

Holomorphic families of Schrödinger operators in L^p

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Abstract. Given two m -sectorial operators in a reflexive Banach space X , a sufficient condition is presented for the family $\{T + \kappa A; \kappa \in \Sigma^c\}$ to be holomorphic of type (A), where Σ is a closed convex region in the κ -plane such that $\partial\Sigma$ is the left branch of a hyperbola. The abstract formulation is almost identical with Kato's when X is a Hilbert space. However, $\partial\Sigma$ is typically reduced to a parabola when X is a Hilbert space. The m -sectoriality is a generalized notion of the nonnegative selfadjointness so that Schrödinger operators with singular potentials in the L^p -spaces ($1 < p < \infty$) are typical examples of the abstract theory. In this connection a detailed analysis is given in the case of $-\Delta + \kappa|x|^{-2}$. This application is an L^p -generalization of Kato's.

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§1. Introduction

This paper is concerned with holomorphic families of the Schrödinger operator $-\Delta + \kappa V(x)$ in $L^p = L^p(\mathbb{R}^N)$, $1 < p < \infty$, $N \in \mathbb{N}$. Here $V(x)$ is a nonnegative potential with singularity at the origin or at infinity, and κ is a complex parameter running outside a closed convex region Σ in the κ -plane. Namely, we ask when $\{-\Delta + \kappa V(x); \kappa \in \Sigma^c\}$, $\Sigma^c := \mathbb{C} \setminus \Sigma$, forms a family of closed linear operators with constant domain in L^p . A solution of this problem with $p = 2$ had already been given by Kato [9] using the abstract theory in the Hilbert space setting. For example, put

$$\Sigma_2 := \{\xi + i\eta \in \mathbb{C}; \eta^2 \leq 4(\beta_0(2) - \xi)\} = \{\xi + i\eta \in \mathbb{C}; \xi \leq \beta_0(2) - \eta^2/4\},$$

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where

$$(1.1) \quad \beta_0(2) := 1 - \frac{(N-2)^2}{4} = -\frac{(N-4)N}{4} \quad \forall N \in \mathbb{N}.$$

Then Kato proved that, in his terms, $\{-\Delta + \kappa|x|^{-2}; \kappa \in \Sigma_2^c\}$ forms a holomorphic family of type (A) in L^2 . His result means that this family consists of closed operators with constant domain in L^2 . Later, Borisov-Okazawa [3] proved that $\{d/dx + \kappa x^{-1}; \kappa \in \Sigma_1^c\}$ forms a holomorphic family of type (A) in $L^p(0, \infty)$ ($1 < p < \infty$), where

$$\Sigma_1 := \left\{ \kappa \in \mathbb{C}; \operatorname{Re} \kappa \leq -\frac{1}{p'} \right\} \quad \left(\frac{1}{p} + \frac{1}{p'} = 1 \right).$$

Recently, Tamura [21] proved that $\{\Delta^2 + \kappa|x|^{-4}; \kappa \in \Sigma_4^c\}$ forms a holomorphic family of type (A) in L^2 , where

$$\Sigma_4 := \{\xi + i\eta \in \mathbb{C}; \eta^2 \leq f(\xi), \xi \leq 112 - 3(N-2)^2\};$$

however, f is too complicated to be stated here.

In this connection we note that the practical use of the holomorphic families of type (A) and other types is exemplified in Kato [8].

Let T and A be two linear m -accretive operators in a Hilbert space H ; the domain and range of T are denoted by $D(T)$ and $R(T)$, respectively. Then Kato gave two abstract theorems [9, Theorems 2.1 and 2.2] on holomorphic families of the form $\{T + \kappa A; \kappa \in \Sigma^c\}$. For a general definition of holomorphic families of type (A) see Definition 1 in Section 2. It seems from the viewpoint of operator semigroups that [9, Theorem 2.1] deals with generators of “general” contraction semigroups on H , while [9, Theorem 2.2] deals with generators of “analytic” contraction semigroups. In [9, Theorem 2.1] he assumes that A^{-1} exists (but not necessarily bounded) and there is a constant $a \in \mathbb{R}$ such that

$$(1.2) \quad \limsup_{\substack{\varepsilon \rightarrow 0 \\ (|\varepsilon|^{-1} \operatorname{Re} \varepsilon \geq \delta > 0)}} \operatorname{Re}((A + \varepsilon)^{-1}v, T^*v) \geq -a\|v\|^2 \quad \forall v \in D(T^*),$$

where T^* is the adjoint of T (and δ may depend on v). Under these conditions he proves among others that $\{T + \kappa A; \operatorname{Re} \kappa > a\}$ forms a holomorphic family of type (A), that is, a typical case of Σ is the left half-plane:

$$(1.3) \quad \Sigma_{acc} = \{\kappa \in \mathbb{C}; \operatorname{Re} \kappa \leq a\}.$$

In this case the family maintains the m -accretivity for only real $\kappa > a_+ := \max\{a, 0\}$. Kato remarks that if A^{-1} is bounded, then (1.2) is equivalent to

$$(1.4) \quad \operatorname{Re}(A^{-1}v, T^*v) \geq -a\|v\|^2 \quad \forall v \in D(T^*)$$

which is identical with the condition introduced by Sohr [19] and [20]. For an interesting characterization of condition (1.4) with $a = 0$ is presented by Miyajima [10]. In his proof the notion of Yosida approximation is employed.

Now let $\{A_\varepsilon; \varepsilon > 0\}$ be the family of Yosida approximation of A :

$$A_\varepsilon := A(1 + \varepsilon A)^{-1} = \varepsilon^{-1}[1 - (1 + \varepsilon A)^{-1}], \quad \varepsilon > 0.$$

Then the second author of the present paper introduced the following condition for $T + A$ (or its closure) to be m -accretive in H : there are constants $a \leq 1$ and $b, c \geq 0$ such that

$$(1.5) \quad \operatorname{Re}(Tu, A_\varepsilon u) \geq -a\|A_\varepsilon u\|^2 - b\|A_\varepsilon u\| \cdot \|u\| - c\|u\|^2 \quad \forall u \in D(T)$$

(see [12, Theorem 4.2 and Corollary 5.5]). As stated in [3, Introduction], we can show that (1.5) is also a generalization of (1.4). It should be noted further that A need not be invertible in condition (1.5). Inequalities of the form (1.5) makes sense even in a (reflexive) Banach space X if we replace the inner product $(Tu, A_\varepsilon u)$ with the semi-inner product $(Tu, F(A_\varepsilon u))$:

$$(1.6) \quad \operatorname{Re}(Tu, F(A_\varepsilon u)) \geq -a_1\|A_\varepsilon u\|^2 - b_1\|A_\varepsilon u\| \cdot \|u\| - c_1\|u\|^2$$

for all $u \in D(T)$, where F is the duality map on X to its adjoint X^* , that is, (f, g) denotes the pairing between $f \in X$ and $g \in X^*$. By virtue of (1.6) we could generalize [9, Theorem 2.1] from the Hilbert space case to the (reflexive) Banach space case (see [3, Theorems 2.2 and 2.4]). As an application we considered the first order singular differential operator $d/dx + \kappa x^{-1}$ in $L^p(0, \infty)$, $1 < p < \infty$. However, a detailed analysis of the second order singular differential operator such as $d^2/dx^2 + \kappa x^{-2}$ in $L^p(0, \infty)$ has been left open over ten years.

In order to generalize [9, Theorem 2.2] let T and A be two linear m -sectorial operators in a (reflexive) Banach space X (in applications in [9, Section 7] Kato assumed in addition that T and A are also selfadjoint in L^2). As introduced by Goldstein [4, Definition 1.5.8] (see also Ouhabaz [17, p. 97]), a linear operator T in X is said to be *sectorial of type* $S(\tan \theta)$ if $(Tu, F(u)) \in S(\tan \theta)$ for $u \in D(T)$, where $S(\tan \theta)$ is the sector defined by

$$(1.7) \quad S(\tan \theta) := \begin{cases} [0, \infty) & (\theta = 0), \\ \{z \in \mathbb{C}; |\operatorname{Im} z| \leq (\tan \theta) \operatorname{Re} z\} & (0 < \theta < \frac{\pi}{2}), \\ \{z \in \mathbb{C}; \operatorname{Re} z \geq 0\} & (\theta = \frac{\pi}{2}); \end{cases}$$

additionally, an accretive operator is sectorial of type $S(\tan \pi/2)$ and a sectorial operator T is m -sectorial if $R(1 + T) = X$. It should be noted that

sectoriality (or more precisely, sectorial-valuedness) was first introduced in a Hilbert space (see Kato [7, Section V.3.10]) as an extension of the notion of nonnegative selfadjointness. In fact, a linear operator T is the generator of an analytic contraction semigroup on X if and only if $-T$ is an m -sectorial operator in (reflexive) X (see [4, Theorem 1.5.9 and Proposition 1.3.9]).

Given two linear m -sectorial operators T and A , we introduce, in addition to (1.6), the bound on $\operatorname{Im}(Tu, F(A_\varepsilon u))$:

$$(1.8) \quad \begin{aligned} & |\operatorname{Im}(Tu, F(A_\varepsilon u))|^2 \\ & \leq d_1 \{ \operatorname{Re}(Tu, F(A_\varepsilon u)) + a_1 \|A_\varepsilon u\|^2 + b_1 \|A_\varepsilon u\| \cdot \|u\| + c_1 \|u\|^2 \} \\ & \quad \times \{ d_2 \operatorname{Re}(Tu, F(A_\varepsilon u)) + a_2 \|A_\varepsilon u\|^2 + b_2 \|A_\varepsilon u\| \cdot \|u\| + c_2 \|u\|^2 \} \end{aligned}$$

for $u \in D(T)$, with $a_1 d_2 \leq a_2$. Letting $\varepsilon \downarrow 0$ in (1.8) with $u \in D(T) \cap D(A)$, we have

$$\begin{aligned} & |\operatorname{Im}(Tu, F(Au))|^2 \\ & \leq d_1 \{ \operatorname{Re}(Tu, F(Au)) + a_1 \|Au\|^2 + b_1 \|Au\| \cdot \|u\| + c_1 \|u\|^2 \} \\ & \quad \times \{ d_2 \operatorname{Re}(Tu, F(Au)) + a_2 \|Au\|^2 + b_2 \|Au\| \cdot \|u\| + c_2 \|u\|^2 \}. \end{aligned}$$

The closedness and m -accretivity of $T + \kappa A$ for a fixed κ is equivalent to that of $t(T + \kappa A) = (tT) + \kappa(tA)$ for every $t > 0$. Therefore the essential parameters in (1.6) and (1.8), with A_ε replaced with A , seems to be a_j and d_j ($j = 1, 2$). In fact, (1.6) and (1.8) yield the “hyperbolic” region (bounded by the left branch of a hyperbola) in terms of a_j and d_j ($j = 1, 2$):

$$(1.9) \quad \Sigma_{\sec} = \{x + iy \in \mathbb{C}; y^2 \leq d_1(x - a_1)(d_2 x - a_2), -\infty < x \leq a_1\}$$

such that $\{T + \kappa A; \kappa = x + iy \in \Sigma_{\sec}^c\}$ forms a holomorphic family of type (A) (see Theorems 3.1 and 3.3 below); note that $\Sigma_{acc}^c \subset \Sigma_{\sec}^c$ (if $a_1 = a$) as a consequence of the difference between accretivity and sectoriality.

Here we want to explain that our result (Theorems 3.1 and 3.3) below is regarded as a generalization of [9, Theorem 2.2] to the (reflexive) Banach space case. To this end we give a brief description of two examples of Σ_{\sec} given by (1.9). Let $T := -\Delta$ be the minus Laplacian in $L^p = L^p(\mathbb{R}^N)$, with domain $D(T) := W^{2,p}(\mathbb{R}^N)$. Then it is well-known recently that T is m -sectorial of type $S(c_p)$ in L^p :

$$|\operatorname{Im}(-\Delta u, F(u))| \leq c_p \operatorname{Re}(-\Delta u, F(u)) \quad \forall u \in W^{2,p}(\mathbb{R}^N),$$

where the best constant $c_p := |p - 2|/(2\sqrt{p - 1})$ was found by Henry [6, p. 32] (see also Okazawa [14] and Voigt [23]). Next, let $A := |x|^{-2}$ be the maximal multiplication operator in L^p . Then $-\Delta + \kappa|x|^{-2}$ is an example of the

Schrödinger operator in L^p for which the hyperbolic region is given by

$$\Sigma_p = \left\{ x + iy \in \mathbb{C}; y^2 \leq (x - \beta_0(p)) \left[\frac{|p-2|^2}{p-1} x - a_2 \right], -\infty < x \leq \beta_0(p) \right\},$$

with

$$a_1 = \beta_0(p) < \frac{a_2}{d_2} = \frac{p-1}{|p-2|^2} a_2$$

(for the details see Theorem 5.1 below). Letting $p \rightarrow 2$, we see that $a_2 \rightarrow 4$ and the “hyperbolic” region Σ_p is reduced to the “parabolic” region (bounded by a parabola)

$$(1.10) \quad \Sigma_2 = \{x + iy \in \mathbb{C}; y^2 \leq 4(\beta_0(2) - x), -\infty < x \leq \beta_0(2)\}$$

as described in [9, Theorem 7.1 and Example 7.4 (a)] (for the constant $\beta_0(2)$ see (1.1)); note that

$$\frac{a_2}{d_2} = \frac{p-1}{|p-2|^2} a_2 \rightarrow \infty \quad (p \rightarrow 2),$$

while $a_1 = \beta_0(p)$ remains bounded. This explains the restriction $a_1 d_2 \leq a_2$ in (1.8). In this connection, if $N \geq 5$ then $-\beta_0(2) = (N-4)N/4 > 0$ is well-known as the constant in the Rellich inequality

$$\frac{(N-4)N}{4} \| |x|^{-2} u \|_{L^2} \leq \| (-\Delta) u \|_{L^2}, \quad u \in H^2(\mathbb{R}^N) \subset D(|x|^{-2})$$

(for a simple proof see, e.g., [16, Lemma 3.2]).

