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Local energy decay for wave equations in exterior domains with regular or fast decaying dissipations

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Abstract. Decaying properties of the local energy for the dissipative wave equations with the Dirichlet boundary conditions in exterior domains are discussed. For the dissipation coefficient, natural conditions ensuring that waves trapped by obstacles may lose their energy are considered. Under this setting, the cases that the dissipation coefficient does not have a compact support, but has some regularities are treated. Further, the cases that the dissipation coefficients decay sufficiently fast at infinity are also discussed.

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§1. Introduction

Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be an exterior domain of a bounded obstacle $\mathcal{O} = \overline{\mathbb{R}^n \setminus \Omega}$ with a smooth and compact boundary $\partial\Omega$. In this paper, we consider the following mixed problem:

$$(1.1) \qquad \begin{cases} (\partial_t^2 - \triangle + a(x)\partial_t)u(t,x) = 0 & \text{in } (0,\infty)\times\Omega, \\ u(t,x) = 0 & \text{on } (0,\infty)\times\partial\Omega, \\ u(0,x) = f_1(x), \quad \partial_t u(0,x) = f_2(x) & \text{on } \Omega. \end{cases}$$

Throughout this paper, we always assume that $a(x) \geq 0$, which means that the term $a(x)\partial_t u(t,x)$ in the equation works as a dissipation. Since $\mathcal{O}$ is compact, we can choose a fixed constant $R_0 > 0$ satisfying $\mathcal{O} \subset B_{R_0}$, where $B_{R_0} = \{ x \in \mathbb{R}^n \,|\, |x| < R_0 \}$. Without loss of generality, we can also assume that the origin is contained in the interior of the obstacle $\mathcal{O}$.

The purpose of this paper is to give decay estimates of the local energy for the solution of (1.1). For the solution u of (1.1), the local energy of u in a domain $D \subset \mathbb{R}^n$ at time t is defined by

$$E(u, D, t) = \frac{1}{2} \int_{\Omega \cap D} \{ |\partial_t u(t, x)|^2 + |\nabla_x u(t, x)|^2 \} dx.$$

Note that the total energy $E(u, \Omega, t)$ satisfies

$$E(u, \Omega, t) + \int_0^t \int_\Omega a(x) |\partial_t u(s, x)|^2 dx ds = E(u, \Omega, 0) \qquad (t \geq 0). \tag{1.2}$$

This identity is given by multiplying $\overline{\partial_t u(t, x)}$ by the equation in (1.1), and using integration by parts. Since $a(x) \geq 0$, it follows that $E(u, \Omega, t) \leq E(u, \Omega, 0)$ $(t \geq 0)$. Thus, the term $a(x)\partial_t u$ in (1.1) may work as a dissipation.

In the case that $a(x) = 0$, if the obstacle is star shaped with respect to the origin, for any $R \geq R_0$, Morawetz [4] gave the following estimate:

$$\begin{aligned} &E(u, \Omega \cap B_R, t) \leq C_R (1 + t)^{-1} E(u, \Omega, 0) \\ &\quad (t \geq 0, (f_1, f_2) \in H_0^1(\Omega) \times L^2(\Omega), supp\, f_1 \cup supp\, f_2 \subset \overline{\Omega} \cap B_{R_0}). \end{aligned} \tag{1.3}$$

This is the starting point of studies on the decaying properties of the local energy. When $a(x) = 0$, local decay estimates like (1.3) suggest that all waves may go out from near the boundary $\partial\Omega$. Since there is no dissipation, waves themselves never decay. Hence the only factor escaping from near the boundary is just the reason why the local energy near the boundary $\partial\Omega$ decays. This means that for the local decay estimate (1.3), geometrical conditions for the obstacle $\mathcal{O}$ are needed. The condition that $\mathcal{O}$ is star shaped is a kind of such geometrical conditions.

If the dissipation term works, i.e. $a(x) > 0$ in some part, it causes the other decaying factor for the local energy. If waves pass on the region $\{x \in \Omega \,|\, a(x) > 0\}$, they may lose their energy. To describe this more precisely, we introduce the set Γ defined by

$$\Gamma = \{ x \in \partial\Omega \,|\, \nu(x) \cdot x > 0 \},$$

which is the invisible part of the boundary from the origin. Waves reflected on Γ may not go out to the outside of the neighborhood B_{R_0} of the obstacle. But, if dissipation terms work for such waves, the local energy may decay even though they do not escape to far field from the obstacles. Thus, we can expect to obtain the local decay estimate (1.3) if we make the following assumptions:

(A.1) $a \in L^\infty(\Omega)$, $a(x) \geq 0$ a.e. $x \in \Omega$.

(A.2) There exists a bounded open set $\omega \subset \mathbb{R}^n$ and a constant $\varepsilon_0 > 0$

such that $\overline{\Gamma} \subset \omega$, and $a(x) \geq \varepsilon_0$ a.e. $x \in \omega \cap \overline{\Omega}$.

Nakao [5] shows the decay estimate given by replacing $C(1+t)^{-1}$ in (1.3) with $C_\delta(1+t)^{-1+\delta}$ for any $\delta > 0$ if the dissipation coefficient $a(x)$ satisfies (A.1) and (A.2), and $supp\, a \subset \overline{\Omega} \cap B_R$ for some $R > 0$ as an additional assumption. Thus, from (A.2), energies for waves remaining near boundary should be damped by the dissipation term.

Although there are many authors contributing the decay estimates of the local energy, we do not introduce them in this paper. For this, see Ikehata [1] and [2] and the references therein.

In [4] and [5], as is in (1.3), compactness of the supports of the initial data is also assumed. This restriction for the initial data is removed in Ikehata [1]. To describe this result and the main theorem of this paper, we introduce the following notations:

$$I_n(f_1, f_2) = \int_\Omega (1+|x|)\{|\nabla_x f_1(x)|^2 + |f_2(x)|^2\}dx + \|d_n(\cdot)(f_2 + a(\cdot)f_1)\|^2_{L^{p_n}(\Omega)} + \|f_1\|^2_{L^2(\Omega)},$$

where p_n and $d_n(x)$ are defined by $p_n = 2n/(n+2)$ $(n \geq 3)$, $p_2 = 2$, $d_n(x) = 1$ $(n \geq 3)$ and $d(x) = |x|\log(B|x|)$ $(n = 2)$ with a constant $B > 0$ satisfying $B \inf_{x\in\Omega} |x| \geq 2$.

