightarrow 0$, i.e,

$$\frac{1}{2}\left[(A-\lambda I)x_n + (A-\lambda I)^* x_n\right] \longrightarrow 0.$$

Since $(A-\lambda I)x_n \longrightarrow 0$, we have $(A-\lambda I)^* x_n \longrightarrow 0$. □

Lemma 3. *For $A \in \mathcal{L}(H)$, $\partial W(A) \cap \sigma(A) \subset \sigma_{ar}(A)$.*

Proof. By the transformation $A \longmapsto \alpha A + \beta$ the hypothesis $\lambda \in \partial W(A) \cap \sigma(A)$ can be replaced by $0 \in \partial W(A) \cap \sigma(A)$ with $\mathcal{R}eA \geq 0$. Since $0 \in \partial\sigma(A) \subset \sigma_a(A)$, it results from the previous lemmas that $0 \in \sigma_{ar}(A)$, hence $\partial W(A) \cap \sigma(A) \subset \sigma_{ar}(A)$. □

§3. Main results

Theorem 1. *Let $A \in \mathcal{L}(H)$ be convexoid. Then A is finite.*

Proof. If A is convexoid, then $\overline{W(A)} = co\sigma(A)$. Hence $\partial W(A) \cap \sigma(A) \neq \phi$. It follows immediately from the previous lemmas that A is finite. □

Remark 1. *It is known that transaloid operators are convexoid operators, and then $\mathcal{F}(H)$ contains all the transaloid operators.*

Theorem 2. *Let $A \in \mathcal{L}(H)$ be spectraloid. Then A is finite.*

Proof. We have $\omega(A) = r(A)$. This implies that there exists $\lambda \in \sigma(A) \subset \overline{W(A)}$ such that $|\lambda| = \omega(A)$, hence $\lambda \in \partial W(A)$, then $\partial W(A) \cap \sigma(A) \neq \phi$, which implies that $A \in \mathcal{F}(H)$. □

As a consequence of the previous theorem we obtain:

Corollary 1. *The following operators are finite.*

(1) Hyponormal operators,

(2) Transaloid operators,

(3) Paranormal operators,

(4) Normaloid operators.

Lemma 4. *[9] For $A \in \mathcal{L}(H)$, the following holds*

$$\overline{W(A)} = co\sigma(A) \Longleftrightarrow \forall \lambda \notin co\sigma(A): \left\| (A - \lambda I)^{-1} \right\| \leq [dist(\lambda, co\sigma(A))].$$

Hence a convexoid element on a C^*-algebra $\mathcal{A}$, may be defined as an element $a \in \mathcal{A}$ satisfying

$$\forall \lambda \notin co\sigma(a): \left\| (a - \lambda e)^{-1} \right\| \leq [dist(\lambda, co\sigma(a))]^{-1}.$$

Theorem 3. *Let $\mathcal{A}$ be a C^*-algebra and let a be a convexoid element on $\mathcal{A}$. Then a is finite.*

Proof. It is known [6, p. 97] that there exist a Hilbert space H and a *-isometric homomorphism φ $(\varphi : \mathcal{A} \longrightarrow \mathcal{L}(H))$. Then $\varphi(a)$ is convexoid. Since φ is isometric it results from Theorem 1 that a is finite. □

Corollary 2. *Let $A \in \mathcal{L}(H)$ be convexoid. Then $T = A + K$ is finite, where K is a compact operator.*

Proof. Since the Calkin algebra $\mathcal{L}(H) \diagup \mathcal{K}(H)$ is a C^*-algebra (where $\mathcal{K}(H)$ is the set of compact operators), $[A] = \{A + K : K \in \mathcal{K}(H)\}$ is convexoid. Hence it follows from Theorem 3 $[A]$ is finite and we have, for all $X \in \mathcal{L}(H)$

$$\begin{aligned} \|TX - XT - I\| &= \|[TX - XT - I]\| \\ &= \|[T][X] - [X][T] - [I]\| \\ &= \|[A][X] - [X][A] - [I]\| \\ &\geq 1. \end{aligned}$$