In the next example the potential is of harmonic oscillator type: $T + \kappa A = -\Delta + \kappa|x|^2$. In this case, computing for $V(x) + \alpha = |x|^2 + \alpha$ instead of $V(x) = |x|^2$ itself, we have the “hyperbolic” region $\Sigma_p(\alpha)$ (depending on $\alpha > 0$) defined by the following inequalities:

$$y^2 \leq \frac{p^2}{4(p-1)} \left(x - \frac{p-1}{4} \rho(\alpha) \right) \left(\frac{|p-2|^2}{p-1} x - \frac{p^2}{4} \rho(\alpha) \right), \quad -\infty < x \leq \frac{p-1}{4} \rho(\alpha),$$

where $\rho(\alpha) := 16/(27\alpha^2)$. Letting $\alpha \rightarrow \infty$, we obtain the sector on the left half-plane:

$$\Sigma_p(\infty) = \left\{ x + iy \in \mathbb{C}; |y| \leq -\frac{p|p-2|}{2(p-1)} x, -\infty < x \leq 0 \right\}.$$

This is a special case of Example 1 in Section 4. In this limiting process it should be noted that the closedness of $-\Delta + \kappa V(x)$ is equivalent to that of $-\Delta + \kappa[V(x) + \alpha]$, while the maximality $R(1 - \Delta + \kappa V(x)) = L^p$ is equivalent to $R(1 - \Delta + \kappa[V(x) + \alpha]) = L^p$ if $\operatorname{Re} \kappa \geq 0$. Thus we see that if $p \rightarrow 2$ then

the sector $\Sigma_p(\infty)$ degenerates into the ray $\Sigma_2(\infty) = (-\infty, 0]$, as described in [9, Section 7, Remark (b)].

Actually, Σ_2 given by (1.10) and $\Sigma_2(\infty)$ are typical examples of the closed convex subset S which was employed by Kato in the second abstract theorem [9, Theorem 2.2]. That is, our result forms a generalization of Kato's to the (reflexive) Banach space case.

This paper is divided into six sections. Section 2 is concerned with the holomorphic family of type (A) of closed linear operators in a Banach space. In Section 3 we prepare an abstract result (Theorems 3.1 and 3.3) for holomorphic family of linear m -sectorial operators in a reflexive Banach space, satisfying (1.6) and (1.8). We consider in Section 4 the general Schrödinger operator $-\Delta + \kappa V(x)$ in L^p . Detailed analysis of the inverse square potential case is carried out in Section 5. The final Section 6 is concerned with the parabolic Cauchy problem for the Schrödinger operator with inverse square potential.

§2. Preliminaries

Let T and A be two closed linear operators from a Banach space X to another Y . Then we consider the operator

$$(2.1) \quad T + \kappa A \quad \text{with domain} \quad D_0 := D(T) \cap D(A),$$

where κ is a complex parameter and D_0 is assumed to be non-trivial. We want to ask if $T + \kappa A$ forms a holomorphic family of type (A) on a non-empty complex region in the sense of Kato. To this end let us recall the general definition of holomorphic family of type (A).

Definition 1 ([7, Section VII.2]). Let G_0 be a domain in \mathbb{C} . Then a family $\{T(\kappa); \kappa \in G_0\}$ of linear operators (from X to Y) is said to be *holomorphic of type (A)* if

- (i) $T(\kappa)$ is closed for every $\kappa \in G_0$, with $D(T(\kappa)) = D_0$ independent of κ ;
- (ii) $T(\kappa)u$ is holomorphic with respect to $\kappa \in G_0$ for every $u \in D_0$.

In particular, if $T(\kappa)$ is a linear function of κ as in (2.1) then only (i) is required. In this case $\kappa = 0$ may be an *exceptional point*; in fact, $D(T) = D(T(0))$ may differ from $D(T) \cap D(A) = D(T(\kappa))$ with $\kappa \neq 0$.

Now we introduce the duality map F on X to X^* , the adjoint of X :

$$(2.2) \quad F(v) := \{f \in X^*; (v, f) = \|v\|^2 = \|f\|^2\} \quad \forall v \in X.$$

Then we have the homogeneity of F : $F(rv) = rF(v)$, $r \geq 0$, and an answer to the above-mentioned question is given in terms of (Tu, f) , $f \in F(Au)$.

Proposition 2.1. *Let T and A be closed linear operators from X to Y . Assume that for every $u \in D_0$ there is $f \in F(Au)$ such that*

$$(2.3) \quad \operatorname{Re}(Tu, f) \geq -a_1\|Au\|^2 - b_1\|Au\| \cdot \|u\| - c_1\|u\|^2,$$

$$(2.4) \quad |\operatorname{Im}(Tu, f)|^2 \leq d_1\{\operatorname{Re}(Tu, f) + a_1\|Au\|^2 + b_1\|Au\| \cdot \|u\| + c_1\|u\|^2\} \\ \times \{d_2\operatorname{Re}(Tu, f) + a_2\|Au\|^2 + b_2\|Au\| \cdot \|u\| + c_2\|u\|^2\},$$

where $a_j \in \mathbb{R}$, $b_j, c_j, d_j \geq 0$ are constants ($j = 1, 2$), with $a_1d_2 \leq a_2$. Let Σ be the closed convex subset of \mathbb{C} given by

$$(2.5) \quad \Sigma := \{x + iy \in \mathbb{C}; y^2 \leq d_1(x - a_1)(d_2x - a_2), -\infty < x \leq a_1\}.$$

Then

(i) $T + \kappa A$ is closed for $\kappa \in \Sigma^c$. In particular, if $b_j = c_j = 0$ ($j = 1, 2$), then

$$(2.6) \quad \|Au\| \leq \operatorname{dist}(\kappa, \Sigma)^{-1} \|(T + \kappa A)u\|, \quad u \in D_0.$$

(ii) $\{T + \kappa A; \kappa \in \Sigma^c, \kappa \neq 0\}$ forms a holomorphic family of type (A); $\kappa = 0$ is an exceptional point even if $a_1 < 0$.

In the next lemma we prove an inequality which yields the $(T + \kappa A)$ -boundedness of A (and hence of T).

Lemma 2.2. *Let T, A be closed operators from X to Y satisfying (2.3) and (2.4). Define the following four functions of θ with $|\theta| < \pi/2$:*

$$(2.7) \quad \delta(\theta) := 1 + \frac{1}{4}d_1d_2\tan^2\theta + i\tan\theta, \quad \alpha(\theta) := a_1 + \frac{1}{4}a_2d_1\tan^2\theta,$$

$$(2.8) \quad \beta(\theta) := b_1 + \frac{1}{4}b_2d_1\tan^2\theta, \quad \gamma(\theta) := c_1 + \frac{1}{4}c_2d_1\tan^2\theta.$$

Then, the following assertions hold.

(i) Let θ be an arbitrary angle with $|\theta| < \pi/2$ and put $\delta = \delta(\theta)$, $\alpha = \alpha(\theta)$, $\beta = \beta(\theta)$ and $\gamma = \gamma(\theta)$ for simplicity. Then for every $u \in D_0$ there is $f \in F(Au)$ such that

$$(2.9) \quad \operatorname{Re}(Tu, \delta f) \geq -\alpha\|Au\|^2 - \beta\|Au\| \cdot \|u\| - \gamma\|u\|^2.$$

(ii) If κ is a complex number with $\operatorname{Re}\{\overline{\delta(\theta)}\kappa\} > \alpha(\theta)$ for some θ with $|\theta| < \pi/2$ ($\Leftrightarrow \kappa \in \Sigma^c$), then A is $(T + \kappa A)$ -bounded: for $u \in D_0$,

$$(2.10) \quad \|Au\| \leq |\delta| (\operatorname{Re}\{\overline{\delta}\kappa\} - \alpha)^{-1} \|(T + \kappa A)u\| \\ + \left[\beta (\operatorname{Re}\{\overline{\delta}\kappa\} - \alpha)^{-1} + \sqrt{\gamma (\operatorname{Re}\{\overline{\delta}\kappa\} - \alpha)^{-1}} \right] \|u\|,$$

and hence T is also $(T + \kappa A)$ -bounded: for $u \in D_0$,

$$(2.11) \quad \|Tu\| \leq [1 + |\kappa| \cdot |\delta| (\operatorname{Re} \{\bar{\delta} \kappa\} - \alpha)^{-1}] \|(T + \kappa A)u\| \\ + |\kappa| \left[\beta (\operatorname{Re} \{\bar{\delta} \kappa\} - \alpha)^{-1} + \sqrt{\gamma (\operatorname{Re} \{\bar{\delta} \kappa\} - \alpha)^{-1}} \right] \|u\|.$$

Proof. Fix θ with $|\theta| < \pi/2$. Then it follows from (2.4) that

$$\begin{aligned} & \operatorname{Re}(Tu, e^{i\theta}f) - (\cos \theta) \operatorname{Re}(Tu, f) = (\sin \theta) \operatorname{Im}(Tu, f) \\ & \geq -d_1^{1/2} |\sin \theta| \{ \operatorname{Re}(Tu, f) + a_1 \|Au\|^2 + b_1 \|Au\| \cdot \|u\| + c_1 \|u\|^2 \}^{1/2} \\ & \quad \times \{ d_2 \operatorname{Re}(Tu, f) + a_2 \|Au\|^2 + b_2 \|Au\| \cdot \|u\| + c_2 \|u\|^2 \}^{1/2} \\ & = -2(\cos \theta)^{1/2} \{ \operatorname{Re}(Tu, f) + a_1 \|Au\|^2 + b_1 \|Au\| \cdot \|u\| + c_1 \|u\|^2 \}^{1/2} \\ & \quad \times \frac{d_1^{1/2} |\sin \theta|}{2(\cos \theta)^{1/2}} \{ d_2 \operatorname{Re}(Tu, f) + a_2 \|Au\|^2 + b_2 \|Au\| \cdot \|u\| + c_2 \|u\|^2 \}^{1/2}. \end{aligned}$$

Applying the inequality $2ab \leq a^2 + b^2$, we have

$$\begin{aligned} \operatorname{Re}(Tu, e^{i\theta}f) & \geq -\frac{(\sin \theta)^2}{4 \cos \theta} d_1 d_2 \operatorname{Re}(Tu, f) - \alpha(\cos \theta) \|Au\|^2 \\ & \quad - \beta(\cos \theta) \|Au\| \cdot \|u\| - \gamma(\cos \theta) \|u\|^2. \end{aligned}$$

Thus we obtain (2.9) with α , β and γ given by (2.7) and (2.8).