For any fixed $0 < \delta < 1$ and $R \geq R_0$, the following local decay estimate is given in Ikehata [1]:

$$E(u, \Omega \cap B_R, t) \leq C_{\delta,R}(1+t)^{-1+\delta} I_n(f_1, f_2) \quad (t \geq 0, (f_1, f_2) \in H^1_0(\Omega) \times L^2(\Omega), I_n(f_1, f_2) < \infty), \tag{1.4}$$

under the additional assumption that $supp\, a$ is compact in Ω. Thus, the restrictions on compactness assumption of the supports for the initial data is removed by Ikehata [1]. Note also that for the case of strong dissipation, i.e. $a(x) \geq \varepsilon_0$ $(|x| \geq R)$ for some fixed $\varepsilon_0 > 0$ and $R \geq R_0$, Nakao [6] shows the following total energy decay:

$$E(u, \Omega, t) \leq C(1+t)^{-1} E(u, \Omega, 0) \quad (t \geq 0, (f_1, f_2) \in H^1_0(\Omega) \times L^2(\Omega)).$$

In this paper, we give an improvement for the restriction on $0 < \delta$ in (1.4) and remove the compactness assumptions of the support for the dissipation coefficient $a(x)$. About this problem, in the Master thesis of Suzuki [7], the case that the dissipation coefficient $a(x)$ has some regularities is considered. According to [7], we introduce the following condition:

(A.3) $a \in W^{2,\infty}(\Omega)$, and there exists a constant $C > 0$ such that

$$a(x)|x| \le C,\ |x|^{-1}x\cdot\nabla_x a(x) \le Ca(x) \text{ and } |x|\triangle a(x) \le Ca(x) \quad \text{a.e. in } x \in \Omega.$$

Instead of (A.3), we can also consider the following one:

(A.4) $a \in W^{1,\infty}(\Omega)$, and there exists a constant $C > 0$ such that $|\nabla_x a(x)| \le Ca(x)|x|^{-1}$ and $a(x)|x| \le C$ a.e. $x \in \Omega$.

For example, for $\delta \ge 1$, $a(x) = (1+|x|)^{-\delta}$ satisfies both (A.3) and (A.4).

The main theorem in this paper is as follows:

Theorem 1. *Let $n \ge 2$ and assume that (A.1), (A.2) and (A.3), or (A.1), (A.2) and (A.4) hold. Then there exists a constant $C > 0$ such that*

$$E(u, \Omega\cap B_R, t) \le \frac{C}{t-R} I_n(f_1, f_2)$$

for any $t > R \ge R_0$ and $(f_1, f_2) \in H_0^1(\Omega) \times L^2(\Omega)$ with $I_n(f_1, f_2) < \infty$.

Remark: From Theorem 1 and (1.2), the local decay estimate (1.4) with $\delta = 0$ is also obtained.

Let us give remarks on assumptions (A.3) and (A.4). At a glance, assumption (A.4) seems to be better than (A.3) since (A.4) needs less regularities. But, we can not conclude so. Instead of this advantage, the dissipation coefficient $a(x)$ satisfying (A.4) should have stronger conditions on the behavior as $|x| \to \infty$. For example, consider $a(x) = e^{-|x|^\delta}$ ($\delta > 0$). For this function, (A.3) holds when $0 < \delta \le 1/2$, while (A.4) does not satisfy for all $\delta > 0$. If $\delta > 1/2$, both (A.3) and (A.4) do not hold. Thus, these assumptions may not handle the cases that the dissipation coefficient $a(x)$ decreases very rapidly or is compactly supported. Note also that strong dissipative cases like as constant dissipation case or the case treated in Nakao [6] do not also satisfy (A.3) and (A.4). Hence, Theorem 1 does not cover the results of Nakao [5], [6] and Ikehata [1]. But, Theorem 1 gives an answer for the excluded cases from the previous works.

Next we consider the following condition on the dissipative coefficient $a(x)$:

(A.5) There exist constants $\delta_0 \ge 0$ and $C > 0$ such that $a(x)|x|^{2+\delta_0} \le C$ a.e. in $x \in \Omega$.

For the dissipation coefficient satisfying (A.5), we have the following decay estimates:

Theorem 2. *Let $n \geq 2$ and assume that (A.1), (A.2) and (A.5) hold. Then there exists a constant $C > 0$ such that for $\delta_0 = 0$*

$$E(u, \Omega \cap B_{c_0(1+t)}, t) \leq \frac{Ce^{C(1-c_0)^{-1}}}{1-c_0} \frac{1+\log(1+t)}{1+t} I_n(f_1, f_2)$$

and for $\delta_0 > 0$

$$E(u, \Omega \cap B_{c_0(1+t)}, t) \leq \frac{Ce^{C(1-c_0)^{-1}}}{1-c_0} \frac{1+\delta_0^{-1} c_0^{-\delta_0}}{1+t} I_n(f_1, f_2)$$

for any $0 < c_0 < 1$, $t \geq 0$ and $(f_1, f_2) \in H_0^1(\Omega) \times L^2(\Omega)$ with $I_n(f_1, f_2) < \infty$.

Note that (A.5) with $\delta_0 > 0$ holds if the support of a is compact. Hence combining Theorem 2 with (1.2), we obtain the local decay estimate (1.4) with $\delta = 0$. Thus, Theorem 2 gives a generalization and an improvement of decay estimates of the local energy of Ikehata [1] and Nakao [5] for the case of compactly supported $a(x)$.