□

Lemma 5. *For $A, T \in \mathcal{L}(H)$, if $A \in \mathcal{Y}$ and T is a normal operator such as $AT = TA$, then for all $\lambda \in \sigma_p(T)$*

$$\|AX - XA - T\| \geq |\lambda|, \text{ for all} X \in \mathcal{L}(H).$$

Proof. Let $\lambda \in \sigma_p(T)$ and M_λ be the eigenspace associated with λ. Since $AT = TA$, we have $AT^* = T^*A$ by the Fuglede's theorem [4]. Hence M_λ reduces both A and T. According to the decomposition $H = M_\lambda \oplus M_\lambda^\perp$, we can write A, T and $X \in \mathcal{L}(H)$ as follows:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \ T = \begin{bmatrix} \lambda & 0 \\ 0 & T_2 \end{bmatrix} \text{ and } X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}.$$

Since the restriction of a class $\mathcal{Y}$ operator to a reduced subspace is a class $\mathcal{Y}$ operator and since $\mathcal{Y} \subset \mathcal{F}(H)$ [2], we have

$$\begin{aligned} \|AX - XA - T\| &= \left\| \begin{bmatrix} A_1X_1 - X_1A_1 - \lambda & * \\ * & * \end{bmatrix} \right\| \\ &\geq \|A_1X_1 - X_1A_1 - \lambda\| \\ &\geq |\lambda| \left\| A_1(\frac{X_1}{\lambda}) - (\frac{X_1}{\lambda})A_1 - I \right\| \\ &\geq |\lambda|. \end{aligned}$$

□

In the sequel, we need the Berberian technique, and it allows us to construct a Hilbert space which contains a given Hilbert space H on which we could speak about "approached eigenvectors" and those as regarded as eigenvectors.

Proposition 1 (Berberian technique)**.** *Let H be a complex Hilbert space, then there exist a Hilbert space $\widehat{H} \supset H$ and an *-isometric homomorphism $\varphi : \mathcal{L}(H) \longrightarrow \mathcal{L}(\widehat{H})$ $(A \longmapsto \widehat{A})$ preserving the order, i.e. for all $A, B \in \mathcal{L}(H)$ and for all $\alpha, \beta \in \mathbb{C}$ we have:*

(1) $\widehat{A^*} = \widehat{A}^*$,

(2) $\widehat{I} = I$,

(3) $\alpha \widehat{A + \beta} B = \alpha\widehat{A} + \varphi\widehat{B}$,

(4) $\widehat{AB} = \widehat{A}\widehat{B}$,

(5) $\left\|\widehat{A}\right\| = \|A\|$,

(6) $\sigma(\widehat{A}) = \sigma(A)$, $\sigma_p(\widehat{A}) = \sigma_a(\widehat{A}) = \sigma_a(A)$,

(7) If A is positive, then $\widehat{A}$ is positive and $\widehat{A^\alpha} = \widehat{A}^\alpha$ for all $\alpha > 0$.

Theorem 4. *Let $A \in \mathcal{Y}$. Then for every normal operator T such that $AT = TA$, we have*

$$\|AX - XA - T\| \geq \|T\|, \text{ for all } X \in \mathcal{L}(H).$$

Proof. Let $\lambda \in \sigma(T) = \sigma_a(T)$ [5]. Then it follows from Proposition 1 that $\widehat{T}$ is normal, $\widehat{A} \in \mathcal{Y}$, $\widehat{A}\widehat{T} = \widehat{T}\widehat{A}$ and $\lambda \in \sigma_p(\widehat{T})$. By applying Lemma 5, we get

$$\|AX - XA - T\| = \left\|\widehat{A}\widehat{X} - \widehat{X}\widehat{A} - \widehat{T}\right\| \geq |\lambda|,$$

for all $X \in \mathcal{L}(H)$. Hence

$$\|AX - XA - T\| \geq \sup_{\lambda\in\sigma(\widehat{T})} |\lambda| = r(\widehat{T}) = \left\|\widehat{T}\right\| = \|T\|,$$

for all $X \in \mathcal{L}(H)$. □

Theorem 5. *Let $A, B \in \mathcal{L}(H)$. If $A, B^* \in \mathcal{Y}$, then*

$$\|AX - XB - T\| \geq \|T\|,$$

for all $X \in \mathcal{L}(H)$ and for all $T \in \ker \delta_{A,B}$.