Now let κ be a complex number with $\operatorname{Re} \{\bar{\delta}(\theta) \kappa\} > \alpha(\theta)$ for some θ with $|\theta| < \pi/2$. Then it follows from (2.9) that

$$\begin{aligned} \operatorname{Re}((T + \kappa A)u, \delta f) & = \operatorname{Re}(Tu, \delta f) + \operatorname{Re} \{\bar{\delta} \kappa\} \|Au\|^2 \\ & \geq (\operatorname{Re} \{\bar{\delta} \kappa\} - \alpha) \|Au\|^2 - \beta \|Au\| \cdot \|u\| - \gamma \|u\|^2 \end{aligned}$$

and hence

$$(\operatorname{Re} \{\bar{\delta} \kappa\} - \alpha) \|Au\|^2 \leq (|\delta| \cdot \|(T + \kappa A)u\| + \beta \|u\|) \|Au\| + \gamma \|u\|^2$$

which implies (2.10) and (2.11). (We shall prove the equivalence $\kappa \in \Sigma^c \Leftrightarrow \operatorname{Re} \{\bar{\delta}(\theta) \kappa\} > \alpha(\theta)$ for some θ with $|\theta| < \pi/2$ in the proof of Proposition 2.1.) \square

Proof of Proposition 2.1. Obviously, (2.10) and (2.11) imply that if κ is a complex number with $\operatorname{Re} \{\bar{\delta}(\theta) \kappa\} > \alpha(\theta)$, then $T + \kappa A$ is closed in X . Thus it remains to prove the equivalence:

$$(2.12) \quad \kappa \in \Sigma^c \Leftrightarrow \operatorname{Re} \{\bar{\delta}(\theta) \kappa\} > \alpha(\theta) \text{ for some } \theta \text{ with } |\theta| < \pi/2.$$

First we have to determine the shape of Σ or its boundary $\partial\Sigma$. The key lies in the equation $\operatorname{Re} \{\bar{\delta} \kappa\} = \alpha$. In fact, we see from (2.7) that the equation

depends on the parameter $\tan \theta$ ($|\theta| < \pi/2$). Thus we have a one-parameter family of lines on the (x, y) -plane:

$$(2.13) \quad \begin{aligned} \operatorname{Re}\{\bar{\delta}\kappa\} - \alpha &= \operatorname{Re}\{\bar{\delta}(x + iy)\} - \alpha \\ &= \left(1 + \frac{1}{4}d_1d_2\tan^2\theta\right)x + (\tan\theta)y - \left(a_1 + \frac{1}{4}a_2d_1\tan^2\theta\right) = 0. \end{aligned}$$

Setting $\lambda := \tan \theta \in \mathbb{R}$, these lines are rewritten as

$$(2.14) \quad F(x, y, \lambda) := \left(1 + \frac{1}{4}d_1d_2\lambda^2\right)x + \lambda y - \left(a_1 + \frac{1}{4}a_2d_1\lambda^2\right) = 0.$$

Now it is easy to understand that $\partial\Sigma$ is nothing but the envelope E of the family (2.14). In fact, E is determined as

$$E = \{x + iy \in \mathbb{C}; F(x, y, \lambda) = 0 = F_\lambda(x, y, \lambda)\},$$

where $F_\lambda(x, y, \lambda) = (1/2)d_1d_2\lambda x + y - (1/2)a_2d_1\lambda$. By a simple computation we obtain

$$\partial\Sigma = \{x + iy \in \mathbb{C}; y^2 = d_1(x - a_1)(d_2x - a_2), \quad -\infty < x \leq a_1\}$$

as required. Incidentally, for a fixed λ , (2.14) is a tangential line of $\partial\Sigma$ at (x_0, y_0) with

$$x_0 = \frac{a_1 - (1/4)a_2d_1\lambda^2}{1 - (1/4)d_1d_2\lambda^2}, \quad y_0 = \frac{(1/2)(a_2 - a_1d_2)d_1\lambda}{1 - (1/4)d_1d_2\lambda^2}.$$

In particular, we have $(x_0, y_0) = (a_1, 0)$ when $\lambda = 0$. Therefore we can conclude that $T + \kappa A$ is closed for $\kappa \in \Sigma^c$ so that $\{T + \kappa A; \kappa \in \Sigma^c, \kappa \neq 0\}$ forms a holomorphic family of type (A).

Finally, we show that if $b_j = c_j = 0$ ($j = 1, 2$), then (2.10) is simplified as (2.6). Given $\kappa \in \Sigma^c$, we can find a supporting line ℓ to Σ such that

$$(2.15) \quad \operatorname{dist}(\kappa, \Sigma) = \operatorname{dist}(\kappa, \ell),$$

where ℓ is one of the lines in the family (2.13). Therefore it suffices to remind of the well-known formula

$$(2.16) \quad \operatorname{dist}(\kappa, \ell) = |\delta|^{-1}(\operatorname{Re}\{\bar{\delta}\kappa\} - \alpha).$$

It then follows from (2.10), (2.15) and (2.16) that

$$\|Au\| \leq \operatorname{dist}(\kappa, \Sigma)^{-1}\|(T + \kappa A)u\|, \quad u \in D_0;$$

note that $\beta = 0 = \gamma$ is a consequence of $b_j = 0 = c_j$ ($j = 1, 2$). This completes the proof of Proposition 2.1. \square

Remark 1. In the perturbation theory of (unbounded) linear operators the relative boundedness is a very popular condition (see [7, Section IV.1.1]). Now suppose that A is T -bounded:

$$\|Au\| \leq a_0\|u\| + b_0\|Tu\|, \quad u \in D(T) \subset D(A).$$

Then for every $g \in F(Au)$ we have

$$(2.17) \quad \operatorname{Re}(Tu, g) \geq -\|Tu\| \cdot \|Au\| \geq -b_0\|Tu\|^2 - a_0\|Tu\| \cdot \|u\|.$$

This is obviously different from (2.3). We note that (2.17) is used by Borisov [2] when X is a Hilbert space.

§3. Holomorphic families of linear m -sectorial operators

Let F be the duality map on a Banach space X to its adjoint X^* (see (2.2)). Then we begin with the definition of a sectorial operator in X .

Definition 2. A linear operator T in X is said to be *sectorial of type $S(\tan \theta_T)$* if there is $\theta_T \in [0, \pi/2]$ such that for all $u \in D(T)$ there is a $f \in F(u)$ satisfying

$$(3.1) \quad (Tu, f) \in S(\tan \theta_T),$$

where $S(\tan \theta_T)$ is the sector defined as (1.7) with $\theta = \theta_T$. In other words, a linear operator T in X is sectorial of type $S(\tan \theta_T)$ if $e^{i\theta}T$ is accretive for every θ with $|\theta| \leq \pi/2 - \theta_T$. In particular, T is *m -sectorial* in X if T is sectorial of type $S(\tan \theta_T)$ and $R(1 + T) = X$.

We say that an operator T in X is simply *sectorial* if T is sectorial of type $S(\tan \theta)$ for some $\theta \in [0, \pi/2]$.

Let T be m -sectorial in a reflexive space X . Then, since $e^{i\theta}T$ is m -accretive, we have

$$\operatorname{Re}(e^{i\theta}Tu, f) \geq 0 \quad \forall f \in F(u), \quad \forall u \in D(T), \quad |\theta| \leq \pi/2 - \theta_T$$

(see Pazy [18, Theorem 1.4.6] and Tanabe [22, Theorem 2.1.5]). Therefore we obtain (3.1) for all $f \in F(u)$ and $u \in D(T)$. Thus we see that as for m -sectorial operators in a reflexive space Definition 2 coincides with that given by Goldstein [4, Definition 1.5.8]. Consequently, if T and A are two m -sectorial operators in a reflexive space, then $T + A$ is also sectorial.

Now let T and A be linear m -sectorial operators in a *reflexive* Banach space X . In this section we consider the m -sectoriality of $T + \kappa A$ with domain $D_0 = D(T) \cap D(A)$. Our basic assumption **(A1)** is stated in terms of the Yosida approximation $\{A_\varepsilon\}$ of A :

(A1) For every $u \in D(T)$ and $\varepsilon > 0$ there is $f_\varepsilon \in F(A_\varepsilon u)$ such that

$$(3.2) \quad \operatorname{Re}(Tu, f_\varepsilon) \geq -a_1 \|A_\varepsilon u\|^2 - b_1 \|A_\varepsilon u\| \cdot \|u\| - c_1 \|u\|^2,$$

$$(3.3) \quad |\operatorname{Im}(Tu, f_\varepsilon)|^2 \leq d_1 \{ \operatorname{Re}(Tu, f_\varepsilon) + a_1 \|A_\varepsilon u\|^2 + b_1 \|A_\varepsilon u\| \cdot \|u\| + c_1 \|u\|^2 \} \\ \times \{ d_2 \operatorname{Re}(Tu, f_\varepsilon) + a_2 \|A_\varepsilon u\|^2 + b_2 \|A_\varepsilon u\| \cdot \|u\| + c_2 \|u\|^2 \},$$

where $a_j \in \mathbb{R}$, $b_j, c_j, d_j \geq 0$ ($j = 1, 2$) are constants, with $a_1 d_2 \leq a_2$.

Then we have the first half of the main theorem in this section.

Theorem 3.1. *Let T and A be linear m -sectorial operators in a reflexive Banach space X . Assume that condition (A1) is satisfied. Let Σ be the closed convex subset of \mathbb{C} given by*

$$(3.4) \quad \Sigma = \{x + iy \in \mathbb{C}; y^2 \leq d_1(x - a_1)(d_2 x - a_2), -\infty < x \leq a_1\}.$$

Then the following assertions hold:

(i) *Let $a_1 \geq 0$. Then $T + \kappa A$ is closed for $\kappa \in \Sigma^c$ and hence $\{T + \kappa A; \kappa \in \Sigma^c\}$ forms a holomorphic family of type (A). In particular, if $b_j = c_j = 0$ ($j = 1, 2$), then for $u \in D_0$,*

$$\|Au\| \leq \operatorname{dist}(\kappa, \Sigma)^{-1} \|(T + \kappa A)u\|.$$

(ii) *Let $a_1 < 0$. Then $0 \in \Sigma^c$ and A is T -bounded with T -bound less than or equal to $(-a_1)^{-1} = \operatorname{dist}(0, \Sigma)^{-1}$: for $u \in D_0 = D(T)$,*

$$(3.5) \quad \|Au\| \leq (-a_1)^{-1} \|Tu\| + [b_1(-a_1)^{-1} + \sqrt{c_1(-a_1)^{-1}}] \|u\|.$$

Consequently, $\{T + \kappa A; \kappa \in \Sigma^c\}$ forms a holomorphic family of type (A).

Proof. First we note that the constants in (3.2) and (3.3) are identical to those in (2.3) and (2.4).

(i) In the same way as in the proof of Lemma 2.2 we can obtain

$$(3.6) \quad \operatorname{Re}(Tu, \delta f_\varepsilon) \geq -\alpha \|A_\varepsilon u\|^2 - \beta \|A_\varepsilon u\| \cdot \|u\| - \gamma \|u\|^2,$$

where α, β, γ and δ are the same as in (2.9). Hence we can obtain (2.10) and (2.11) with A replaced with A_ε . Since $A_\varepsilon u \rightarrow Au$ for $u \in D_0$, we have (2.10) and (2.11) themselves. Therefore it follows from Proposition 2.1 that $\{T + \kappa A; \kappa \in \Sigma^c\}$ forms a holomorphic family of type (A).

(ii) Since $-a_1 > 0$, it follows from (3.2) that

$$(-a_1) \|A_\varepsilon u\|^2 \leq (\|Tu\| + b_1 \|u\|) \|A_\varepsilon u\| + c_1 \|u\|^2,$$

which implies

$$\|A_\varepsilon u\| \leq (-a_1)^{-1} \|Tu\| + [b_1(-a_1)^{-1} + \sqrt{c_1(-a_1)^{-1}}] \|u\|, \quad u \in D(T).$$

Therefore we conclude that $u \in D(A)$ and $A_\varepsilon u \rightarrow Au$ as $\varepsilon \downarrow 0$, with (3.5). \square

Remark 2. The m -sectoriality of T and A was not used in the proof of Theorem 3.1. However, the notion is essential in applications in Sections 4 and 5.

Lemma 3.2. *Assume that (A1) above and (A2) below are satisfied:*

(A2) *Let $v \in X$ and $\varepsilon > 0$. Then $\operatorname{Im}(A_\varepsilon v, g) = 0$ for every $g \in F(v)$, and*

$$(v, g_\varepsilon) \geq 0 \text{ so that } \operatorname{Im}(v, g_\varepsilon) = 0 \quad \forall g_\varepsilon \in F(A_\varepsilon v).$$

Then (3.2) and (3.3) hold, with T replaced with $T + \lambda$ ($\lambda > 0$).

In fact, we have $\operatorname{Re}(Tu, f_\varepsilon) \leq \operatorname{Re}((T + \lambda)u, f_\varepsilon)$ for $f_\varepsilon \in F(A_\varepsilon u)$.

Here we note that condition (A2) means that A is symmetric and nonnegative if X is a Hilbert space.

Theorem 3.3. *Let T be a linear m -sectorial operator of type $S(\tan \theta_T)$ in a reflexive Banach space X and let A be a linear m -sectorial operator in X . Assume that conditions (A1) and (A2) are satisfied. Let Σ be the region as in (3.4). Then in addition to (i), (ii) in Theorem 3.1 one has*

(iii) *$T + \kappa A$ is m -accretive in X for $\kappa \in \Sigma^c$ with $\operatorname{Re} \kappa \geq 0$.*

(iv) *$T + \kappa A$ is m -sectorial of type $S(\tan \theta)$ in X for $\theta \in [\theta_T, \pi/2)$ and $\kappa \in \Sigma^c$ with $|\arg \kappa| \leq \theta$.*

Assume further that $b_j = c_j = 0$ ($j = 1, 2$) in (A1). Then one has

(v) *$\mathbb{R}_- := (-\infty, 0)$ belongs to the resolvent set of $T + \kappa A$ for $\kappa \in \Sigma^c$.*

(vi) *In particular, if there exists a constant $\gamma \geq 0$ such that*

$$(3.7) \quad \operatorname{Re}(Tu, f) \geq \gamma(A_\varepsilon u, f), \quad u \in D(T), \quad f \in F(u),$$

then $T + \kappa A$ is m -accretive in X for $\kappa \in \Sigma^c$ with $\operatorname{Re} \kappa \geq -\gamma$.