§2. Proof of Theorem 1

As is in Ikehata [1] and Nakao [5], we also use various integral identities for solutions of (1.1). Before describing them, let us note that it suffices to show every identity and estimate only for the solutions u of problem (1.1) with $f_1, f_2 \in C_0^\infty(\Omega)$. Every solution u of (1.1) in the space $\cap_{j=0}^1 C^{1-j}([0,\infty); H_0^j(\Omega))$ can be approximated by a sequence of solutions u_j $(j = 1, 2, \ldots)$ of (1.1) with $u_j(0,\cdot)$, $\partial_t u_j(0,\cdot) \in C_0^\infty(\Omega)$. Hence, in what follows, we always assume that f_1, $f_2 \in C_0^\infty(\Omega)$. From usual existence theorem of the solutions of wave equation, we can see that $u \in \cap_{j=0}^2 C^{2-j}([0,\infty); H^j(\Omega))$ and *supp* u is compact in $[0,T] \times \overline{\Omega}$ for any $T > 0$ if the initial data f_1 and f_2 belong to $C_0^\infty(\Omega)$.

We choose any $\eta \in C^1(\bar{\Omega})$ of real-valued. Multiplying $t\partial_t \overline{u}$, $\eta\overline{u}$, $x \cdot \partial_x \overline{u}$ by the equation in (1.1) respectively, and integrating by parts, we obtain the following identities:

$$tE(u, \Omega, t) + \int_0^t \int_\Omega sa(x) \left|\partial_t u\right|^2 dxds = \int_0^t E(u, \Omega, s) ds, \tag{2.1}$$

$$\begin{aligned} &\int_0^t \int_\Omega \eta \Big(|\nabla_x u|^2 - |\partial_t u|^2\Big) dxds + \mathrm{Re} \left[\int_\Omega \eta \overline{u} \partial_t u dx\right]_0^t \\ &\qquad = -\frac{1}{2}\left[\int_\Omega a(x)\eta|u|^2 dx\right]_0^t - \mathrm{Re} \int_0^t \int_\Omega (\nabla_x \eta \cdot \nabla_x u)\, \overline{u} dxds, \end{aligned} \tag{2.2}$$

$$\frac{n}{2} \int_0^t \int_\Omega \left(|\partial_t u|^2 - |\nabla_x u|^2\right) dxds + \int_0^t \int_\Omega |\nabla_x u|^2 dxds \tag{2.3}$$

$$+\text{Re}\left[\int_\Omega \partial_t u(x\cdot\nabla_x\overline{u})dx\right]_0^t$$
$$= -\text{Re}\int_0^t\int_\Omega a(x)\partial_t u(x\cdot\nabla_x\overline{u})dxds + \frac{1}{2}\int_0^t\int_{\partial\Omega} x\cdot\nu(x)\,|\partial_\nu u|^2\,dSds,$$

where $[f]_0^t = f(t) - f(0)$. Note that the identity (2.2) holds even for $\eta \in W^{1,\infty}_{loc}(\Omega)$. For any f_1 and $f_2 \in C_0^\infty(\Omega)$, there exists $R > 0$ such that $supp\, u \subset [0,t]\times(\overline{\Omega}\cap B_{R+t})$ for any $t \geq 0$ since the propagation speed for solution u of (1.1) is less than 1. We put $q_n = 2n/(n-1)$ $(n \geq 2)$. Noting $W^{1,\infty}_{loc}(\Omega) \subset W^{1,n}_{loc}(\Omega)$, we can choose a sequence $\{\eta_j\}$ in $C^1(\overline{\Omega})$ satisfying $\eta_j \to \eta$ in $W^{1,n}(\Omega\cap B_R)$ as $j\to\infty$. From $2 < q_n < 2n/(n-2)$, the Sobolev imbedding theorem implies that all u, $\partial_t u$ and $\nabla_x u$ belong to $C([0,\infty);H^1(\Omega)) \subset C([0,\infty);L^{q_n}(\Omega))$. Combining this fact with Hörder inequality, for $1/q_n + 1/q_n + 1/n = 1$, we can show that each integral in (2.2) for η_j converges to corresponding integrals for η. Thus, we obtain the above mentioned fact.

To show Theorem 1, we basically follow the argument given in Ikehata [1]. But, to handle dissipation coefficient $a(x)$ with non-compact support, even some of basic parts should be changed. To explain these parts, we give the argument from the beginning even though some of them are overlapped with the ones developed in Ikehata [1].

Adding (2.3) to the equality (2.2) with $\eta = \frac{n-1}{2}$, we have

$$\begin{aligned}\int_0^t E(u,\Omega,s)ds =& -\frac{n-1}{2}\text{Re}\left[\int_\Omega \overline{u}\partial_t u dx\right]_0^t - \frac{n-1}{4}\left[\int_\Omega a(x)|u|^2dx\right]_0^t\\ &- \text{Re}\left[\int_\Omega \partial_t u(x\cdot\nabla_x\overline{u})dx\right]_0^t - \text{Re}\int_0^t\int_\Omega a(x)\partial_t u(x\cdot\nabla_x\overline{u})dxds\\ &+\frac{1}{2}\int_0^t\int_{\partial\Omega} x\cdot\nu(x)\,|\partial_\nu u|^2\,dS_xds.\end{aligned} \tag{2.4}$$

As is in Ikehata [1] and Nakao [5], using (A.2), we can handle the boundary integral in (2.4), and obtain the following estimate:

$$\begin{aligned}\int_0^t\int_{\partial\Omega} x\cdot\nu(x)\left|\frac{\partial u}{\partial\nu}\right|^2 dS_xds \leq& CE(u,\Omega,0)\\ &+C\int_0^t\int_\omega a(x)|u|^2dxds + \frac{1}{2}\|u(t,\cdot)\|^2_{L^2(\Omega)} + C\,\|u(0,\cdot)\|^2_{L^2(\Omega)}\,.\end{aligned} \tag{2.5}$$

For the paper to be self-contained, we give a proof of (2.5) in Appendix.

Next, we need to control the L^2-norm $\|u(t,\cdot)\|^2_{L^2(\Omega)}$. The following lemma is just Lemma 2.6 in Ikehata [1] originated in Ikehata and Matsuyama [3].

