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On a braid monoid analogue of a theorem of Tits

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Abstract. We extend a theorem of Tits about the fundamental groups of graphs of Coxeter groups to those of braid monoids. More precisely, we show that every self-homotopy of a word decomposes into self-homotopies each of which is inessential, a cube, a prism or a permutohedron.

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§1. Introduction

This paper grew out of my attempt [2] to prove a coherence theorem for braided monoidal 2-categories. This theorem is categorical in nature but the essential part is combinatorial and can be viewed as a theorem about homotopies of words defined by braid relations. In the context of Coxeter groups, Tits [5] showed that every self-homotopy decomposes into self-homotopies each of which is inessential or lies in a rank 3 residue. This means that nontrivial self-homotopies of galleries in Coxeter complexes only occur in finite stars of simplices of codimension 3. To obtain a similar result for braid monoids, we first prove a variant of a result in [1] which asserts that every positive braid has a unique factorization with respect to a given set of generators. Using this factorization we then show that every self-homotopy decomposes into self-homotopies each of which is inessential, a *cube*, a *prism* or a *permutohedron*. This result is an important step toward the coherence theorem, and it seems to be of independent interest as well.

§2. Positive braids

In this section, we consider positive braids and show that every positive braid has a unique factorization with respect to a given subset of $\{1, 2, ..., n-1\}$. This is a variant of a result in [1].

For $n \geq 1$, denote by B_n^+ the monoid generated by $n-1$ generators $\sigma_1, \sigma_2, ..., \sigma_{n-1}$ and the relations

$$\begin{aligned} \sigma_i\sigma_j &= \sigma_j\sigma_i \quad &&\text{if } |i-j| \geq 2, \\ \sigma_i\sigma_j\sigma_i &= \sigma_j\sigma_i\sigma_j \quad &&\text{if } |i-j| = 1, \end{aligned}$$

where $i, j = 1, 2, ..., n-1$. The elements of B_n^+ are called positive braids on n strings. Throughout this paper, e denotes the unit in B_n^+ and l denotes the length function on B_n^+.

Definition 1. For a positive braid P, an element $i \in \{1, 2, ..., n-1\}$ is called a starting element of P if there exists $Q \in B_n^+$ such that $P = \sigma_i Q$. Similarly, an element $i \in \{1, 2, ..., n-1\}$ is called a finishing element of P if there exists $Q \in B_n^+$ such that $P = Q\sigma_i$. For a positive braid P, we denote by $S(P)$ the set of starting elements of P. Similarly, we denote by $F(P)$ the set of finishing elements of P.

Definition 2. A positive braid is called a positive permutation braid if it can be drawn as a geometric braid in which every pair of strings crosses at most once.

In other words, positive permutation braids are the image of the map $\rho : S_n \to B_n^+$ defined by $\rho(w) = \sigma_{i_1}\sigma_{i_2}...\sigma_{i_r}$ for some reduced expression $w = s_{i_1}s_{i_2}...s_{i_r}$ in the symmetric group S_n.

For a subset J of $\{1, 2, ..., n-1\}$, let S_J^+ be the set consisting of the unit and all permutation braids generated by the set $\{\sigma_i : i \in J\}$ in B_n^+.

Definition 3. Set

$$\sigma_i * \sigma_j = \begin{cases} \sigma_i & \text{if } i = j, \\ \sigma_i\sigma_j\sigma_i & \text{if } |i-j| = 1, \\ \sigma_i\sigma_j & \text{if } |i-j| \geq 2. \end{cases}$$

We frequently use the following lemmas:

Lemma 1. *For elements i, j in J and for $A \in S_J^+$ we have*

$$i \notin S(A) \Leftrightarrow \sigma_i A \in S_J^+,$$

$$i, j \notin S(A) \Leftrightarrow (\sigma_i * \sigma_j)A \in S_J^+.$$

Proof. These follow from the Exchange Property of Coxeter groups and the characterization of the permutation braids by the map $\rho : S_n \to B_n^+$ above. □

Lemma 2. *If $P = AB$ with $P \in S_J^+$ then we have $A, B \in S_J^+$.*

Proof. Straightforward. □

We also use the following lemma of Garside [3].