To prove Theorem 3.3 we need the following

Lemma 3.4. *Let A be a linear m -accretive operator in a reflexive Banach space X .*

(i) *Let $\{u_\varepsilon\}$ be a family in $D(A)$ such that $u_\varepsilon \rightharpoonup u \in X$ ($\varepsilon \downarrow 0$). If $\|Au_\varepsilon\|$ are bounded as ε tends to 0, then $u \in D(A)$ and $Au_\varepsilon \rightharpoonup Au$ ($\varepsilon \downarrow 0$).*

(ii) *Let $\{v_\varepsilon\}$ be a family in X such that $v_\varepsilon \rightharpoonup u \in X$ ($\varepsilon \downarrow 0$). If $\|A_\varepsilon v_\varepsilon\|$ are bounded as ε tends to 0, then $u \in D(A)$ and $A_\varepsilon v_\varepsilon \rightharpoonup Au$ ($\varepsilon \downarrow 0$).*

Note that (ii) is reduced to (i) in Lemma 3.4. In fact, put $u_\varepsilon := (1 + \varepsilon A)^{-1} v_\varepsilon$. Then $Au_\varepsilon = A_\varepsilon v_\varepsilon$ is bounded and $u_\varepsilon = v_\varepsilon - \varepsilon A_\varepsilon v_\varepsilon \rightharpoonup u$ ($\varepsilon \downarrow 0$).

Proof of Theorem 3.3. (iii) Since $(A_\varepsilon u, f)$ is real-valued (see (A2)), we have

$$\operatorname{Re}((T + \kappa A_\varepsilon)u, f) = \operatorname{Re}(Tu, f) + (A_\varepsilon u, f) \operatorname{Re} \kappa, \quad u \in D(T), \quad f \in F(u).$$

Therefore $T + \kappa A_\varepsilon$ (and hence $T + \kappa A$) is accretive in X if $\kappa \in \Sigma^c$ with $\operatorname{Re} \kappa \geq 0$. To prove the assertion it remains to show that

$$R(1 + T + \kappa A) = X \quad \text{for } \kappa \in \Sigma^c \text{ with } \operatorname{Re} \kappa \geq 0.$$

Let $\kappa \in \Sigma^c$ with $\operatorname{Re} \kappa \geq 0$. Since κA_ε is bounded and accretive in X , $T + \kappa A_\varepsilon$ is m -accretive in X , that is, for every $v \in X$ and $\varepsilon > 0$ there is $u_\varepsilon \in D(T)$ such that

$$u_\varepsilon + Tu_\varepsilon + \kappa A_\varepsilon u_\varepsilon = v,$$

with $\|u_\varepsilon\| \leq \|v\|$. Thus we see from (2.10), (2.15) and (2.16) that

$$\|A_\varepsilon u_\varepsilon\| \leq \left[(2 + \beta|\delta|^{-1}) \operatorname{dist}(\kappa, \Sigma)^{-1} + \sqrt{\gamma|\delta|^{-1} \operatorname{dist}(\kappa, \Sigma)^{-1}} \right] \|v\|,$$

where δ is given by (2.7). This implies that $v \in R(1 + T + \kappa A)$ (cf. [11, Proposition 2.2]; see [4, Exercises 1.6.12 (7)]).

(iv) Let θ be an angle with $\theta_T \leq \theta < \pi/2$ and $\kappa \in \mathbb{C}$ with $|\arg \kappa| \leq \theta$. Here we may assume that $\kappa \neq 0$. Then condition (A2) yields that for every $u \in D_0$ and $f \in F(u)$,

$$\begin{aligned} |\operatorname{Im}((T + \kappa A)u, f)| &\leq |\operatorname{Im}(Tu, f)| + (Au, f) |\operatorname{Im} \kappa| \\ &\leq (\tan \theta_T) \operatorname{Re}(Tu, f) + \frac{|\operatorname{Im} \kappa|}{\operatorname{Re} \kappa} \operatorname{Re}(\kappa Au, f) \\ &= (\tan \theta_T) \operatorname{Re}(Tu, f) + |\tan(\arg \kappa)| \operatorname{Re}(\kappa Au, f) \\ &\leq (\tan \theta) \operatorname{Re}((T + \kappa A)u, f). \end{aligned}$$

Since $T + \kappa A$ is m -accretive (see (iii) above), we see that $T + \kappa A$ is m -sectorial of type $S(\tan \theta)$ in X .

(v) Let $\kappa \in \Sigma^c$. Then $\operatorname{Re}\{\overline{\delta(\theta)}\kappa\} > \alpha(\theta)$ for some θ with $|\theta| < \pi/2$ (see (2.12)), where $\delta = \delta(\theta)$ and $\alpha = \alpha(\theta)$ are given by (2.7). Next let

$$t > \alpha_+ = \alpha_+(\theta) := \max\{\alpha, 0\}.$$

Then we see that $t/\bar{\delta} \in \Sigma^c$ with $\operatorname{Re}\{t/\bar{\delta}\} > 0$ and hence $(T + (t/\bar{\delta})A + \lambda)^{-1}$ exists for $\lambda > 0$ as a consequence of the m -accretivity of $T + (t/\bar{\delta})A$. Therefore $(T + \kappa A + \lambda)^{-1}$ also exists for $\lambda > 0$. In fact, we can show that if $\lambda > 0$, then the resolvent of $T + \kappa A$ is given by the Neumann series:

$$\begin{aligned} (3.8) \quad (T + \kappa A + \lambda)^{-1} \\ = (T + (t/\bar{\delta})A + \lambda)^{-1} \sum_{n=0}^{\infty} ((t/\bar{\delta}) - \kappa)^n [A(T + (t/\bar{\delta})A + \lambda)^{-1}]^n. \end{aligned}$$

To this end we have to show that

$$(3.9) \quad \|A(T + (t/\bar{\delta})A + \lambda)^{-1}\| \leq |\delta|(t - \alpha)^{-1}, \quad \lambda > 0.$$

Now, it follows from (2.10) with T replaced with $T + \lambda$ that

$$\|Au\| \leq |\delta|(t - \alpha)^{-1}\|(T + (t/\bar{\delta})A + \lambda)u\|, \quad u \in D_0, \quad \lambda > 0$$

(see Lemma 3.2); note that $\beta = \gamma = 0$. Since $T + (t/\bar{\delta})A$ is m -accretive, we obtain (3.9). Hence the resolvent (3.8) exists if κ satisfies

$$|(t/\bar{\delta}) - \kappa| \cdot |\delta|(t - \alpha)^{-1} < 1 \Leftrightarrow |t - \bar{\delta}\kappa| < t - \alpha.$$

Noting that

$$\begin{aligned} & \{\kappa \in \mathbb{C}; \exists \theta \text{ such that } \operatorname{Re} \{\bar{\delta}(\bar{\theta})\kappa\} > \alpha(\theta)\} \\ &= \bigcup_{t > \alpha_+(\theta)} \{\kappa \in \mathbb{C}; \exists \theta \text{ such that } |t - \bar{\delta}(\bar{\theta})\kappa| < t - \alpha(\theta)\}, \end{aligned}$$

we can obtain the conclusion.

(**vi**) Let $\kappa \in \Sigma^c$. Then, since (**v**) yields $R(1 + T + \kappa A) = X$, it suffices to note that if $\operatorname{Re} \kappa + \gamma \geq 0$,

$$\operatorname{Re}((T + \kappa A)u, f) = \lim_{\varepsilon \downarrow 0} \operatorname{Re}((T + \kappa A_\varepsilon)u, f) \geq 0, \quad u \in D_0.$$

This completes the proof of Theorem 3.3. \square

Remark 3. Borisov-Okazawa [3, Theorem 2.4 (ii)] is false unless $b = 0 = c$ in (2.2).

§4. Application to Schrödinger operators in $L^p(\mathbb{R}^N)$

Let $V \geq 0$ be a function in $C^1(\mathbb{R}^N)$ (or in $C^1(\mathbb{R}^N \setminus \{0\})$), $N \in \mathbb{N}$. Then we consider the Schrödinger operator $-\Delta + \kappa V(x)$ in $L^p = L^p(\mathbb{R}^N)$, $1 < p < \infty$, with domain

$$W^{2,p}(\mathbb{R}^N) \cap \{u \in L^p; Vu \in L^p\}$$

(for the domain characterization of $-\Delta$ see, e.g., Hempel-Voigt [5]). We shall figure out the region Σ for which $\{-\Delta + \kappa V(x); \kappa \in \Sigma^c\}$ forms a holomorphic family of type (A). To do so we shall use Theorem 3.3 with $T := -\Delta$ and $A := A(\alpha) := V(x) + \alpha$ ($\alpha > 0$) instead of $V(x)$ itself. As stated in Introduction, T is m -sectorial of type $S(c_p)$:

$$|\operatorname{Im}(-\Delta u, F(u))| \leq c_p \operatorname{Re}(-\Delta u, F(u)) \quad \forall u \in W^{2,p}(\mathbb{R}^N).$$

Here (f, g) denotes the pairing between $f \in L^p$ and $g \in L^{p'}$ ($p^{-1} + p'^{-1} = 1$), and (f, g) is linear in f and conjugate linear in g . Thus the computation depends on the single-valued duality map on L^p : $F(0) = 0$ and

$$F(v)(x) := \|v\|^{2-p} |v(x)|^{p-2} v(x), \quad 0 \neq v \in L^p.$$

As for $A(\alpha)$ its m -sectoriality is a simple consequence of the positivity of $V(x) + \alpha$. In fact, it is easy to see that $R(1 + A(\alpha)) = L^p$, with

$$|\operatorname{Im}(A(\alpha)u, F(u))| = 0 \leq \operatorname{Re}(A(\alpha)u, F(u)) = \|u\|^{2-p} \int_{\mathbb{R}^N} [V(x) + \alpha] |u(x)|^p dx.$$

Next we introduce the Yosida approximation $A(\alpha)_\varepsilon$ of $A(\alpha)$: for $v \in L^p$ and $\varepsilon > 0$,

$$(4.1) \quad \begin{aligned} A(\alpha)_\varepsilon v(x) &:= (V + \alpha)_\varepsilon(x) v(x) \\ &= (V(x) + \alpha) [1 + \varepsilon(V(x) + \alpha)]^{-1} v(x). \end{aligned}$$

Then we see that $A(\alpha)_\varepsilon$ satisfies condition **(A2)**:

$$\operatorname{Im}(A(\alpha)_\varepsilon v, F(v)) = 0, \quad (v, F(A(\alpha)_\varepsilon v)) \geq 0 \quad \forall v \in L^p.$$

In this section we do not use the Yosida approximation of $V(x)$ itself.

Now we assume that $A(\alpha) = V(x) + \alpha$ satisfies

(V) $_\varepsilon$ $(V + \alpha)_\varepsilon \in C^1(\mathbb{R}^N)$ and for every $\alpha > 0$ and $\varepsilon > 0$ there is a constant $\rho(\alpha) \geq 0$ (independent of $\varepsilon > 0$) such that

$$(4.2) \quad |\nabla(V + \alpha)_\varepsilon(x)|^2 \leq \rho(\alpha) [(V + \alpha)_\varepsilon(x)]^3 \quad \forall x \in \mathbb{R}^N.$$

Before stating the main theorem we want to give a sufficient condition for **(V) $_\varepsilon$** .

Proposition 4.1. *Assume that $V \in C^1(\mathbb{R}^N)$ and for every $\alpha > 0$ there is a constant $\rho(\alpha) \geq 0$ such that*

$$(4.3) \quad |\nabla V(x)|^2 \leq \rho(\alpha) [V(x) + \alpha]^3 \quad \forall x \in \mathbb{R}^N.$$

*Then $V(x)$ satisfies condition **(V) $_\varepsilon$** .*

Proof. Since $V \in C^1(\mathbb{R}^N)$, we see from (4.1) that

$$\nabla(V + \alpha)_\varepsilon(x) = \frac{1}{\varepsilon} \nabla \left[1 - \frac{1}{1 + \varepsilon(V(x) + \alpha)} \right] = \frac{\nabla V(x)}{[1 + \varepsilon(V(x) + \alpha)]^2}.$$

Then it follows from (4.3) that

$$|\nabla(V + \alpha)_\varepsilon(x)|^2 \leq \rho(\alpha) \frac{[(V + \alpha)_\varepsilon(x)]^3}{1 + \varepsilon(V(x) + \alpha)} \leq \rho(\alpha) [(V + \alpha)_\varepsilon(x)]^3.$$

This completes the proof. □

Let $\rho(\alpha)$ be as introduced in condition $(\mathbf{V})_\varepsilon$. Then we assume further that

$$(4.4) \quad \rho := \lim_{\alpha \rightarrow \infty} \rho(\alpha) < \infty$$

(for this class of potentials cf. Kato [9, Section 6]).