Lemma 1. *There exists a constant $C > 0$ such that every solution $u(t,x)$ of (1.1) with the initial data $f_1, f_2 \in C_0^\infty(\Omega)$ satisfies*

$$\|u(t,\cdot)\|_{L^2(\Omega)}^2 + \int_0^t\int_\Omega a(x)\,|u(s,x)|^2\,dxds \le C\left(\|d_n(\cdot)(f_2 + a(\cdot)f_1)\|_{L^{pn}(\Omega)}^2 + \|f_1\|_{L^2(\Omega)}^2\right) \qquad (t \ge 0).$$

The identities (1.2) and (2.4), the estimate (2.5) and Lemma 1 imply that there exists a constant $C > 0$ depending only on the space dimension n, Ω and $a(x)$ satisfying

$$\int_0^t E(u,\Omega,s)ds + \frac{n-1}{4}\int_\Omega a(x)|u(t,x)|^2dx \le CI_n(f_1,f_2) + I(t;u) \quad (t \ge 0), \tag{2.6}$$

where

$$\begin{aligned} I(t;u) &= -\mathrm{Re}\,\Big[\int_\Omega \partial_t u(t,x)x\cdot\nabla_x u(t,x)dx + \int_0^t\int_\Omega a(x)\partial_t u(s,x)x\cdot\nabla_x u(s,x)dxds\Big] \\ &\le \int_\Omega |x|e(t,x;u)dx + \int_0^t\int_\Omega a(x)|x|e(s,x;u)dxds. \end{aligned}$$

In the above, we put $e(t,x;u) = 2^{-1}\{|\partial_t u(t,x)|^2 + |\nabla_x u(t,x)|^2\}$.

Next, we use the following estimates:

$$\int_\Omega |x|e(t,x;u)dx \le CI_n(f_1,f_2) + tE(u,\Omega,t) - (t-R)E(u,\Omega\cap B_R,t) \qquad \text{for any } t, R \ge 0, \tag{2.7}$$

$$\int_0^t\int_\Omega a(x)|x|\,|\partial_t u|^2\,dxds \le CI_n(f_1,f_2) + \int_0^t\int_\Omega a(x)s\,|\partial_t u|^2\,dxds \quad \text{for any } t \ge 0. \tag{2.8}$$

The estimates (2.7) and (2.8) are given in Ikehata [1] and Suzuki [7] respectively. To show Theorems 1 and 2, we need these estimates.

To obtain (2.7) and (2.8), we need the following estimates for $|x| \ge t$, i.e. the outside of the propagation cone:

$$\int_\Omega (|x|-t)_+ e(t,x;u)dx \le \int_\Omega (1+|x|)e(0,x;u)dx, \tag{2.9}$$

$$\int_0^t\int_\Omega a(x)(|x|-s)_+\,|\partial_t u(s,x)|^2\,dxds \le \int_\Omega (1+|x|)e(0,x;u)dx, \tag{2.10}$$

where $(|x|-t)_+ = \max\{|x|-t, 0\}$. As is in Ikehata [1], (2.9) is obtained by using the idea showing weighted estimates due to Todorova and Yordanov [8]. To show Theorem 1, we also need (2.10), which has been implicitly given in the proof of (2.9) in Ikehata [1]. For the paper to be self-contained, we give proofs of (2.9) and (2.10) in Appendix.

Since $|x| \le t + (|x|-t)_+$, we have

$$\int_\Omega |x| e(t,x;u)dx \le RE(u,\Omega\cap B_R,t) + t\int_{\Omega\setminus B_R} e(t,x;u)dx + \int_\Omega (|x|-t)_+ e(t,x;u)dx.$$

Combining this and (2.9) with $\int_{\Omega\setminus B_R} e(t,x;u)dx = E(u,\Omega,t) - E(u,\Omega\cap B_R,t)$, we obtain (2.7). The estimate (2.8) also follows from (2.10) and

$$\int_0^t\int_\Omega a(x)|x|\,|\partial_t u|^2\,dxds \le \int_0^t\int_\Omega a(x)s\,|\partial_t u|^2\,dxds + \int_0^t\int_\Omega a(x)(|x|-s)_+\,|\partial_t u|^2\,dxds.$$

Thus we get (2.7) and (2.8).

From (2.6), (2.7) and (2.8), it follows that

$$\int_0^t E(u,\Omega,s)ds \le CI_n(f_1,f_2) + tE(u,\Omega,t) - (t-R)E(u,\Omega\cap B_R,t) + \int_0^t\int_\Omega a(x)s\,|\partial_t u|^2\,dxds + \frac{1}{2}\int_0^t\int_\Omega a(x)|x|(|\nabla_x u|^2 - |\partial_t u|^2)dxds.$$

Combining this estimate with (2.1), we obtain

$$(t-R)E(u,\Omega\cap B_R,t) \le CI_n(f_1,f_2) + \frac{1}{2}\int_0^t\int_\Omega a(x)|x|(|\nabla_x u|^2 - |\partial_t u|^2)dxds.$$

Hence, to finish the proof of Theorem 1, it suffices to show the following estimate:

Lemma 2. *Assume that (A.1) and (A.3), or (A.1) and (A.4) hold. Then, there exists a constant $C>0$ such that*

$$\int_0^t\int_\Omega a(x)|x|(|\nabla_x u|^2 - |\partial_t u|^2)dxds \le CI_n(f_1,f_2)$$

for any $t \ge 0$, and f_1, $f_2 \in C_0^\infty(\Omega)$ with $I_n(f_1,f_2) < \infty$.