Lemma 3. *(Garside) Let $P = P_1\sigma_i = P_2\sigma_j$ in B_n^+. Then $P = P_3(\sigma_i * \sigma_j)$ for some P_3 in B_n^+.*

Definition 4. Given a subset J of $\{1, 2, ..., n-1\}$, a factorization $P = AB$ with $A, B \in B_n^+$ is called J-weighted if $B \in S_J^+$ and $F(A) \cap J \subset S(B)$. For X, Y in S_J^+, X is called a subfactor of Y if $Y = QX$ for some Q.

Proposition 1. *With J above, every positive braid P has a unique J-weighted factorization $P = A_1B_1$. If $P = AB$ is another factorization with $B \in S_J^+$, B becomes a subfactor of B_1.*

Proof. We first show the existence of a J-weighted factorization $P = A_1B_1$. Consider all factorizations $P = AB$ with $B \in S_J^+$, and select one in which $l(B)$ is maximal. If $F(A) \cap J \not\subset S(B)$ then we can find $i \in F(A) \cap J$ with $i \notin S(B)$. Then we can write $A = A'\sigma_i$ for some A', and by Lemma 1 $\sigma_i B$ becomes an element of S_J^+. Set $B' = \sigma_i B$. Then $P = A'B'$ with $l(B') \geq l(B)$, which is a contradiction.

We now show that every other factorization $P = AB$ with $B \in S_J^+$ satisfies $B_1 = QB$ for some Q. Otherwise there exist factorizations

$$P = A'\sigma_i C$$

with $\sigma_i C \in S_J^+$ such that C is a subfactor of B_1 but $\sigma_i C$ is not. Choose such a factorization with largest possible length C, and write $B_1 = QC$. If $Q = e$ then $P = A'\sigma_i B_1$ with $\sigma_i B_1 \in S_J^+$, which contradicts the maximality of $l(B_1)$. Thus $Q \neq e$, and we can choose $j \in F(Q) \cap J$ to write $Q = Q'\sigma_j$ for some Q'. Then $P = A_1B_1 = A_1Q'\sigma_j C$, and by setting $A'' = A_1Q'$, we have

$$P = A''\sigma_j C.$$

From the identity $P = A'\sigma_i C = A''\sigma_j C$, it follows that $A'\sigma_i = A''\sigma_j$, and by Lemma 3, we have $A'\sigma_i = A'''(\sigma_i * \sigma_j)$ for some A''' in B_n^+. As a result we have

$$P = A'''(\sigma_i * \sigma_j)C.$$

Since $\sigma_i C \in S_J^+$ we have $i \notin S(C)$. Also, since $B_1 = QC = Q'\sigma_j C$ and $B_1 \in S_j^+$ we have $\sigma_j C \in S_J^+$ by Lemma 2, and hence $j \notin S(C)$. Applying

Lemma 1 to these facts that $i \notin S(C)$ and $j \notin S(C)$, we have $(\sigma_i * \sigma_j)C \in S_J^+$. Since $B_1 = QC = Q'\sigma_j C$, $\sigma_j C$ is a subfactor of B_1. On the other hand, $\sigma_i C$ is not a subfactor of B_1, so that $i \neq j$. Now suppose $|i-j| \geq 2$. In this case, we have

$$P = A'''(\sigma_i * \sigma_j)C = A'''\sigma_i\sigma_j C.$$

Since $\sigma_i C$ is not a subfactor of B_1, $\sigma_j\sigma_i C = \sigma_i\sigma_j C$ is not a subfactor of B_1. So the factor $\sigma_j C$ satisfies the condition of C in the factorization $P = A'\sigma_i C$ but $l(\sigma_j C) \geq l(C)+1$, which contradicts the maximality of the length of $l(C)$. We next consider the case $|i-j| = 1$. In this case, we have