In terms of ρ we can obtain a generalization of Kato [9, Theorem 7.1].

Theorem 4.2. *For the potential $V(x)$ assume that condition $(\mathbf{V})_\varepsilon$ is satisfied, together with (4.4). Define the hyperbolic region $\Sigma_p(\infty)$ as*

$$(4.5) \quad \Sigma_p(\infty) := \left\{ x + iy \in \mathbb{C}; y^2 \leq f(x), -\infty < x \leq \frac{p-1}{4}\rho \right\},$$

where f is defined as

$$f(x) := \frac{p^2}{4(p-1)} \left(x - \frac{p-1}{4}\rho \right) \left(\frac{|p-2|^2}{p-1}x - \frac{p^2}{4}\rho \right).$$

Then

- (a) $\{-\Delta + \kappa V(x); \kappa \in \Sigma_p(\infty)^c\}$ forms a holomorphic family of type (A).
- (b) $-\Delta + \kappa V(x)$ is m -accretive in L^p for $\kappa \in \Sigma_p(\infty)^c$ with $\operatorname{Re} \kappa \geq 0$.
- (c) $-\Delta + \kappa V(x)$ is m -sectorial of type $S(\tan \theta)$ in L^p for $\theta \in [\tan^{-1} c_p, \pi/2)$ and $\kappa \in \Sigma_p(\infty)^c$ with $|\arg \kappa| \leq \theta$.

Proof. Let $\Sigma_p(\infty)$ be the region defined by (4.5). Then for $\kappa \in \Sigma_p(\infty)^c$ we prove the assertions (a), (b) and (c). Since $\Sigma_p(\infty)^c$ is open, we see from (4.4) that $\kappa \in \Sigma_p(\alpha)^c$ for sufficiently large $\alpha > 0$, where $\Sigma_p(\alpha)$ is defined by (4.5) with f and ρ replaced with f_α and $\rho(\alpha)$, respectively:

$$\begin{aligned} \Sigma_p(\alpha) &:= \left\{ x + iy \in \mathbb{C}; y^2 \leq f_\alpha(x), -\infty < x \leq \frac{p-1}{4}\rho(\alpha) \right\}, \\ f_\alpha(x) &:= \frac{p^2}{4(p-1)} \left(x - \frac{p-1}{4}\rho(\alpha) \right) \left(\frac{|p-2|^2}{p-1}x - \frac{p^2}{4}\rho(\alpha) \right). \end{aligned}$$

It is expected that $\Sigma_p(\alpha)$ is the region associated with $T = -\Delta$ and $A(\alpha) = V + \alpha$. Therefore it suffices to prove the closedness, m -accretivity and m -sectoriality of $-\Delta + \kappa[V(x) + \alpha]$ for $\kappa \in \Sigma_p(\alpha)^c$ instead of (a), (b) and (c). Since (A2) is already verified above, we are going to show that $T = -\Delta$ and $A(\alpha)_\varepsilon = (V + \alpha)_\varepsilon$ satisfy condition (A1): for $u \in C_0^\infty(\mathbb{R}^N)$,

$$(4.6) \quad \operatorname{Re}(Tu, F(A(\alpha)_\varepsilon u)) \geq -\frac{p-1}{4}\rho(\alpha)\|A(\alpha)_\varepsilon u\|^2,$$

$$(4.7) \quad |\operatorname{Im}(Tu, F(A(\alpha)_\varepsilon u))|^2 \leq \frac{p^2}{4(p-1)} \left\{ \operatorname{Re}(Tu, F(A(\alpha)_\varepsilon u)) + \frac{p-1}{4}\rho(\alpha)\|A(\alpha)_\varepsilon u\|^2 \right\} \times \left\{ \frac{|p-2|^2}{p-1} \operatorname{Re}(Tu, F(A(\alpha)_\varepsilon u)) + \frac{p^2}{4}\rho(\alpha)\|A(\alpha)_\varepsilon u\|^2 \right\};$$

note that $C_0^\infty(\mathbb{R}^N)$ is a core for T . The proof is divided into three steps.

Step 1 [Proof of (4.6)]. It suffices to show that for $u \in C_0^\infty(\mathbb{R}^N)$,

$$(4.8) \quad \operatorname{Re}(Tu, |A(\alpha)_\varepsilon u|^{p-2} A(\alpha)_\varepsilon u) \geq -\frac{p-1}{4} \rho(\alpha) \|A(\alpha)_\varepsilon u\|^p.$$

First we note that if $1 < p < 2$, then

$$\operatorname{Re}(Tu, |A(\alpha)_\varepsilon u|^{p-2} A(\alpha)_\varepsilon u) = \lim_{\delta \downarrow 0} \operatorname{Re} I(\delta),$$

where

$$I(\delta) := - \int_{\mathbb{R}^N} [(V + \alpha)_\varepsilon(x)]^{p-1} (|u(x)|^2 + \delta)^{(p-2)/2} \overline{u(x)} \Delta u(x) dx;$$

if $p \geq 2$ then we may take $\delta = 0$ so that the computation will be simpler. Thus we restrict ourselves to the case of $1 < p < 2$. Put $Q(x) := [(V + \alpha)_\varepsilon(x)]^{p-1}$. Then integration by parts gives

$$(4.9) \quad I(\delta) = \int_{\mathbb{R}^N} f(x, \delta) dx + \int_{\mathbb{R}^N} g(x, \delta) dx,$$

where $f(x, \delta) := Q(x)(|u(x)|^2 + \delta)^{(p-2)/2} |\nabla u(x)|^2$ and

$$g(x, \delta) := \left\{ \frac{p-2}{2} \nabla |u(x)|^2 + \frac{\nabla Q(x)}{Q(x)} (|u(x)|^2 + \delta) \right\} \\ \cdot Q(x)(|u(x)|^2 + \delta)^{(p-4)/2} \overline{u(x)} \nabla u(x).$$

This yields that

$$(4.10) \quad \operatorname{Re} I(\delta) - \int_{\mathbb{R}^N} f(x, \delta) dx = \operatorname{Re} \int_{\mathbb{R}^N} g(x, \delta) dx \\ = (p-2) \int_{\mathbb{R}^N} Q(x) |u(x)|^2 (|u(x)|^2 + \delta)^{(p-4)/2} |\nabla |u(x)||^2 dx \\ + \int_{\mathbb{R}^N} |u(x)| (|u(x)|^2 + \delta)^{(p-2)/2} \nabla Q(x) \cdot \nabla |u(x)| dx.$$

Now suppose that $\operatorname{supp} u \subset B = B(0, R)$ for some $R > 0$. Then, applying the Cauchy-Schwarz and geometric-arithmetic mean inequalities to

$$(|u(x)|^2 + \delta)^{(p-2)/2} |u(x)| \nabla Q(x) \cdot \nabla |u(x)|,$$

we can obtain

$$(4.11) \quad \operatorname{Re} I(\delta) + \frac{1}{p^2} J(\delta) - \delta \int_{\mathbb{R}^N} Q(x) (|u(x)|^2 + \delta)^{(p-4)/2} |\nabla u(x)|^2 dx \\ \geq \int_{\mathbb{R}^N} Q(x) |u(x)|^2 (|u(x)|^2 + \delta)^{(p-4)/2} \left(|\nabla u(x)|^2 - |\nabla |u(x)||^2 \right) dx \geq 0$$

(note that $|\nabla|u(x)|| \leq |\nabla u(x)|$), where

$$(4.12) \quad J(\delta) := \frac{p^2}{4(p-1)} \int_{B(0,R)} \frac{|\nabla Q(x)|^2}{Q(x)} (|u(x)|^2 + \delta)^{p/2} dx.$$

Here (4.2) in condition $(\mathbf{V})_\varepsilon$ applies to give

$$\frac{|\nabla Q(x)|^2}{Q(x)} \leq (p-1)^2 \rho(\alpha) [(V + \alpha)_\varepsilon(x)]^p.$$

Therefore we have

$$(4.13) \quad -\operatorname{Re} I(\delta) \leq \frac{1}{p^2} J(\delta) \leq \frac{p-1}{4} \rho(\alpha) \|A(\alpha)_\varepsilon (|u|^2 + \delta)^{1/2}\|_{L^p(B)}^p.$$

Letting $\delta \downarrow 0$, we obtain (4.8); the computation stated above is essentially the same as in [13, Proof of Theorem 2.1].

Step 2. Before computing the $\operatorname{Im} I(\delta)$, we note that (4.11) is written as

$$(4.14) \quad \begin{aligned} & |p-2|^2 \left[\operatorname{Re} I(\delta) - \int_{\mathbb{R}^N} f(x, \delta) dx \right] + \frac{|p-2|^2}{p^2} J(\delta) \\ & \geq -|p-2|^2 \int_{\mathbb{R}^N} Q(x) |u(x)|^2 (|u(x)|^2 + \delta)^{(p-4)/2} |\nabla|u(x)||^2 dx. \end{aligned}$$

Setting

$$\begin{aligned} h(x, \delta) &:= \left| \frac{p-2}{2} \nabla|u(x)|^2 + \frac{\nabla Q(x)}{Q(x)} (|u(x)|^2 + \delta) \right|^2 \\ &\quad \times Q(x) (|u(x)|^2 + \delta)^{(p-4)/2}, \end{aligned}$$

we have $|g(x, \delta)| \leq \sqrt{f(x, \delta)} \sqrt{h(x, \delta)}$; note that

$$f(x, \delta) \geq Q(x) |u(x)|^2 (|u(x)|^2 + \delta)^{(p-4)/2} |\nabla u(x)|^2.$$

Therefore the Cauchy-Schwarz inequality yields that

$$(4.15) \quad \left| \int_{\mathbb{R}^N} g(x, \delta) dx \right|^2 \leq \left(\int_{\mathbb{R}^N} f(x, \delta) dx \right) \left(\int_{B(0,R)} h(x, \delta) dx \right).$$

Now we see that (4.10) can be written in terms of $J(\delta)$ and $h(x, \delta)$:

$$\begin{aligned} & 2(p-2) \left[\operatorname{Re} I(\delta) - \int_{\mathbb{R}^N} f(x, \delta) dx \right] + \frac{4(p-1)}{p^2} J(\delta) \\ &= \int_{B(0,R)} h(x, \delta) dx \\ &\quad + |p-2|^2 \int_{\mathbb{R}^N} Q(x) |u(x)|^2 (|u(x)|^2 + \delta)^{(p-4)/2} |\nabla|u(x)||^2 dx. \end{aligned}$$

Combining this equality with (4.14), we obtain

$$(4.16) \quad \int_{B(0,R)} h(x, \delta) dx \leq J(\delta) + p(p-2)[\operatorname{Re} I(\delta) - K(\delta)],$$

where we have set

$$K(\delta) := \int_{\mathbb{R}^N} f(x, \delta) dx.$$

Step 3 [Proof of (4.7)]. First by virtue of (4.9) and (4.15) we have that

$$\begin{aligned} |\operatorname{Im} I(\delta)|^2 &= \left| \operatorname{Im} \int_{\mathbb{R}^N} g(x, \delta) dx \right|^2 = \left| \int_{\mathbb{R}^N} g(x, \delta) dx \right|^2 - \left| \operatorname{Re} \int_{\mathbb{R}^N} g(x, \delta) dx \right|^2 \\ &\leq \left(\int_{B(0,R)} h(x, \delta) dx \right) K(\delta) - |\operatorname{Re} I(\delta) - K(\delta)|^2. \end{aligned}$$

Then we see from (4.16) that

$$\begin{aligned} |\operatorname{Im} I(\delta)|^2 &\leq \{J(\delta) + p(p-2)[\operatorname{Re} I(\delta) - K(\delta)]\} K(\delta) - |\operatorname{Re} I(\delta) - K(\delta)|^2 \\ &= -(p-1)^2 [K(\delta)]^2 \\ &\quad + \{J(\delta) + (p^2 - 2p + 2)\operatorname{Re} I(\delta)\} K(\delta) - |\operatorname{Re} I(\delta)|^2. \end{aligned}$$

By completing the square we have

$$\begin{aligned} |\operatorname{Im} I(\delta)|^2 &\leq \frac{1}{4(p-1)^2} \{J(\delta) + (p^2 - 2p + 2)\operatorname{Re} I(\delta)\}^2 - |\operatorname{Re} I(\delta)|^2 \\ &= \frac{p^2 |p-2|^2}{4(p-1)^2} \left\{ \operatorname{Re} I(\delta) + \frac{1}{p^2} J(\delta) \right\} \left\{ \operatorname{Re} I(\delta) + \frac{1}{|p-2|^2} J(\delta) \right\}. \end{aligned}$$

It then follows from (4.13) that

$$\begin{aligned} |\operatorname{Im} I(\delta)|^2 &\leq \frac{p^2}{4(p-1)} \left\{ \operatorname{Re} I(\delta) + \frac{p-1}{4} \rho(\alpha) \|A(\alpha)_\varepsilon (|u|^2 + \delta)^{1/2}\|_{L^p(B)}^p \right\} \\ &\quad \times \left\{ \frac{|p-2|^2}{p-1} \operatorname{Re} I(\delta) + \frac{p^2}{4} \rho(\alpha) \|A(\alpha)_\varepsilon (|u|^2 + \delta)^{1/2}\|_{L^p(B)}^p \right\}. \end{aligned}$$

Letting $\delta \downarrow 0$, we obtain (4.7). Thus for T and $A(\alpha)_\varepsilon$, **(A1)** is satisfied. \square

Example 1. $V(x) := |x|^{2l}$ ($l \in (-\infty, -1) \cup [1, \infty)$). Then Theorem 4.2 yields that

(a) $\{-\Delta + \kappa |x|^{2l}; \kappa \in \Sigma_p(\infty)^c\}$ forms a holomorphic family of type (A), where $\Sigma_p(\infty)$ is given by the following sector of the κ -plane:

$$\Sigma_p(\infty) = \left\{ x + iy \in \mathbb{C}; |y| \leq -\frac{p|p-2|}{2(p-1)}x, -\infty < x \leq 0 \right\};$$

- (b) $-\Delta + \kappa|x|^{2l}$ is m -accretive in L^p for κ with $\operatorname{Re} \kappa \geq 0$;
 (c) $-\Delta + \kappa|x|^{2l}$ is m -sectorial of type $S(\tan \theta)$ in L^p for $\theta \in [\tan^{-1} c_p, \pi/2)$ and κ with $|\arg \kappa| \leq \theta$.