Proof. Let $T \in \ker \delta_{A,B}$. Then $T \in \ker \delta_{A^*,B^*}$ [11, Theorem 2]. Therefore, $ATT^* = TBT^* = TT^*A$. Since $A \in \mathcal{Y}$, TT^* is normal and $A(TT^*) = (TT^*)A$, the previous theorem implies that

$$\begin{aligned} \|T\|^2 = \|TT^*\| &\leq \|TT^* - (AXT^* - XT^*A)\| \\ &= \|TT^* - (AXT^* - XBT^*)\| \\ &\leq \|T^*\| \, \|T - (AX - XB)\| . \end{aligned}$$

Thus

$$\|AX - XB - T\| \geq \|T\| .$$

□

Theorem 6. *Let $A, B \in \mathcal{L}(H)$ be $A = \overset{n}{\underset{i=1}{\oplus}} A_i$, $B = \overset{n}{\underset{i=1}{\oplus}} B_i$. If there exists $j \leq n$ such that $(A_j, B_j) \in \mathcal{GF}(H_j)$, then $(A, B) \in \mathcal{GF}(H)$.*

Proof. Let $j \leq n$ such that $(A_j, B_j) \in \mathcal{GF}(H_j)$. Then for all $X \in \mathcal{L}(H)$

$$\begin{aligned} \|AX - XB - I\| &= \left\| \begin{pmatrix} * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & A_jX_{jj} - X_{jj}B_j - I_j & * & * & * \\ * & * & * & * & . & * & * \\ * & * & * & * & * & . & * \\ * & * & * & * & * & * & * \end{pmatrix} \right\| \\ &\geq \|A_jX_{jj} - X_{jj}B_j - I_j\| \\ &\geq 1. \end{aligned}$$

□

Proposition 2. *For $(A, B) \in \mathcal{GF}(H)$, the following assertions hold:*

(1) $(\alpha A + \beta, \alpha B + \beta) \in \mathcal{GF}(H)$, *for each* $\alpha, \beta \in \mathbb{C}$.

(2) $\left(A^{-1}, B^{-1}\right) \in \mathcal{GF}(H)$, *if A and B are invertible.*

(3) $(R, T) \in \mathcal{GF}(H)$, *if R and T are simultaneously unitarily equivalent to A and B respectively.*

(4) $(B^*, A^*) \in \mathcal{GF}(H)$.

(5) $\left(A^{2^m}, B^{2^m}\right) \in \mathcal{GF}(H)$, *for all* $m \in \mathbb{N}$.

(6) $\sigma(A) \cap \sigma(B) \neq \phi$.

Proof. (1) If $(A,B) \in \mathcal{GF}(H)$ then [7, Theorem 18] there exists a state f on $\mathcal{L}(H)$ such that $f(AX) = f(XB)$ for all $X \in \mathcal{L}(H)$. As a consequence of the linearity of f,

$$\forall \alpha, \beta \in \mathbb{C} : f((\alpha A + \beta) X) = f(X(\alpha B + \beta))$$

for all $X \in \mathcal{L}(H)$.

(2) Let f be a state on $\mathcal{L}(H)$ such that $f(AX) = f(XB)$ for all $X \in \mathcal{L}(H)$. Then we have

$$f\left(A^{-1}X\right) = f\left(\left(A^{-1}XB^{-1}\right)B\right) = f\left(A\left(A^{-1}XB^{-1}\right)\right) = f\left(XB^{-1}\right),$$

for all $X \in \mathcal{L}(H)$.