$$P = A'''(\sigma_i * \sigma_j)C = A'''\sigma_j\sigma_i\sigma_j C.$$

Since $\sigma_i C$ is not a subfactor of B_1, $\sigma_j\sigma_i\sigma_j C = \sigma_i\sigma_j\sigma_i C$ is not a subfactor of B_1. Further, if $\sigma_i\sigma_j C$ is a subfactor of B_1, this factor satisfies the condition above but $l(\sigma_i\sigma_j C) \geq l(C)+2$, which contradicts the maximality of the length of $l(C)$. If $\sigma_i\sigma_j C$ is not a subfactor of B_1, the factor $\sigma_j C$ satisfies the condition above but $l(\sigma_j C) \geq l(C)+1$, which contradicts the maximality of the length of $l(C)$. In each of these cases we have a contradiction, so the claim is proved.

We next show the uniqueness of the factorization. Suppose that $P = AB$ is another J-weighted factorization. Then we can write $B_1 = QB$ with Q in B_n^+. If $Q = e$ then $B_1 = B$, so we can assume $Q \neq e$. In this case we can find $i \in F(Q) \cap J$ so that $Q = Q'\sigma_i$ for some Q'. Since $B_1 \in S_J^+$, we have $\sigma_i B \in S_J^+$ and hence $i \notin S(B)$. On the other hand, since i is an element of $F(Q)$ and the identity $A = A_1 Q$ holds, i is an element of $F(A) \cap J$. Thus $F(A) \cap J \not\subset S(B)$, which is a contradiction. □

§3. Words and homotopies

In this section we consider homotopies between two words and prove the main theorem in this paper. Given a word $f = i_1 ... i_k$ in the free monoid on $\{1, ..., n-1\}$, we set $r(f) = \sigma_{i_1} ... \sigma_{i_k}$ in B_n^+. Let $\pi : B_n^+ \to S_n$ be the natural map from B_n^+ to the symmetric group S_n. A word $f = i_1 ... i_k$ is called *reduced* if k is minimal among all such expressions for $\pi \circ r(f)$ in S_n. Two words f and g are called *equivalent* if $r(f) = r(g)$. For distinct i and j in $\{1, ..., n-1\}$, write

$$p(i,j) = \begin{cases} jij & \text{if } |i-j| = 1, \\ ij & \text{if } |i-j| \geq 2. \end{cases}$$

An elementary homotopy is an alteration from a word of the form $f_1 p(i,j) f_2$ to the word $f_1 p(j,i) f_2$ where $i, j \in \{1, ..., n-1\}$ and f_1, f_2 are some words. We denote by $f \simeq g$ an elementary homotopy between f and g.

Two words are called homotopic if there exists a sequence of elementary homotopies between them. Obviously, two words are equivalent if and only if they are homotopic or identical. A self-homotopy is a sequence of elementary homotopies beginning and ending with the same word. In particular, a *cube* is a self-homotopy of the following form:

$$\begin{array}{ccccc} f_1ijkf_2 & \simeq & f_1ikjf_2 & \simeq & f_1kijf_2 \\ \simeq & & & & \simeq \\ f_1jikf_2 & \simeq & f_1jkif_2 & \simeq & f_1kjif_2 \end{array} .$$

A *prism* is a self-homotopy of the following form:

$$\begin{array}{ccccccc} f_1jijkf_2 & \simeq & f_1jikjf_2 & \simeq & f_1jkijf_2 & \simeq & f_1kjijf_2 \\ \simeq & & & & & & \simeq \\ f_1ijikf_2 & \simeq & f_1ijkif_2 & \simeq & f_1ikjif_2 & \simeq & f_1kijif_2 \end{array} .$$