The proof is divided into two cases.

Case 1: $l \geq 1$ (**Harmonic oscillator type potential**). Put

$$\rho(\alpha) := \frac{4}{27l}(l+1)^{1+1/l}(2l-1)^{2-1/l} \frac{1}{\alpha^{1+1/l}}.$$

Then we have

$$\frac{|\nabla V(x)|^2}{[V(x) + \alpha]^3} = \frac{4l^2|x|^{2(2l-1)}}{(|x|^{2l} + \alpha)^3} \leq \rho(\alpha).$$

In fact, we can obtain $\rho(\alpha)$ as the maximum of $g(t) := 4l^2 t^{2l-1}/(t^l + \alpha)^3$ ($t := |x|^2 \geq 0$). Therefore $\rho(\alpha) \rightarrow 0$ ($\alpha \rightarrow \infty$).

Case 2: $l < -1$. Put $k := -l > 1$. Then the Yosida approximation of $V(x) + \alpha$ is given by

$$(V + \alpha)_\varepsilon(x) = \frac{1 + \alpha|x|^{2k}}{(1 + \alpha\varepsilon)|x|^{2k} + \varepsilon} \quad (\varepsilon > 0).$$

Put

$$\rho(\alpha) := \frac{4}{27k}(k-1)^{1-1/k}(2k+1)^{2+1/k} \frac{1}{\alpha^{1-1/k}}.$$

Then we have

$$\frac{|\nabla(V + \alpha)_\varepsilon(x)|^2}{[(V + \alpha)_\varepsilon(x)]^3} \leq \frac{4k^2|x|^{2(k-1)}}{(1 + \alpha|x|^{2k})^3} \leq \rho(\alpha).$$

In fact, we can obtain $\rho(\alpha)$ as the maximum of $g(t) := 4k^2 t^{k-1}/(1 + \alpha t^k)^3$ ($t := |x|^2 \geq 0$). Therefore $\rho(\alpha) \rightarrow 0$ ($\alpha \rightarrow \infty$). Thus we obtain the same conclusion both in Case 1 and in Case 2.

Example 2 (Inverse square potential). $V(x) := |x|^{-2}$. Then we have

$$\frac{|\nabla(V + \alpha)_\varepsilon(x)|^2}{[(V + \alpha)_\varepsilon(x)]^3} \leq \frac{4}{(1 + \alpha|x|^2)^3} \leq \rho(\alpha) := 4 \rightarrow 4 \quad (\alpha \rightarrow \infty).$$

Hence we see from Theorem 4.2 that

(a) $\{-\Delta + \kappa|x|^{-2}; \kappa \in \Sigma_p(\infty)^c\}$ forms a holomorphic family of type (A), where $\Sigma_p(\infty)$ is given by the following hyperbolic region of the κ -plane:

$$\Sigma_p(\infty) = \{x + iy \in \mathbb{C}; y^2 \leq f(x), \quad -\infty < x \leq p-1\},$$

and f is defined as

$$f(x) := \frac{p^2}{4(p-1)} \left(x - (p-1) \right) \left(\frac{|p-2|^2}{p-1} x - p^2 \right);$$

(b) $-\Delta + \kappa|x|^{-2}$ is m -accretive in L^p for $\kappa \in \Sigma_p(\infty)^c$ with $\operatorname{Re} \kappa \geq 0$;

(c) $-\Delta + \kappa|x|^{-2}$ is m -sectorial of type $S(\tan \theta)$ in L^p for $\theta \in [\tan^{-1} c_p, \pi/2)$ and $\kappa \in \Sigma_p(\infty)^c$ with $|\arg \kappa| \leq \theta$.

§5. More about the case of $-\Delta + \kappa|x|^{-2}$ in L^p

In this section we shed new light on the Schrödinger operator $-\Delta + \kappa|x|^{-2}$ in $L^p = L^p(\mathbb{R}^N)$ through detailed computation ($1 < p < \infty$, $N \in \mathbb{N}$). As stated above, $T = -\Delta$ with domain $D(T) = W^{2,p}(\mathbb{R}^N)$ is m -sectorial in L^p , while $A = |x|^{-2}$ is also m -sectorial as a maximal multiplication operator in L^p and its Yosida approximation is given by

$$A_\varepsilon = A(1 + \varepsilon A)^{-1} = (|x|^2 + \varepsilon)^{-1}, \quad \varepsilon > 0.$$

It is easy to see that condition **(A2)** is satisfied (see the previous section). On the other hand, to derive **(A1)** we have utilized only the quotient

$$|\nabla(|x|^{-2} + \alpha)_\varepsilon|^2 / [(|x|^{-2} + \alpha)_\varepsilon]^3.$$

As its consequence the boundary of $\Sigma_p(\infty)$ was given by the function with constants depending only on p . However, some of those constants are already known to depend also on N . Thus, we shall derive **(A1)** and (3.7) with sharp constants: for $u \in C_0^\infty(\mathbb{R}^N)$,

$$(5.1) \quad \operatorname{Re}(Tu, F(A_\varepsilon u)) \geq -\beta_0(p) \|A_\varepsilon u\|^2,$$

$$(5.2) \quad |\operatorname{Im}(Tu, F(A_\varepsilon u))|^2 \leq \{ \operatorname{Re}(Tu, F(A_\varepsilon u)) + \beta_0(p) \|A_\varepsilon u\|^2 \} \\ \times \left\{ \frac{|p-2|^2}{p-1} \operatorname{Re}(Tu, F(A_\varepsilon u)) + \beta_1(p) \|A_\varepsilon u\|^2 \right\},$$

$$(5.3) \quad \operatorname{Re}(Tu, F(u)) \geq \gamma_0(p)(Au, F(u)) \geq \gamma_0(p)(A_\varepsilon u, F(u)),$$

where $\beta_0(p)$, $\beta_1(p)$ and $\gamma_0(p)$ depend not only on p but also on N :

$$\beta_0(p) := p^{-2}(p-1)(2p-N)N \quad (1 < p < \infty),$$

$$\beta_1(p) := \begin{cases} 4 + 2N(p-2) & (2 \leq p < \infty), \\ \frac{4}{p-1} + \frac{2N}{p}(2-p)^2 & (1 < p < 2), \end{cases}$$

$$\gamma_0(p) := p^{-2}(p-1)(N-2)^2 \quad (N \geq 3).$$

Note that $\beta_0(p)$ and $\gamma_0(p)$ are optimal (see [15, Section 5.2]).

Then as a consequence of Theorem 3.3 we can state the following

Theorem 5.1. *Let $T = -\Delta$, $A = |x|^{-2}$ and*

$$\Sigma_p = \left\{ x + iy \in \mathbb{C}; y^2 \leq (x - \beta_0(p)) \left[\frac{|p-2|^2}{p-1} x - \beta_1(p) \right], -\infty < x \leq \beta_0(p) \right\}.$$

(i) *Let $2p \geq N$. Then $T + \kappa A = -\Delta + \kappa|x|^{-2}$ is closed for $\kappa \in \Sigma_p^c$, with*

$$\| |x|^{-2} u \| \leq \text{dist}(\kappa, \Sigma_p)^{-1} \| (-\Delta + \kappa|x|^{-2}) u \|, \quad u \in W^{2,p}(\mathbb{R}^N) \cap D(|x|^{-2}),$$

and hence $\{T + \kappa A; \kappa \in \Sigma_p^c\}$ forms a holomorphic family of type (A).

(ii) *Let $2p < N$. Then $-\beta_0(p) > 0$ so that A is T -bounded with T -bound $[-\beta_0(p)]^{-1}$:*

$$\| |x|^{-2} u \| \leq \frac{p^2}{(p-1)(N-2p)N} \| (-\Delta) u \|, \quad u \in W^{2,p}(\mathbb{R}^N),$$

and hence $\{T + \kappa A; \kappa \in \Sigma_p^c\}$ forms a holomorphic family of type (A).

(iii) *If $N = 1, 2$, then $T + \kappa A$ is m -accretive in L^p for $\kappa \in \Sigma_p^c$ with $\text{Re } \kappa \geq 0$.*

(iv) *$T + \kappa A$ is m -sectorial of type $S(\tan \theta)$ in L^p for $\theta \in [\tan^{-1} c_p, \pi/2)$ and $\kappa \in \Sigma_p^c$ with $|\arg \kappa| \leq \theta$.*

(v) *$\mathbb{R}_- = (-\infty, 0)$ belongs to the resolvent set of $T + \kappa A$ for $\kappa \in \Sigma_p^c$.*

(vi) *If $N \geq 3$, then $T + \kappa A$ is m -accretive in L^p for $\kappa \in \Sigma_p^c$ with $\text{Re } \kappa \geq -\gamma_0(p)$.*

The conclusion of Theorem 5.1 is a considerable improvement of Example 2. In fact, let Σ_p and $\Sigma_p(\infty)$ be as in Theorem 5.1 and Example 2, respectively. Then, since $\beta_0(p) - (p-1) = -p^{-2}(p-1)(p-N)^2 \leq 0$, we see from the computation of asymptotes that

$$\Sigma_p(\infty)^c \subset \Sigma_p^c.$$

Actually, both (5.1) and (5.3) had already been proved in [15, Lemma 3.5 and Theorem 2.4]. As for (5.1) we shall give a simplified proof in Lemma 5.3 below. In this connection we note that for $N \geq 3$,

$$\begin{aligned} -\gamma_0(p) &\leq \beta_0(p) \quad \text{when } 2 - \frac{2}{N} \leq p < \infty, \\ -\gamma_0(p) &\geq \beta_0(p) \quad \text{when } 1 < p \leq 2 - \frac{2}{N}. \end{aligned}$$

Thus, in order to prove Theorem 5.1 it remains to prove the most complicated inequality (5.2). This will be done in Lemma 5.6 below.

The next lemma is a new version of Hardy's inequality.

Lemma 5.2. *Let $v \in C_0^1(\mathbb{R}^N)$ and $\delta > 0$. Suppose that $\text{supp } v \subset B(0, R)$ for some $R > 0$. Then one has*

$$(5.4) \quad \frac{N^2}{p^2} \int_{B(0,R)} (|v(x)|^2 + \delta)^{p/2} dx + \frac{2N^2\delta}{p(2-p)} \int_{B(0,R)} (|v(x)|^2 + \delta)^{(p-2)/2} dx \\ \leq \int_{\mathbb{R}^N} |x|^2 |v(x)|^2 (|v(x)|^2 + \delta)^{(p-4)/2} |\nabla v(x)|^2 dx.$$

Consequently, if $1 < p < 2$ then

$$(5.5) \quad \int_{\mathbb{R}^N} |x|^2 |v(x)|^2 (|v(x)|^2 + \delta)^{(p-4)/2} |\nabla v(x)|^2 dx \\ \geq \frac{N^2}{p^2} \int_{B(0,R)} (|v(x)|^2 + \delta)^{p/2} dx.$$

On the other hand, if $p \geq 2$ then one can set $\delta = 0$ in (5.4):

$$(5.6) \quad \int_{\mathbb{R}^N} |x|^2 |v(x)|^{p-2} |\nabla v(x)|^2 dx \geq \frac{N^2}{p^2} \int_{\mathbb{R}^N} |v(x)|^p dx.$$

These inequalities remain true even if v is replaced with $|v| \in W^{1,p}(\mathbb{R}^N) \cap C_0(\mathbb{R}^N)$.