Proof. First we assume that (A.1) and (A.4). In this case, we use the identity (2.2) with $\eta(x) = a(x)|x|$. Since (2.2) is still valid even for $\eta \in W^{1,\infty}_{loc}(\Omega)$, we obtain

$$\int_0^t\int_\Omega a(x)|x|(|\nabla_x u|^2 - |\partial_t u|^2)dxds + \mathrm{Re}\ \Big[\int_\Omega a(x)|x|u\overline{\partial_t u}dx\Big]_0^t$$
$$= -\frac{1}{2}\Big[\int_\Omega (a(x))^2|x||u|^2dx\Big]_0^t$$
$$- \mathrm{Re}\ \int_0^t\int_\Omega \Big\{a(x)\Big(\frac{x}{|x|}\cdot\nabla_x u\Big)\overline{u} + |x|(\nabla_x a\cdot\nabla_x u)\overline{u}\Big\}dxds.$$

Hence it follows that

$$\int_0^t\int_\Omega a(x)|x|(|\nabla_x u|^2 - |\partial_t u|^2)dxds \le CI_n(f_1, f_2) \tag{2.11}$$
$$- \mathrm{Re}\ \int_0^t\int_\Omega \Big\{a(x)\Big(\frac{x}{|x|}\cdot\nabla_x u\Big)\overline{u} + |x|(\nabla_x a\cdot\nabla_x u)\overline{u}\Big\}dxds$$
$$- \Big\{\mathrm{Re}\ \int_\Omega a(x)|x|u(t,x)\overline{\partial_t u(t,x)}dx + \frac{1}{2}\int_\Omega (a(x))^2|x||u(t,x)|^2dx\Big\}$$

since definition of $I_n(f_1, f_2)$ and assumption (A.1) imply that

$$\Big|\int_\Omega a(x)|x|u(0,x)\overline{\partial_t u(0,x)}dx\Big| + \Big|\int_\Omega (a(x))^2|x||u(0,x)|^2dx\Big| \le MI_n(f_1, f_2) \tag{2.12}$$

with $M = \|a\|_{L^\infty(\Omega)} + \|a\|^2_{L^\infty(\Omega)}$. Using $a(x)|x| \le C$ in assumption (A.4), and noting (1.2) and Lemma 1, we have

$$\Big|\int_\Omega a(x)|x|u(t,x)\overline{\partial_t u(t,x)}dx\Big| \le C\int_\Omega \big\{|u(t,x)|^2 + |\partial_t u(t,x)|^2\big\}dx \le CI_n(f_1, f_2). \tag{2.13}$$

For the second term in (2.11), we use $|\nabla_x a(x)| \le Ca(x)|x|^{-1}$ in assumption (A.4), and get

$$\Big|\int_0^t\int_\Omega \Big\{a(x)\Big(\frac{x}{|x|}\cdot\nabla_x u\Big)\overline{u} + |x|(\nabla_x a\cdot\nabla_x u)\overline{u}\Big\}dxds\Big|$$
$$\le C\Big\{\int_0^t\int_\Omega a(x)|u(s,x)|^2dxds + \int_0^t\int_\Omega a(x)|\nabla_x u(s,x)|^2dxds\Big\}.$$

Combining the above estimate, (2.11) and (2.13) with Lemma 1, we obtain

$$\int_0^t\int_\Omega a(x)|x|(|\nabla_x u|^2 - |\partial_t u|^2)dxds \le CI_n(f_1, f_2)$$

$$+\int_0^t \int_\Omega a(x)|\nabla_x u(s,x)|^2 dxds.$$

Hence, to obtain the estimate stated in Lemma 2, it suffices to show

$$\int_0^t \int_\Omega a(x)|\nabla_x u(s,x)|^2 dxds \leq C I_n(f_1, f_2) \tag{2.14}$$

when (A.1) and (A.4) are assumed.

For this purpose, we use the identity (2.2) with $\eta(x) = a(x)$, and obtain

$$\begin{aligned}\int_0^t\int_\Omega a(x)(|\nabla_x u|^2 - |\partial_t u|^2)dxds &= -\mathrm{Re}\ \left[\int_\Omega a(x)u\overline{\partial_t u}dx\right]_0^t \\ &\quad -\frac{1}{2}\left[\int_\Omega (a(x))^2|u|^2dx\right]_0^t - \mathrm{Re}\ \int_0^t\int_\Omega(\nabla_x a\cdot\nabla_x u)\overline{u}dxds \\ \leq \left|\left[\int_\Omega a(x)u\overline{\partial_t u}dx\right]_0^t\right| &+ \frac{1}{2}\int_\Omega (a(x))^2|u(0,x)|^2dx + \int_0^t\int_\Omega |\nabla_x a||\nabla_x u||u|dxds.\end{aligned}$$

From (1.2), Lemma 1 and definition of $I_n(f_1, f_2)$, it follows that

$$\left|\left[\int_\Omega a(x)u\overline{\partial_t u}dx\right]_0^t\right| + \frac{1}{2}\int_\Omega (a(x))^2|u(0,x)|^2dx \leq C I_n(f_1, f_2).$$

Combining these estimates, we have

$$\begin{aligned}\int_0^t\int_\Omega a(x)\left|\nabla_x u\right|^2 dxds \leq \int_0^t\int_\Omega a(x)\left|\partial_t u\right|^2 dxds + C I_n(f_1, f_2) \\ + \int_0^t\int_\Omega |\nabla_x a||\nabla_x u||u|dxds.\end{aligned} \tag{2.15}$$

Since $d_0 = \inf_{x\in\Omega}|x| > 0$, assumption (A.4) implies that $|\nabla_x a(x)| \leq Ca(x)|x|^{-1} \leq Cd_0^{-1}a(x)$ a.e. $x \in \Omega$, which yields

$$\begin{aligned}\int_0^t\int_\Omega |\nabla_x a||\nabla_x u||u|dxds &\leq Cd_0^{-1}\int_0^t\int_\Omega a(x)|\nabla_x u||u|dxds \\ &\leq \frac{1}{2}\int_0^t\int_\Omega a(x)|\nabla_x u|^2dxds + \frac{(Cd_0^{-1})^2}{2}\int_0^t\int_\Omega a(x)|u|^2dxds.\end{aligned}$$

Combining the above estimate and (2.15) with Lemma 1 again, we obtain

$$\int_0^t\int_\Omega a(x)\left|\nabla_x u\right|^2 dxds \leq 2\int_0^t\int_\Omega a(x)\left|\partial_t u\right|^2 dxds + C I_n(f_1, f_2).$$

Hence, we get (2.14) assuming (A.1) and (A.4) if we note

$$\int_0^t \int_\Omega a(x)\,|\partial_t u|^2\,dxds \le E(u,\Omega,0),$$

which is given by (1.2). Thus, we obtain the estimate in Lemma 2 assuming that (A.1) and (A.4) hold.