A *permutohedron* is a self-homotopy of the following form:

$$\begin{array}{ccccccccc} f_1ijikjif_2 & \simeq & f_1ijkijif_2 & \simeq & f_1ijkjijf_2 & \simeq & f_1ikjkijf_2 & \simeq & f_1ikjikjf_2 \\ \simeq & & & & & & & & \simeq \\ f_1jijkjif_2 & & & & & & & & f_1kijikjf_2 \\ \simeq & & & & & & & & \simeq \\ f_1jikjkif_2 & & & & & & & & f_1kjijkjf_2 \\ \simeq & & & & & & & & \simeq \\ f_1jkijkif_2 & \simeq & f_1jkijikf_2 & \simeq & f_1jkjijkf_2 & \simeq & f_1kjkijkf_2 & \simeq & f_1kjikjkf_2 \end{array} .$$

A self-homotopy is *inessential* if it is of the form

$$f = f_0 \simeq f_1 \simeq ... \simeq f_{k-1} \simeq f_k \simeq f_{k-1} \simeq .... \simeq f_1 \simeq f_0 = f;$$

or if it is of the form

$$\begin{array}{ccc} f_1p(i,j)f_2p(k,l)f_3 & \simeq & f_1p(j,i)f_2p(k,l)f_3 \\ \simeq & & \simeq \\ f_1p(i,j)f_2p(l,k)f_3 & \simeq & f_1p(j,i)f_2p(l,k)f_3 \end{array} .$$

Given a word f, let $H(f)$ denote the graph whose vertices are words homotopic to f and whose edges are elementary homotopies. A self-homotopy is a circuit in this graph. We shall say that a circuit τ in a graph decomposed in two circuits $\tau_1\tau_2$ and $\tau_2^{-1}\tau_3$ if $\tau = \tau_1\tau_3$. In the context of Coxeter groups, Tits [5] proved that every self-homotopy decomposes into self-homotopies each of which is inessential or lies in a rank 3 residue.

The main result of this paper is the following

Theorem 1. *Every self-homotopy decomposes into self-homotopies each of which is inessential, a cube, a prism or a permutohedron.*

Proof. We consider everything modulo inessential self-homotopies of the first type, and use induction on the length of the words appearing in a self-homotopy. If all the vertices in a self-homotopy end in i for some i, then we can use the induction hypothesis to conclude that the self-homotopy decomposes as required. Otherwise, we can find a sequence of elementary homotopies of the form

$$fi \simeq f'j \simeq ...j \simeq .... \simeq ...j \simeq g'j \simeq gk,$$

where $i,j,k \in \{1,2,...,n-1\}$ with $i \neq j$, $j \neq k$, and f, f', g, g' are some words. Let $w = r(fi) = r(gk)$ in B_n^+. By applying Proposition 1 to w and $J = \{i,j,k\}$ we obtain a unique factorization $w = w_1w_2$ such that w_2 has maximal length in S_J^+. Choose words h and h' so that $r(h) = w_1$ and $r(h') = w_2$. The word h' can be chosen to be reduced and to end in i,j or k. Since S_J^+ can be identified with the symmetric group generated by $\{\pi(\sigma_i); i \in J\}$, we can apply a technique used in [4] to see that there are suitable words h_k, h_i, h_j such that h' becomes $h_kp(j,i)$, $h_ip(j,k)$, and $h_jp(k,i)$. This means, in particular, that fi is homotopic to $hh_kp(j,i)$. The word fi can be written as $fi = \varphi p(j,i)$ with a word φ, and we can take as a sequence of elementary homotopies from fi to $hh_kp(j,i)$ a sequence which increases the length of reduced words containing $p(j,i)$. Thus, we have a sequence of elementary homotopies from φ to hh_k so that the original sequence from fi to $hh_kp(j,i)$ is obtained from the sequence by putting $p(j,i)$ to all the vertices in the sequence. The word gk is homotopic to $hh_ip(j,k)$ with the word h used in common with fi. As a result, we obtain a circuit of the following form:

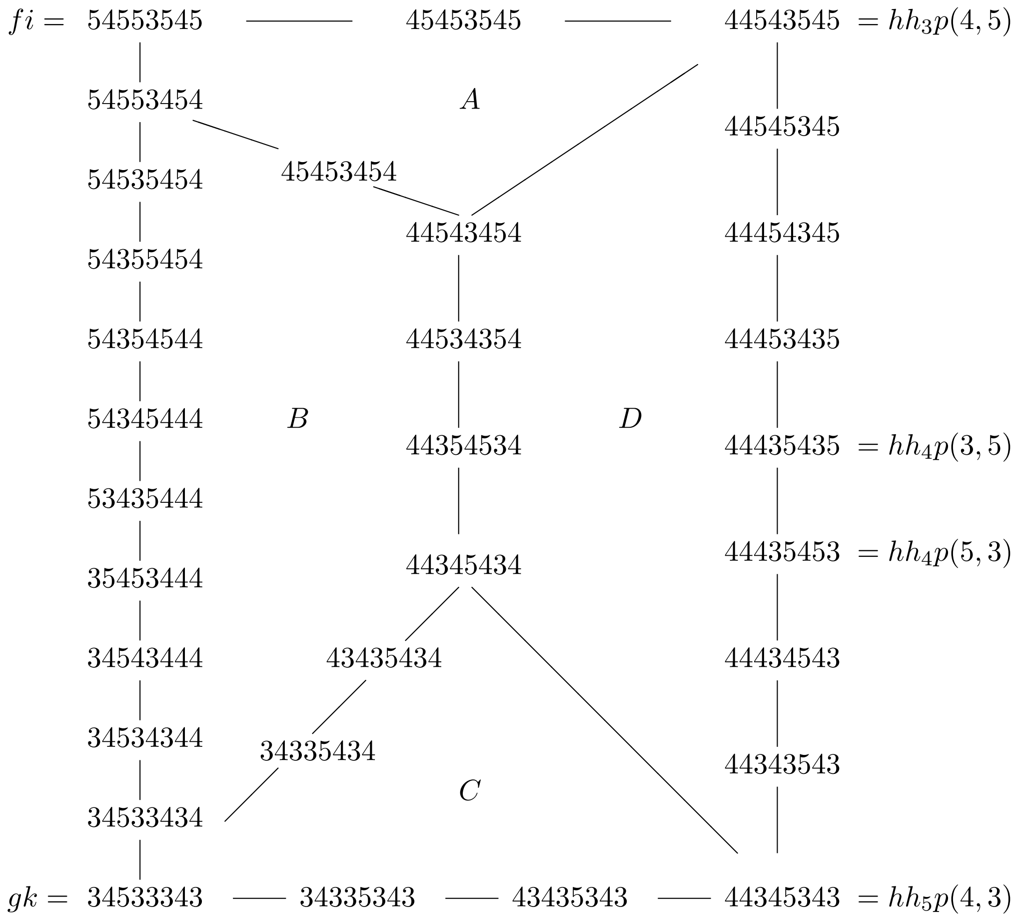
$$
\begin{array}{ccccccccc}
 & fi & \simeq & f'j & \ldots \quad \text{all end in } j \quad \ldots & g'j & \simeq & gk & \\
\begin{array}{l}\text{all}\\ \text{end}\\ \text{in } i\end{array} & \vdots & A & \vdots\ \begin{array}{l}\text{all}\\ \text{end}\\ \text{in } j\end{array} & B & \begin{array}{l}\text{all}\\ \text{end}\\ \text{in } j\end{array}\ \vdots & C & \vdots & \begin{array}{l}\text{all}\\ \text{end}\\ \text{in } k\end{array} \\
 & hh_kp(j,i) & \simeq & hh_kp(i,j) \ \ldots & \text{all end in } j & \ldots\ hh_ip(k,j) & \simeq & hh_ip(j,k) & \\
 & \vdots & & & D & & & \vdots & \\
 & \text{all end in } i & \ldots & hh_jp(k,i) & \simeq & hh_jp(i,k) & \ldots & \text{all end in } k &
\end{array}
$$