Remark 4. The right-hand side of (5.5) can be replaced with $(N/p)^2 \|v\|^p$. However, we need (5.5) itself at the end of the proof of Lemma 5.3.

Proof of Lemma 5.2. Let $v \in C_0^1(\mathbb{R}^N)$ and $\delta > 0$. Then we have

$$0 \leq \int_{\mathbb{R}^N} |x|^2 |v(x)|^2 (|v(x)|^2 + \delta)^{(p-4)/2} \left| \nabla v(x) + \frac{N}{p} \frac{x}{|x|^2} v(x) \right|^2 dx \\ \leq \int_{\mathbb{R}^N} |x|^2 |v(x)|^2 (|v(x)|^2 + \delta)^{(p-4)/2} |\nabla v(x)|^2 dx \\ + \frac{N^2}{p^2} \int_{B(0,R)} (|v(x)|^2 + \delta)^{p/2} dx \\ + \frac{N}{p} \int_{\mathbb{R}^N} |v(x)|^2 (|v(x)|^2 + \delta)^{(p-4)/2} (x \cdot \nabla) |v(x)|^2 dx.$$

The integrand of the third term on the right-hand side is rewritten as

$$|v(x)|^2 (|v(x)|^2 + \delta)^{(p-4)/2} (x \cdot \nabla) |v(x)|^2 \\ = (|v(x)|^2 + \delta)^{(p-2)/2} (x \cdot \nabla) |v(x)|^2 \\ - \delta (|v(x)|^2 + \delta)^{(p-4)/2} (x \cdot \nabla) |v(x)|^2 \\ = \frac{2}{p} (x \cdot \nabla) (|v(x)|^2 + \delta)^{p/2} - \frac{2\delta}{p-2} (x \cdot \nabla) (|v(x)|^2 + \delta)^{(p-2)/2}.$$

Therefore integration by parts gives

$$\begin{aligned} & \frac{N}{p} \int_{\mathbb{R}^N} |v(x)|^2 (|v(x)|^2 + \delta)^{(p-4)/2} (x \cdot \nabla) |v(x)|^2 dx \\ &= -\frac{2N^2}{p^2} \int_{B(0,R)} (|v(x)|^2 + \delta)^{p/2} dx \\ & \quad - \frac{2N^2\delta}{p(2-p)} \int_{B(0,R)} (|v(x)|^2 + \delta)^{(p-2)/2} dx. \end{aligned}$$

Thus we obtain (5.4). \square

Lemma 5.3. *Let $T = -\Delta$ and $A = |x|^{-2}$. Then (5.1) holds:*

$$\operatorname{Re}(Tu, F(A_\varepsilon u)) \geq -\beta_0(p) \|A_\varepsilon u\|^2, \quad u \in C_0^\infty(\mathbb{R}^N), \quad \varepsilon > 0.$$

Proof. For $u \in C_0^\infty(\mathbb{R}^N)$ and $\varepsilon > 0$ let $v := (|x|^2 + \varepsilon)^{-1}u = A_\varepsilon u \in C_0^\infty(\mathbb{R}^N)$. Then we can express Tu in terms of v as follows:

$$Tu = TA_\varepsilon^{-1}v = -2Nv(x) - 2(x \cdot \nabla)v(x) - \operatorname{div}[(|x|^2 + \varepsilon)\nabla v(x)].$$

Therefore, instead of (5.1), we shall show that

$$(5.7) \quad \operatorname{Re} I_0(\delta) \geq -\beta_0(p) \|(|v|^2 + \delta)^{1/2}\|_{L^p(B)}^p,$$

where we assume that $\operatorname{supp} v \subset B = B(0, R)$ for some $R > 0$ and

$$(5.8) \quad I_0(\delta) := (TA_\varepsilon^{-1}v, (|v|^2 + \delta)^{(p-2)/2}v).$$

In addition $\delta > 0$ if $1 < p < 2$, while $\delta = 0$ if $p \geq 2$. First we have

$$\begin{aligned} (5.9) \quad & I_0(\delta) + 2N \int_{\mathbb{R}^N} |v(x)|^2 (|v(x)|^2 + \delta)^{(p-2)/2} dx \\ &= \int_{\mathbb{R}^N} (f_{01}(x, \delta) + g_0(x, \delta)) dx \\ & \quad + \delta \int_{\mathbb{R}^N} (|x|^2 + \varepsilon) (|v(x)|^2 + \delta)^{(p-4)/2} |\nabla v(x)|^2 dx, \end{aligned}$$

where $f_{0j}(x, \delta)$ ($j = 1, 2$) and $g_0(x, \delta)$ are defined as follows:

$$\begin{aligned} f_{01}(x, \delta) &:= (|x|^2 + \varepsilon) |v(x)|^2 (|v(x)|^2 + \delta)^{(p-4)/2} |\nabla v(x)|^2 \\ &\geq (|x|^2 + \varepsilon) |v(x)|^2 (|v(x)|^2 + \delta)^{(p-4)/2} |\nabla |v(x)||^2 =: f_{02}(x, \delta), \\ g_0(x, \delta) &:= \left\{ -2(|v(x)|^2 + \delta)x + \frac{p-2}{2}(|x|^2 + \varepsilon)\nabla |v(x)|^2 \right\} \\ &\quad \cdot (|v(x)|^2 + \delta)^{(p-4)/2} \overline{v(x)} \nabla v(x). \end{aligned}$$

The definition of $g_0(x, \delta)$ and (5.9) respectively yield

$$(5.10) \quad \operatorname{Re} \int_{\mathbb{R}^N} g_0(x, \delta) dx = \frac{2N}{p} \|(|v|^2 + \delta)^{1/2}\|_{L^p(B)}^p \\ + (p-2) \int_{\mathbb{R}^N} f_{02}(x, \delta) dx,$$

$$(5.11) \quad \operatorname{Im} \int_{\mathbb{R}^N} g_0(x, \delta) dx = \operatorname{Im} I_0(\delta).$$

Now it follows from (5.9) and (5.10) that

$$\begin{aligned} & \operatorname{Re} I_0(\delta) + \frac{2(p-1)}{p} N \|(|v|^2 + \delta)^{1/2}\|_{L^p(B)}^p \\ & \geq \int_{\mathbb{R}^N} (f_{01}(x, \delta) + (p-2)f_{02}(x, \delta)) dx \\ & = (p-1) \int_{\mathbb{R}^N} f_{01}(x, \delta) dx - (p-2) \int_{\mathbb{R}^N} (f_{01}(x, \delta) - f_{02}(x, \delta)) dx \\ & = (p-1) \int_{\mathbb{R}^N} f_{02}(x, \delta) dx + \int_{\mathbb{R}^N} (f_{01}(x, \delta) - f_{02}(x, \delta)) dx. \end{aligned}$$

Thus (5.5) and (5.6) with v replaced with $|v|$ apply to give

$$\begin{aligned} & \operatorname{Re} I_0(\delta) + \frac{2(p-1)}{p} N \|(|v|^2 + \delta)^{1/2}\|_{L^p(B)}^p \\ & \quad - \int_{\mathbb{R}^N} (f_{01}(x, \delta) - f_{02}(x, \delta)) dx \\ & \geq (p-1) \int_{\mathbb{R}^N} |x|^2 |v(x)|^2 (|v(x)|^2 + \delta)^{(p-4)/2} |\nabla |v(x)||^2 dx \\ & \geq \frac{p-1}{p^2} N^2 \|(|v|^2 + \delta)^{1/2}\|_{L^p(B)}^p. \end{aligned}$$

Therefore we obtain a useful inequality

$$\operatorname{Re} I_0(\delta) + \beta_0(p) \|(|v|^2 + \delta)^{1/2}\|_{L^p(B)}^p \geq \int_{\mathbb{R}^N} (f_{01}(x, \delta) - f_{02}(x, \delta)) dx \geq 0,$$

which is more precise than (5.7). Letting $\delta \downarrow 0$, we have

$$\operatorname{Re} (TA_\varepsilon^{-1}v, |v|^{p-2}v) = \lim_{\delta \downarrow 0} \operatorname{Re} I_0(\delta) \geq -\beta_0(p) \|v\|^p.$$

This is equivalent to (5.1). □

Now set

$$\begin{aligned} J_0(\delta) &:= \operatorname{Re} I_0(\delta) + \beta_0(p) \|(|v|^2 + \delta)^{1/2}\|_{L^p(B)}^p, \\ K_0(\delta) &:= \int_{\mathbb{R}^N} (f_{01}(x, \delta) - f_{02}(x, \delta)) \, dx \\ &= \int_{\mathbb{R}^N} (|x|^2 + \varepsilon) |v(x)|^2 (|v(x)|^2 + \delta)^{(p-4)/2} (|\nabla v(x)|^2 - |\nabla |v(x)||^2) \, dx. \end{aligned}$$

In the proof of the previous lemma we have established the following

Lemma 5.4. *For $v \in C_0^\infty(\mathbb{R}^N)$ let $f_{0j}(x, \delta)$ be as in the proof of Lemma 5.3 ($j = 1, 2$). Then one has*

$$(5.12) \quad \int_{\mathbb{R}^N} f_{01}(x, \delta) \, dx - \left(\frac{N}{p}\right)^2 \|(|v|^2 + \delta)^{1/2}\|_{L^p(B)}^p \leq \frac{J_0(\delta)}{p-1} + \frac{p-2}{p-1} K_0(\delta);$$

$$(5.13) \quad \frac{K_0(\delta)}{p-1} + \int_{\mathbb{R}^N} f_{02}(x, \delta) \, dx \leq \frac{\operatorname{Re} I_0(\delta)}{p-1} + \frac{2N}{p} \|(|v|^2 + \delta)^{1/2}\|_{L^p(B)}^p;$$

$$(5.14) \quad 0 \leq K_0(\delta) \leq J_0(\delta).$$

Next we can prepare a lemma in terms of $f_{0j}(x, \delta)$ and $K_0(\delta)$ that will be basic for the precise estimate of $\operatorname{Im}(TA_\varepsilon^{-1}v, |v|^{p-2}v)$.

Lemma 5.5. *For $v \in C_0^\infty(\mathbb{R}^N)$ let $f_{0j}(x, \delta)$, $g_0(x, \delta)$ be as in the proof of Lemma 5.3 ($j = 1, 2$). Set*

$$\begin{aligned} h_0(x, \delta) &:= \left| \frac{p-2}{2} (|x|^2 + \varepsilon)^{1/2} \nabla |v(x)|^2 - 2 \frac{|v(x)|^2 + \delta}{(|x|^2 + \varepsilon)^{1/2}} x \right|^2 \\ &\quad \times (|v(x)|^2 + \delta)^{(p-4)/2}. \end{aligned}$$

Assume that $\operatorname{supp} v \subset B(0, R)$ for some $R > 0$. Then one has

$$\begin{aligned} (5.15) \quad &\int_{B(0, R)} h_0(x, \delta) \, dx \\ &\leq \left[4 + \frac{4N}{p} (p-2) \right] \|(|v|^2 + \delta)^{1/2}\|_{L^p(B)}^p + |p-2|^2 \int_{\mathbb{R}^N} f_{02}(x, \delta) \, dx; \end{aligned}$$

$$\begin{aligned} (5.16) \quad &\left| \operatorname{Re} \int_{\mathbb{R}^N} g_0(x, \delta) \, dx \right|^2 \\ &\leq 4 \|(|v|^2 + \delta)^{1/2}\|_{L^p(B)}^p \left[\int_{\mathbb{R}^N} f_{01}(x, \delta) \, dx - \left(\frac{N}{p}\right)^2 \|(|v|^2 + \delta)^{1/2}\|_{L^p(B)}^p \right] \\ &\quad + \left[\frac{4N}{p} (p-2) \|(|v|^2 + \delta)^{1/2}\|_{L^p(B)}^p + |p-2|^2 \int_{\mathbb{R}^N} f_{02}(x, \delta) \, dx \right] K_0(\delta), \end{aligned}$$

where we have set

$$(F_1 H)(\delta) := \left(\int_{\mathbb{R}^N} f_{01}(x, \delta) dx \right) \left(\int_{B(0, R)} h_0(x, \delta) dx \right).$$

Proof. It follows from the definition of $h_0(x, \delta)$ that

$$\begin{aligned} h_0(x, \delta) &= |p-2|^2 f_{02}(x, \delta) \\ &= \frac{4|x|^2}{|x|^2 + \varepsilon} (|v(x)|^2 + \delta)^{p/2} - \frac{4}{p} (p-2)(x \cdot \nabla)(|v(x)|^2 + \delta)^{p/2}. \end{aligned}$$

Integrating this equality over the ball $B(0, R)$, we obtain (5.15). Then it follows from (5.15) and the definition of $K_0(\delta)$ that

$$\begin{aligned} &(F_1 H)(\delta) \\ &\leq \left[4 + \frac{4N}{p}(p-2) \right] \|(|v|^2 + \delta)^{1/2}\|_{L^p(B)}^p \left(\int_{\mathbb{R}^N} f_{01}(x, \delta) dx \right) \\ &\quad + |p-2|^2 \left(\int_{\mathbb{R}^N} f_{01}(x, \delta) dx \right) \left(\int_{\mathbb{R}^N} f_{02}(x, \delta) dx \right) \\ &= 4 \|(|v|^2 + \delta)^{1/2}\|_{L^p(B)}^p \left(\int_{\mathbb{R}^N} f_{01}(x, \delta) dx \right) - \left(\frac{2N}{p} \right)^2 \|(|v|^2 + \delta)^{1/2}\|_{L^p(B)}^{2p} \\ &\quad + \left[\frac{4N}{p}(p-2) \|(|v|^2 + \delta)^{1/2}\|_{L^p(B)}^p + |p-2|^2 \int_{\mathbb{R}^N} f_{02}(x, \delta) dx \right] K_0(\delta) \\ &\quad + \left| \frac{2N}{p} \|(|v|^2 + \delta)^{1/2}\|_{L^p(B)}^p + (p-2) \left(\int_{\mathbb{R}^N} f_{02}(x, \delta) dx \right) \right|^2. \end{aligned}$$

Therefore (5.16) is a consequence of (5.10). \square

Now we are in a position to derive the estimate of $\text{Im}(TA_\varepsilon^{-1}v, |v|^{p-2}v)$ in terms of $\text{Re}(TA_\varepsilon^{-1}v, |v|^{p-2}v)$.