(3) Let U be a unitary operator. Then by [7, Theorem 18] we have

$$\begin{aligned}
(A,B) \in \mathcal{GF}(H) &\Longleftrightarrow 0 \in \overline{W(AX - XB)}, \forall X \in \mathcal{L}(H) \\
&\Longleftrightarrow 0 \in \overline{W(U^*(AX - XB)U)}, \forall X \in \mathcal{L}(H) \\
&\Longleftrightarrow 0 \in \overline{W(U^*(AUU^*X - XUU^*B)U)}, \forall X \in \mathcal{L}(H) \\
&\Longleftrightarrow 0 \in \overline{W((U^*AU)Y - Y(U^*BU))}, \forall Y \in \mathcal{L}(H) \\
&\Longleftrightarrow (U^*AU, U^*BU) \in \mathcal{GF}(H).
\end{aligned}$$

(4) Let f be a state on $\mathcal{L}(H)$ such that $f(AX) = f(XB)$ for all $X \in \mathcal{L}(H)$. Then we have

$$\begin{aligned}
f^*(B^*X) &= \overline{f(B^*X)^*} = \overline{f(X^*B)} \\
&= \overline{f(AX^*)} = \overline{f(XA^*)^*} \\
&= f^*(XA^*),
\end{aligned}$$

for all $X \in \mathcal{L}(H)$. Since the adjoint of a state is a state, we have $(B^*, A^*) \in \mathcal{GF}(H)$.

(5) Let f be a state on $\mathcal{L}(H)$ such that $f(AX - XB) = 0$ for all $X \in \mathcal{L}(H)$. By recurrence we have:

For $m = 0$, $\left(A^{2^0}, B^{2^0}\right) = (A,B) \in \mathcal{GF}(H)$. Suppose that, for all $m \in \mathbb{N}$, there exists a state f on $\mathcal{L}(H)$ such that

$$f\left(A^{2^m}X - XB^{2^m}\right) = 0, \text{ for all } X \in \mathcal{L}(H).$$

Then

$$f\left(A^{2^m}\left(A^{2^m}X\right) - \left(A^{2^m}X\right)B^{2^m}\right) = 0 \text{ and } f\left(A^{2^m}\left(XB^{2^m}\right) - \left(XB^{2^m}\right)B^{2^m}\right) = 0,$$

hence

$$f\left(A^{2^{m+1}}X - XB^{2^{m+1}}\right) = 0.$$

(6) Suppose that $\sigma(A) \cap \sigma(B) = \phi$. In [10] M. Rosenblum proved that $\sigma(\delta_{A,B}) \subset \sigma(A) - \sigma(B)$, and then $\delta_{A,B}$ is invertible, hence there exists $X \in \mathcal{L}(H)$ for which $\|\delta_{A,B}(X) - I\| < 1$. □

Theorem 7. *Let $A, B \in \mathcal{L}(H)$. If there exist a normed sequence $(f_n)_{n\geq 1}$ in H and a scalar λ such that*

$$\|(A-\lambda I)^* f_n\| \longrightarrow 0 \text{ and } \|(B-\lambda I) f_n\| \longrightarrow 0,$$

then $(A, B) \in \mathcal{GF}(H)$.

Proof. If $X \in \mathcal{L}(H)$. Then

$$\begin{aligned} \|AX - XB - I\| &= \|(A-\lambda I) X - X(B-\lambda I) - I\| \\ &\geq |([(A-\lambda I) X - X(B-\lambda I) - I] f_n, f_n)| \\ &= |(Xf_n, (A-\lambda I)^* f_n) - ((B-\lambda I) f_n, X^* f_n) - 1|. \end{aligned}$$

By passage to the limit, we get $\|AX - XB - I\| \geq 1$, for all $X \in \mathcal{L}(H)$. □