Next, we consider the case that (A.1) and (A.3) are assumed. Since $a \in W^{2,\infty}(\Omega)$, in (2.2) with $\eta(x) = a(x)|x|$, we obtain by integration by parts

$$\begin{aligned}\int_0^t \int_\Omega & a(x)|x|(|\nabla_x u|^2 - |\partial_t u|^2)dxds \\ &= -\operatorname{Re}\Big[\int_\Omega a(x)|x|u\overline{\partial_t u}dx\Big]_0^t - \frac{1}{2}\Big[\int_\Omega (a(x))^2|x||u|^2dx\Big]_0^t \\ &\quad + \operatorname{Re}\frac{1}{2}\int_0^t\int_\Omega \Big\{\operatorname{div}\big(a(x)\frac{x}{|x|}\big) + \operatorname{div}\big(|x|\nabla_x a\big)\Big\}|u|^2dxds.\end{aligned}$$

From (A.3), it follows that $a(x)|x| \le C$, $|x|^{-1}x\cdot\nabla_x a(x) \le Ca(x)$ and $|x|\triangle a(x) \le Ca(x)$ a.e. in $x \in \Omega$ for some fixed constant $C > 0$. Hence, we have

$$\begin{aligned}\Big\{\operatorname{div}\big(a(x)&\frac{x}{|x|}\big) + \operatorname{div}\big(|x|\nabla_x a\big)\Big\}|u|^2 \\ &\le \Big(|x|\triangle a(x) + 2\frac{x}{|x|}\cdot\nabla_x a(x) + a(x)\frac{n-1}{|x|}\Big)|u|^2 \\ &\le C(3+(n-1)d_0^{-1})a(x)|u|^2 \qquad \text{a.e. in } x \in \overline{\Omega}.\end{aligned}$$

From this estimate and the identity above, (2.12) and (2.13), it follows that

$$\begin{aligned}\int_0^t\int_\Omega a(x)|x|(|\nabla_x u|^2 - |\partial_t u|^2)dxds &\le CI_n(f_1,f_2) \\ &\quad + C'\int_0^t\int_\Omega a(x)|u(t,x)|^2dxds\end{aligned}$$

for some other constant $C' > 0$. Hence noting Lemma 1 again, we also obtain the estimate in Lemma 2 assuming that (A.1) and (A.3) hold. This completes the proof of Lemma 2 and also Theorem 1. ■

§3. Proof of Theorem 2

Here we show Theorem 2. Note that the basic idea for the proof of Theorem 2 just comes from the case of compactly supported $a(x)$ in Ikehata [1]. Since $a(x)$ satisfies (A.1) and (A.2), we can use (2.6). Noting that the estimate

(2.7) is uniform with respect to $R \geq 0$, from these estimates, we obtain a fixed constant $C > 0$ such that

$$\text{(3.1)} \quad \int_0^t E(u,\Omega,s)ds \leq CI_n(f_1,f_2) + tE(u,\Omega,t) - (t-R)E(u,\Omega\cap B_R,t) + \int_0^t\int_\Omega a(x)|\partial_t u(s,x)x\cdot\nabla_x u(s,x)|dxds \qquad (t,R\geq 0).$$

We take $0 < c_0 < 1$ arbitrary, and put $R_{c_0}(t) = c_0(1+t)$. For the integral in the right hand side of (3.1), it follows that

$$\int_0^t\int_\Omega a(x)|\partial_t u(s,x)x\cdot\nabla_x u(s,x)|dxds \leq \frac{1}{2}\int_0^t\int_\Omega \frac{a(x)|x|^2}{1+s}|\nabla_x u(s,x)|^2dxds + \frac{1}{2}\int_0^t\int_\Omega (1+s)a(x)|\partial_t u(s,x)|^2dxds.$$

Since assumption (A.5) yields

$$\frac{a(x)|x|^2}{1+s} = \frac{a(x)|x|^{2+\delta_0}}{(1+s)|x|^{\delta_0}} \leq \frac{C}{c_0^{\delta_0}(1+s)^{1+\delta_0}} \qquad (x\in\Omega, |x|\geq R_{c_0}(s)),$$

$$\frac{a(x)|x|^2}{1+s} = \frac{a(x)|x|^{2+\delta_0}}{(1+s)|x|^{\delta_0}} \leq \frac{Cd_0^{-\delta_0}}{1+s} \qquad (x\in\Omega, |x|\leq R_{c_0}(s)),$$

where $d_0 = \inf_{x\in\Omega}|x| > 0$, it follows that

$$\begin{aligned}\int_0^t\int_\Omega \frac{a(x)|x|^2}{1+s}|\nabla_x u(s,x)|^2dxds &\leq \frac{C}{d_0^{\delta_0}}\int_0^t\int_{\Omega\cap B_{R_{c_0}(s)}}\frac{|\nabla_x u(s,x)|^2}{1+s}dxds \\ &\quad + \frac{C}{c_0^{\delta_0}}\int_0^t\int_{\Omega\setminus B_{R_{c_0}(s)}}\frac{|\nabla_x u(s,x)|^2}{(1+s)^{1+\delta_0}}dxds \\ &\leq \frac{C}{d_0^{\delta_0}}\int_0^t\frac{E(u,\Omega\cap B_{R_{c_0}(s)},s)}{1+s}ds + \frac{C}{c_0^{\delta_0}}\int_0^t\frac{E(u,\Omega,s)}{(1+s)^{1+\delta_0}}ds.\end{aligned}$$