In the circuit A, $fi = \varphi p(j,i)$ and $f'j = \varphi p(i,j)$, and we can use the sequence from φ to hh_k to obtain a sequence from $f'j$ to $hh_kp(i,j)$. Hence the circuit A decomposes into inessential ones. The same is true for the circuit C. In the circuit B, all the vertices end in j, so we can use the induction hypothesis to conclude that B decomposes as required. If $i = k$ then D reduces to a point modulo inessential self-homotopies of the first type. If $|i - j| \geq 2$, $|j - k| \geq 2$, and $|k - i| \geq 2$, then D becomes a cube. If $\{i, j, k\} = \{a, a+1, b\}$ for some a, b with $b \leq a - 2$ or $b \geq a + 3$, then D becomes a prism. Finally if $\{i, j, k\} = \{a, a+1, a+2\}$ for some a, then D becomes a permutohedron. Besides, all the vertices in the altered sequence end in i or k, so we can repeat this procedure until we obtain a circuit whose all vertices end in i for some i. This completes the proof. □

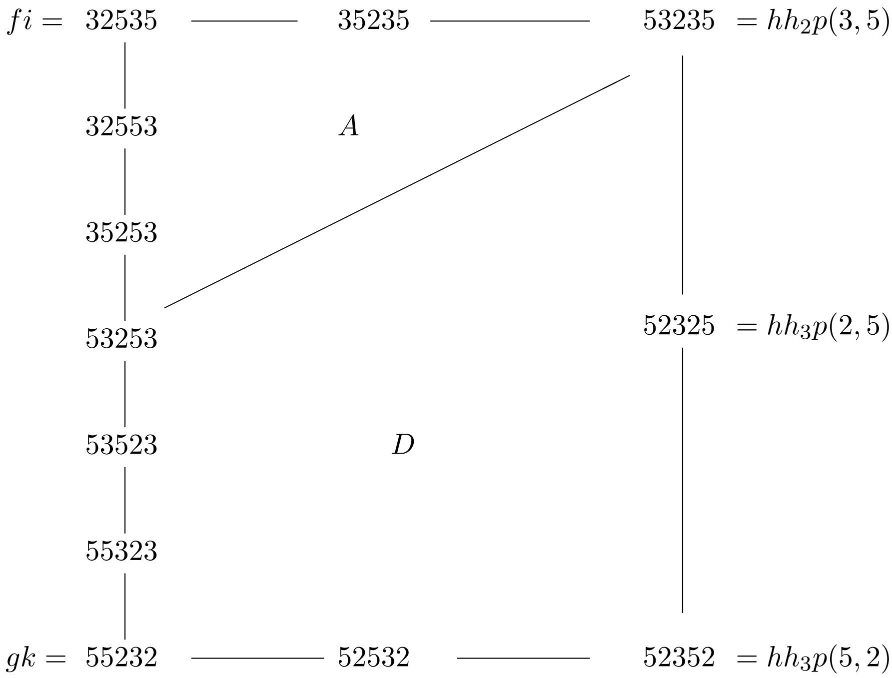
Of course, this result should be generalized to braid monoids corresponding to more general Coxeter groups. But our intention was to construct a step toward the coherence theorem [2], so we content ourselves with the case discussed above.

§4. Examples

In this section we illustrate by examples how the theorem holds. The following example shows a case where $i = 5$, $j = 4$, $k = 3$, $h = 44$ and $h' = 543545, 543454, 345343$, etc.



In this case, we obtain a permutohedron in D. The next is a case where $i = 5$, $j = 3$, $k = 2$, $h = 5$ and $h' = 3235, 3253, 3523$, etc.



In this case, we obtain a prism in D.

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