Corollary 3. *Let $A \in \mathcal{L}(H)$. Then, for all $\lambda \in \sigma_a(A)$ and for all $C \in \mathcal{L}(H)$,*

$$((A-\lambda I)^*, C(A-\lambda I)) \in \mathcal{GF}(H).$$

Proof. Let $\lambda \in \sigma_a(A)$, then there exists a normed sequence $(f_n)_{n\geq 1}$ in H such that $\|(A-\lambda I) f_n\| \longrightarrow 0$. If $T = A - \lambda I$ and $R = CT$ with $C \in \mathcal{L}(H)$, then

$$\left\|[(T-0)^*]^* f_n\right\| = \|(A-\lambda I) f_n\| \longrightarrow 0$$

and

$$\|(R-0) f_n\| = \|C(A-\lambda I) f_n\| \longrightarrow 0,$$

hence

$$((A-\lambda I)^*, C(A-\lambda I)) = (T^*, R) \in \mathcal{GF}(H).$$

□

Corollary 4. *For all $A \in \mathcal{L}(H)$, there exists $B \in \mathcal{L}(H)$ for which (A, B) is a generalized finite operator.*

Proof. We say that the approximate spectrum is never empty. Let $\lambda \in \sigma_a(A^*)$, hence it follows from the previous corollary that

$$((A^*-\lambda I)^*, C(A^*-\lambda I)) = \left(\left(A-\overline{\lambda} I\right), C(A^*-\lambda I)\right) \in \mathcal{GF}(H),$$

for all $C \in \mathcal{L}(H)$, and by applying (1) of Proposition 2 we get

$$(A, B) \in \mathcal{GF}(H),$$

where $B = C(A^*-\lambda I) + \overline{\lambda} I$. □

Corollary 5. *$\mathcal{F}(H)$ contains the following class:*

$$S(H) = \left\{ A \in \mathcal{L}(H) : \ A - \overline{\lambda} I = C(A^* - \lambda I) \ \text{ with } \lambda \in \sigma_a(A^*) \ \text{ and } C \in \mathcal{L}(H) \ \right\}.$$

Proof. It follows from the previous corollary that, if $A \in S(H)$, then $(A, A) \in \mathcal{GF}(H)$ i.e. $A \in \mathcal{F}(H)$. □

Acknowledgement

I would like to thank the referee for his/her careful reading of the paper. The valuable suggestions, critical remarks, and pertinent comments made numerous improvements throughout.

References

[1] J. H. Anderson, *On normal derivation*, Proc. Amer. Math. Soc, 38(1973), 135-140.

[2] A. Bachir and S. Mecheri, *Some Properties of* ($\mathcal{Y}$) *Class Operators*, KYUNGPOOK Math. J. 49(2009), 203-209.

[3] S. K. Berberian, *Approximate proper vectors*, Proc. Amer. Math. Soc, 13(1962), 111-114.

[4] B. Fuglede, *A commutativity theorem for normal operators*, Proc. Nat. Acad. Sci. U. S. A., 36(1950), 35-40.

[5] P. R. Halmos, *Hibert space problem book*, Springer, Verlag, New York, (1962).

[6] D. A. Herrero, *Approximation of Hilbert space operators I,* Pitmann advanced publishing program, Boston-London, Melbourne (1982).

[7] S. Mecheri, *Finite operators*, Demonstratio Math, 35(2002), 355-366.

[8] S. Mecheri, *The numerical range of linear operators.* Filomat 22 No. 2, 1-8 (2008).

[9] G. H. Orland, On a class of operators, Proc. Amer. Math. Soc. 15 (1964), 75-79.

[10] M. Rosenblum, *On the operator equation $BX - XY = Q$*, Duke Math. J. 23 (1956), 263-269

[11] A. Uchiyama and T. Yoshino, *On the class $\mathcal{Y}$ operators*, Nihonkai. Math. J. Vol. 8 (1997), 179-194.

[12] J. P. Williams, *Finite operators*, Proc. Amer. Math. Soc, 26(1970), 129-135.

Smail Bouzenada
Department of Mathematics, University of Tebessa, 12002 Tebessa, Algeria
E-mail: bouzenadas@gmail.com