Combining these estimates with (1.2), we obtain

$$\begin{aligned}\int_0^t\int_\Omega a(x)|\partial_t u(s,x)x\cdot\nabla_x u(s,x)|dxds &\leq C'\int_0^t\frac{E(u,\Omega\cap B_{R_{c_0}(s)},s)}{1+s}ds \\ &\quad + C'g(t)E(u,\Omega,0) + \frac{1}{2}\int_0^t\int_\Omega sa(x)|\partial_t u(s,x)|^2dxds,\end{aligned}$$

where $g(t) = 1 + c_0^{-\delta_0}\int_0^t(1+s)^{-(1+\delta_0)}ds$. Note that in the above, $C' > 0$ is a constant independent of $0 < c_0 < 1$ and $t \geq 0$. From the above estimate and (3.1), it follows that

$$\int_0^t E(u,\Omega,s)ds \leq Cg(t)I_n(f_1,f_2) + tE(u,\Omega,t) - (t-R)E(u,\Omega\cap B_R,t)$$

$$+C\int_0^t \frac{E(u,\Omega\cap B_{R_{c_0}(s)},s)}{1+s}ds+\frac{1}{2}\int_0^t\int_\Omega sa(x)|\partial_t u(s,x)|^2dxds,$$

the identity (2.1) implies that there exists a constant $C>0$ such that

$$(3.2)\qquad (t-R)E(u,\Omega\cap B_R,t)+\frac{1}{2}\int_0^t\int_\Omega sa(x)|\partial_t u(s,x)|^2dxds$$
$$\leq Cg(t)I_n(f_1,f_2)+C\int_0^t \frac{E(u,\Omega\cap B_{R_{c_0}(s)},s)}{1+s}ds \quad (t,R\geq 0, 0<c_0<1).$$

Now we put $R=R_{c_0}(t)$ in (3.2). Noting that the identity (1.2) yields $E(u,\Omega\cap B_{R_{c_0}(t)},t)\leq E(u,\Omega,0)$ $(t\geq 0)$, we obtain

$$(3.3)\qquad (1-c_0)(1+t)E(u,\Omega\cap B_{R_{c_0}(t)},t)+\frac{1}{2}\int_0^t\int_\Omega sa(x)|\partial_t u(s,x)|^2dxds$$
$$\leq Cg(t)I_n(f_1,f_2)+C\int_0^t \frac{E(u,\Omega\cap B_{R_{c_0}(s)},s)}{1+s}ds \qquad (t\geq 0).$$

We set $\phi(t)=(1+t)^{-1}E(u,\Omega\cap B_{R_{c_0}(t)},t)$. Then (3.3) implies that

$$(3.4)\qquad \phi(t)\leq \frac{C_1g(t)}{(1+t)^2}I_n(f_1,f_2)+\frac{C_1}{(1+t)^2}\int_0^t\phi(s)ds,$$

where $C_1=C/(1-c_0)$. Since $\frac{d}{dt}\big(e^{C_1(1+t)^{-1}}\big)=-C_1(1+t)^{-2}e^{C_1(1+t)^{-1}}$, it follows that

$$\frac{d}{dt}\Big(e^{C_1(1+t)^{-1}}\int_0^t\phi(s)ds\Big)\leq -I_n(f_1,f_2)g(t)\frac{d}{dt}\big(e^{C_1(1+t)^{-1}}\big),$$

which yields

$$\begin{aligned}\int_0^t\phi(s)ds &\leq I_n(f_1,f_2)e^{-C_1(1+t)^{-1}}\Big\{-\big[g(s)e^{C_1(1+s)^{-1}}\big]_0^t \\ &\qquad +\int_0^t e^{C_1(1+s)^{-1}}c_0^{-\delta_0}(1+s)^{-(1+\delta_0)}ds\Big\} \\ &\leq e^{C_1}g(t)I_n(f_1,f_2) \qquad (t\geq 0).\end{aligned}$$

Combining the above estimate with (3.4), we obtain

$$E(u,\Omega\cap B_{R_{c_0}(t)},t)\leq 2C_1e^{C_1}(1+t)^{-1}\Big\{1+\frac{1}{c_0^{\delta_0}}\int_0^t\frac{ds}{(1+s)^{1+\delta_0}}\Big\}I_n(f_1,f_2).$$

This completes the proof of Theorem 2. ■

Remark: From (3.3), it also follows that

$$\int_0^t \int_\Omega sa(x)|\partial_t u(s,x)|^2 dxds \le Ce^{C(1-c_0)^{-1}}(1+\delta_0^{-1}c_0^{-\delta_0})I_n(f_1,f_2)$$

for $\delta_0 > 0$ and $t \ge 0$, and

$$\int_0^t \int_\Omega sa(x)|\partial_t u(s,x)|^2 dxds \le Ce^{C(1-c_0)^{-1}}(1+\log(1+t))I_n(f_1,f_2)$$

for $\delta_0 = 0$ and $t \ge 0$.

§4. Appendix

4.1. Estimate of the boundary integral

Here we show the estimate (2.5), which is given in Ikehata [1]. As is in Ikehata [1] and Nakao [5], to handle the boundary integral in (2.4), for the bounded open set $\omega \subset \mathbb{R}^n$ in (A.2), we choose a real vector valued function $h(x) = (h_1(x), \dots, h_n(x)) \in C^1(\overline{\Omega})$ satisfying $h(x) \cdot \nu(x) \ge 0$ $(x \in \partial\Omega)$, $h(x) = \nu(x)$ $(x \in \Gamma)$ and $supp\, h \subset \omega$. For this function $h \in C_0^1(\overline{\Omega})$, multiplying $h(x) \cdot \nabla_x u$ by the equation in (1.1), we get the following identity:

$$\begin{aligned}\frac{1}{2}\int_0^t\int_\Omega \nabla_x{\cdot}h(x)\left(|\partial_t u(s,x)|^2 - |\nabla u(s,x)|^2\right)dxds \\ + \mathrm{Re}\int_0^t\int_\Omega \sum_{i,j=1}^n \partial_{x_i}u\partial_{x_j}\overline{u}\partial_{x_i}h_j(x)dxds \\ = \frac{1}{2}\int_0^t\int_{\partial\Omega}|\partial_\nu u|^2\,\nu(x){\cdot}h(x)dSds - \mathrm{Re}\int_0^t\int_\Omega a(x)\partial_t u(h(x)\cdot\nabla_x u)dxds \\ - \mathrm{Re}\left[\int_\Omega \partial_t u(h\cdot\nabla_x\overline{u})dx\right]_0^t.\end{aligned}$$

Since $x \cdot \nu(x) \le 0$ $(x \in \partial\Omega \setminus \Gamma)$, $|x| < R_0$ $(x \in \partial\Omega)$, the above identity, (1.2) and assumptions (A.1) and (A.2) imply

$$\begin{aligned}\int_0^t\int_{\partial\Omega} x\cdot\nu(x)\,|\partial_\nu u|^2\,dS_x ds &\le R_0\int_0^t\int_{\partial\Omega} h(x)\cdot\nu(x)\,|\partial_\nu u|^2\,dS_x ds \\ &\le C\Big\{\int_0^t\int_{\tilde\omega} e(s,x;u)dxds + E(u,\Omega,0)\Big\}, \end{aligned} \tag{4.1}$$

where $\tilde\omega \subset \mathbb{R}^n$ is an open set satisfying $supp\, h \subset \tilde\omega$ and $\overline{\tilde\omega} \subset \omega$. Note that the constant $C > 0$ in (4.1) depends only on $\partial\Omega$, ω and $a(x)$ since $R_0 > 0$ and $\sup_{x\in\omega}(|h(x)| + |\nabla_x h(x)|) < \infty$ is chosen corresponding to them.