Lemma 5.6. *Let $T = -\Delta$ and $A = |x|^{-2}$. Then the inequality (5.2) holds: for all $u \in C_0^\infty(\mathbb{R}^N)$, $\varepsilon > 0$,*

$$\begin{aligned} |\text{Im}(Tu, F(A_\varepsilon u))|^2 &\leq \{ \text{Re}(Tu, F(A_\varepsilon u)) + \beta_0(p) \|A_\varepsilon u\|^2 \} \\ &\quad \times \left\{ \frac{|p-2|^2}{p-1} \text{Re}(Tu, F(A_\varepsilon u)) + \beta_1(p) \|A_\varepsilon u\|^2 \right\}. \end{aligned}$$

Proof. As in the proof of Lemma 5.3 we set $v := (|x|^2 + \varepsilon)^{-1}u = A_\varepsilon u \in C_0^\infty(\mathbb{R}^N)$ for $u \in C_0^\infty(\mathbb{R}^N)$. Let $I_0(\delta)$ be as defined in (5.8). Then we shall show that for v with $\text{supp } v \subset B = B(0, R)$ for some $R > 0$,

$$\begin{aligned} (5.17) \quad |\text{Im } I_0(\delta)|^2 &\leq \{ \text{Re } I_0(\delta) + \beta_0(p) \|(|v|^2 + \delta)^{1/2}\|_{L^p(B)}^p \} \\ &\quad \times \left\{ \frac{|p-2|^2}{p-1} \text{Re } I_0(\delta) + \beta_1(p) \|(|v|^2 + \delta)^{1/2}\|_{L^p(B)}^p \right\}. \end{aligned}$$

Since $|g_0(x, \delta)| \leq \sqrt{f_{01}(x, \delta)}\sqrt{h_0(x, \delta)}$, the Cauchy-Schwarz inequality yields

$$(5.18) \quad \left| \int_{\mathbb{R}^N} g_0(x, \delta) dx \right|^2 \leq (F_1 H)(\delta).$$

It then follows from (5.12) and (5.16) that

$$(5.19) \quad \begin{aligned} & (F_1 H)(\delta) - \left| \operatorname{Re} \int_{\mathbb{R}^N} g_0(x, \delta) dx \right|^2 \\ & \leq \frac{4}{p-1} \|(|v|^2 + \delta)^{1/2}\|_{L^p(B)}^p J_0(\delta) \\ & \quad + \left[4 \left(\frac{N}{p} + \frac{1}{p-1} \right) (p-2) \|(|v|^2 + \delta)^{1/2}\|_{L^p(B)}^p \right. \\ & \quad \left. + |p-2|^2 \int_{\mathbb{R}^N} f_{02}(x, \delta) dx \right] K_0(\delta). \end{aligned}$$

In this way we can compute $|\operatorname{Im} I_0(\delta)|$. In fact, (5.11) and (5.18) yield that

$$\begin{aligned} |\operatorname{Im} I_0(\delta)|^2 &= \left| \int_{\mathbb{R}^N} g_0(x, \delta) dx \right|^2 - \left| \operatorname{Re} \int_{\mathbb{R}^N} g_0(x, \delta) dx \right|^2 \\ &\leq (F_1 H)(\delta) - \left| \operatorname{Re} \int_{\mathbb{R}^N} g_0(x, \delta) dx \right|^2. \end{aligned}$$

Therefore we see from (5.19) that

$$(5.20) \quad \begin{aligned} |\operatorname{Im} I_0(\delta)|^2 &\leq \frac{4}{p-1} \|(|v|^2 + \delta)^{1/2}\|_{L^p(B)}^p J_0(\delta) \\ &\quad + 4 \left(\frac{N}{p} + \frac{1}{p-1} \right) (p-2) \|(|v|^2 + \delta)^{1/2}\|_{L^p(B)}^p K_0(\delta) \\ &\quad + |p-2|^2 \left\{ \frac{K_0(\delta)}{p-1} + \int_{\mathbb{R}^N} f_{02}(x, \delta) dx \right\} K_0(\delta) \\ &\quad - \frac{|p-2|^2}{p-1} [K_0(\delta)]^2. \end{aligned}$$

Applying (5.13) to the third term (and dropping the fourth term) on the right-hand side of (5.20), we have

$$(5.21) \quad \begin{aligned} |\operatorname{Im} I_0(\delta)|^2 &\leq \frac{4}{p-1} \|(|v|^2 + \delta)^{1/2}\|_{L^p(B)}^p J_0(\delta) \\ &\quad + 4 \left(\frac{N}{p} + \frac{1}{p-1} \right) (p-2) \|(|v|^2 + \delta)^{1/2}\|_{L^p(B)}^p K_0(\delta) \\ &\quad + |p-2|^2 \left\{ \frac{\operatorname{Re} I_0(\delta)}{p-1} + \frac{2N}{p} \|(|v|^2 + \delta)^{1/2}\|_{L^p(B)}^p \right\} K_0(\delta). \end{aligned}$$

Now in order to use (5.14) we have to divide our computation into two cases. In fact, the second term on the right-hand side of (5.21) contains the factor $(p-2)$ so that the computation depends on its sign.

Case 1: $2 \leq p < \infty$. Here we can set $\delta = 0$ and apply (5.14) to (or replace $K_0(0)$ with $J_0(0)$ in) the second and third terms on the right-hand side of (5.21). Then $|\operatorname{Im} I_0(0)|^2$ is less than or equal to $J_0(0)$ multiplied by

$$\begin{aligned} & \frac{4}{p-1} \|v\|^p + \frac{|p-2|^2}{p-1} \operatorname{Re} I_0(0) \\ & + 2 \left(\frac{2N}{p} + \frac{2}{p-1} + \frac{N}{p} (p-2) \right) (p-2) \|v\|^p \\ & = \frac{(p-2)^2}{p-1} \operatorname{Re} I_0(0) + [4 + 2N(p-2)] \|v\|^p. \end{aligned}$$

Thus we obtain (5.17) (with $p \geq 2$ and $\delta = 0$) which is equivalent to (5.2).

Case 2: $1 < p < 2$. In this case we note that the second term on the right-hand side of (5.21) is nonpositive. So we have

$$\begin{aligned} |\operatorname{Im} I_0(\delta)|^2 & \leq \frac{4}{p-1} \|(|v|^2 + \delta)^{1/2}\|_{L^p(B)}^p J_0(\delta) \\ & + |p-2|^2 \left\{ \frac{\operatorname{Re} I_0(\delta)}{p-1} + \frac{2N}{p} \|(|v|^2 + \delta)^{1/2}\|_{L^p(B)}^p \right\} K_0(\delta). \end{aligned}$$

Now (5.14) yields that $|\operatorname{Im} I_0(\delta)|^2$ is less than or equal to $J_0(\delta)$ multiplied by

$$\begin{aligned} & \frac{4}{p-1} \|(|v|^2 + \delta)^{1/2}\|_{L^p(B)}^p + \frac{|p-2|^2}{p-1} \operatorname{Re} I_0(\delta) \\ & + \frac{2N}{p} |p-2|^2 \|(|v|^2 + \delta)^{1/2}\|_{L^p(B)}^p \\ & = \frac{(2-p)^2}{p-1} \operatorname{Re} I_0(\delta) + \left[\frac{4}{p-1} + \frac{2N}{p} (2-p)^2 \right] \|(|v|^2 + \delta)^{1/2}\|_{L^p(B)}^p. \end{aligned}$$

Letting $\delta \downarrow 0$, we can obtain the inequality which is equivalent to (5.2) with $p \in (1, 2)$. \square

§6. Concluding remark

Here we want to give a rough description of the solvability of the Cauchy problem for $N \geq 3$:

$$(6.1) \quad \begin{cases} \frac{\partial u}{\partial t} - \Delta u + \frac{\kappa}{|x|^2} u = 0, & (x, t) \in \mathbb{R}^N \times (0, \infty), \\ u(x, 0) = u_0(x), & x \in \mathbb{R}^N \end{cases}$$

together with a conjecture. Let Σ_p be the region as in Section 5. Then, since $T + \kappa A = -\Delta + \kappa|x|^{-2}$ is m -accretive in L^p for $\kappa \in \Sigma_p^c$ with $\operatorname{Re} \kappa \geq -\gamma_0(p)$, $-(T + \kappa A)$ generates a contraction semigroup $\{\exp(-t(T + \kappa A)); t \geq 0\}$ on L^p . Therefore for every $u_0 \in W^{2,p}(\mathbb{R}^N) \cap D(|x|^{-2})$ a unique solution to (6.1) is given by $\exp(-t(T + \kappa A))u_0$. Here we note that

$$-\gamma_0(2) = -\frac{(N-2)^2}{4} \leq -\frac{(N-2)^2}{pp'} = -\gamma_0(p) \quad \forall p \in [1, \infty).$$

In fact, we have

$$\frac{1}{pp'} \leq \frac{p-1}{p^2} + \left(\frac{1}{p} - \frac{1}{2}\right)^2 = \frac{1}{4}.$$

Thus the lower bound of $\operatorname{Re} \kappa$ for which (6.1) is solvable may be given by $-\gamma_0(2)$. Therefore we may restrict ourselves to the case of $p = 2$.

Let $\kappa \in \Sigma_2$ with $\operatorname{Re} \kappa \geq -\gamma_0(2)$ and assume that there exists θ with $|\theta| < \pi/2$ such that

$$|\operatorname{Im} \kappa| \leq (\tan \theta)(\operatorname{Re} \kappa + \gamma_0(2)).$$

Then we can show that $T + \kappa A$ is sectorial in L^2 . In fact, it follows from (5.3) that

$$\begin{aligned} |\operatorname{Im}((T + \kappa A)u, u)| &= |\operatorname{Im} \kappa|(Au, u) \\ &\leq (\tan \theta)(\operatorname{Re} \kappa + \gamma_0(2))(Au, u) \\ &\leq (\tan \theta)((Au, u)\operatorname{Re} \kappa + (Tu, u)) \\ &= (\tan \theta)\operatorname{Re}((T + \kappa A)u, u). \end{aligned}$$

Now let F be the Friedrichs extension of $T + \kappa A$ (see, e.g., [7] or [17]). Then F is m -sectorial in L^2 and hence $-F$ generates an analytic contraction semigroup $\{\exp(-tF); t \geq 0\}$ on L^2 . Therefore a unique solution to (6.1) is given by $\exp(-tF)u_0$ for every $u_0 \in L^2$. Though the argument is not perfect, the above observation roughly explains the solvability of (6.1) for κ with $\operatorname{Re} \kappa \geq -\gamma_0(2)$.

On the other hand, Baras-Goldstein [1] have shown that there exists no solution to (6.1) with $u_0 \geq 0$ ($u_0 \not\equiv 0$) for $\kappa \in \mathbb{R}$ with $\kappa < -\gamma_0(2)$. Thus we may conjecture that there exists no solution to (6.1) with $u_0 \geq 0$ ($u_0 \not\equiv 0$) for $\kappa \in \mathbb{C}$ with $\operatorname{Re} \kappa < -\gamma_0(2)$.

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