Next along with the idea in [1] and [5], we take $\chi \in C_0^\infty(\omega)$ with $\chi(x) = 1$ near $\overline{\tilde{\omega}}$, $0 \le \chi \le 1$, and use (2.2) with $\eta = \chi^2$. Since the Schwarz inequality implies

$$\Big|\int_0^t\int_\Omega (\nabla_x\eta\cdot\nabla_x u)\,\overline{u}dxds\Big| \le \int_0^t\int_\Omega 2|\chi||\nabla_x\chi||\nabla_x u||u|dxds$$
$$\le \int_0^t\int_\Omega \Big\{\frac{\chi^2|\nabla_x u|}{2} + 2|\nabla_x\chi|^2|u|^2\Big\}dxds,$$

from (1.2), we obtain

$$\int_0^t\int_\Omega \eta|\nabla_x u|^2dxds \le \int_0^t\int_\Omega \big\{\eta|\partial_t u|^2 + 2|\nabla_x\chi|^2|u|^2\big\}dxds$$
$$+\frac{1}{2}\int_0^t\int_\Omega \eta|\nabla_x u|^2dxds + 2E(u,\Omega,0) + C\big\{\|u(t,\cdot)\|^2_{L^2(\Omega)} + \|u(0,\cdot)\|^2_{L^2(\Omega)}\big\}.$$

Combining the above estimate, (4.1) with the fact that $\eta = 1$ near $\overline{\tilde{\omega}}$, we obtain

$$\int_0^t\int_{\partial\Omega} x\cdot\nu(x)\,|\partial_\nu u|^2\,dS_xds \le C\int_0^t\int_\Omega \big\{\eta|\partial_t u|^2 + |\nabla_x\chi|^2|u|^2\big\}dxds$$
$$+ C\big\{E(u,\Omega,0) + \|u(t,\cdot)\|^2_{L^2(\Omega)} + \|u(0,\cdot)\|^2_{L^2(\Omega)}\big\}.$$

Since assumption (A.2) implies that $\varepsilon_0^{-1}a(x) \ge 1$ a.e. in $\omega\cap\overline{\Omega}$, it follows that $0 \le \eta(x) \le \varepsilon_0^{-1}a(x)$ and $|\nabla_x\chi(x)|^2 \le \max_{x\in\omega}|\nabla_x\chi(x)|^2\varepsilon_0^{-1}a(x)$ a.e. in $x\in\overline{\Omega}$. Hence we get

$$\int_0^t\int_\Omega \big\{\eta|\partial_t u|^2 + |\nabla_x\chi|^2|u|^2\big\}dxds \le C\int_0^t\int_\omega \big\{a(x)|\partial_t u|^2 + a(x)|u|^2\big\}dxds$$
$$\le CE(u,\Omega,0) + C\int_0^t\int_\omega a(x)|u|^2dxds,$$

where we used (1.2) and $supp\,\eta \cup supp\,\nabla_x\chi \subset \omega$. From these estimates, we obtain (2.5).

4.2. Proof of the estimates (2.9) and (2.10)

As is in Ikehata [1], we introduce a weight function $\psi(t,x) \in C^1([0,\infty)\times\overline{\Omega})$ defined by

$$\psi(t,x) = \begin{cases} 1+|x|-t & (|x| \ge t\ ,\ x\in\overline{\Omega}),\\ (1-|x|+t)^{-1} & (|x| \le t\ ,\ x\in\overline{\Omega}).\end{cases}$$

For the solution u of (1.1) with $f_1, f_2 \in C_0^\infty(\Omega)$, it follows that

$$
\begin{aligned}
0 &= \mathrm{Re}\ \left[\psi\overline{\partial_t u}(\partial_t^2 u - \triangle u + a(x)\partial_t u)\right] \\
&= \frac{d}{dt}\Big\{\psi e(t,x;u)\Big\} - \mathrm{div}(\mathrm{Re}\ (\psi\overline{\partial_t u}\nabla_x u)) + a(x)\psi|\partial_t u|^2 \\
&\qquad - \frac{1}{2\partial_t\psi}\Big|\partial_t\psi\nabla_x u - \partial_t u\nabla_x\psi\Big|^2 + \frac{1}{\partial_t\psi}\Big(|\nabla_x\psi|^2 - |\partial_t\psi|^2\Big)|\partial_t u|^2,
\end{aligned}
$$

where $e(t,x;u) = 2^{-1}\big\{|\partial_t u(t,x)|^2 + |\nabla_x u(t,x)|^2\big\}$. Since the weight function ψ satisfies $|\nabla_x\psi(s,x)|^2 - |\partial_t\psi(s,x)|^2 = 0$ and $\partial_t\psi(s,x) < 0$ in $[0,t]\times\Omega$ for any fixed $t \geq 0$, integrating the above equality in $[0,t]\times\Omega$, we get

$$
\int_\Omega \psi(t,x)e(t,x;u)dx + \int_0^t\int_\Omega \psi(s,x)a(x)|\partial_t u(s,x)|^2 dxds \leq \int_\Omega \psi(0,x)e(0,x;u)dx.
$$

Thus, noting that $\psi(0,x) = 1+|x|$, $\psi(t,x) \geq (|x|-t)_+ \geq 0$, from the above estimate, we obtain (2.9) and (2.10).

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