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3-dimensional quasi-Sasakian manifolds and Ricci solitons

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Abstract. The object of the present paper is to obtain a necessary and sufficient condition for a 3-dimensional quasi-Sasakian manifold to be an $\eta-$Einstein manifold. An example is given to verify the theorem. Finally Ricci solitons and gradient Ricci solitons have been studied.

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§1. Introduction

The notion of quasi-Sasakian structure was introduced by D. E. Blair [4] to unify Sasakian and cosymplectic structures. S. Tanno [24] also added some remarks on quasi-Sasakian structures. The properties of quasi-Sasakian manifolds have been studied by several authors, viz., J. C. Gonzalez and D. Chinea [13], S. Kanemaki [16], [17] and J. A. Oubina [22], De and Sarkar [9], De and Mondal [8]. B. H. Kim [18] studied quasi-Sasakian manifolds and proved that every fibred Riemannian spaces with invariant fibres normal to the structure vector field do not admit nearly Sasakian or contact structure but a quasi-Sasakian or cosymplectic structure. Recently, quasi-Sasakian manifolds have been the subject of growing interest in view of finding the significant applications to physics, in particular to super gravity and magnetic theory [1], [2]. Quasi-Sasakian structures have wide applications in the mathematical analysis of string theory [3], [12]. Motivated by the roles of curvature tensor and Ricci tensor of quasi-Sasakian manifolds in string theory [3] we would like to study some curvature properties and Ricci soliton in a 3-dimensional quasi-Sasakian manifold. On a 3-dimensional quasi-Sasakian manifold, the structure function β was defined by Z. Olszak [20] and with the help of this function he has obtained necessary and sufficient conditions for the manifold to be conformally flat [21]. Next he has proved that if the manifold is additionally conformally flat with $\beta =$ constant, then (a) the manifold is locally a product of R and a two-dimensional Kaehlerian space of constant Gauss curvature (the cosymplectic case), or, (b) the manifold is of constant positive curvature (the non-cosymplectic case, here the quasi-Sasakian structure is homothetic to a Sasakian structure). It is also known that $D-$homothetic and homothetic deformations of (quasi-) Sasakian structures lead to quasi-Sasakian structures [19]. In dimension 3, certain $D-$conformal deformations of Sasakian structures yield quasi-Sasakian and non-Sasakian structures of rank 3 [24].

A Ricci soliton is a generalization of an Einstein metric. We recall the notion of Ricci soliton according to [6]. On the manifold M, a Ricci soliton is a triple (g, V, λ) with g, a Riemannian metric, V a vector field and λ a real scalar such that

$$\pounds_V g + 2S + 2\lambda g = 0 \tag{1.1}$$

where $\pounds$ is the Lie derivative. The Ricci soliton is said to be shrinking, steady and expanding according as λ is negative, zero and positive. If the vector field V is the gradient of a potential function $-f$, then g is called a gradient Ricci soliton and equation (1.1) takes the form

$$\nabla\nabla f = S + \lambda g, \tag{1.2}$$

where ∇ denotes the Riemannian connection.

A Ricci soliton on a compact manifold has constant curvature in dimension 2 (Hamilton [14]), and also in dimension 3 (Ivey [15]). For details we refer to Chow and Knoff [7] and Derdzinski [11]. Recently in [6] C. Calin and M. Crasmareanu have studied Ricci solitons in $f-$Kenmotsu manifolds. We also recall the following significant result of Perelman [23]: A Ricci soliton on a compact manifold is a gradient Ricci soliton.

On the other hand, the roots of contact geometry lie in differential equations as in 1872 Sophus Lie introduced the notion of contact transformation as a geometric tool to study systems of differential equations. This subject has manifold connections with the other fields of pure mathematics , and substantial applications in applied areas such as mechanics, optic, phase space of dynamical system, thermodynamics and control theory.

The paper is organized as follows: After preliminaries in section 3 we obtain a necessary and sufficient condition for a 3-dimensional quasi-Sasakian manifold to be an $\eta-$Einstein manifold and also verify the result by a concrete example. In the last section we study Ricci solitons and gradient Ricci solitons in 3-dimensional quasi-Sasakian manifold and prove that in a 3-dimensional non-cosymplectic quasi-Sasakian manifold, the Ricci soliton (g, ξ, λ) is expanding provided β is constant and the manifold is of constant curvature. Finally we prove that if the metric g of a 3-dimensional quasi-Sasakian manifold with constant structure function β is a gradient Ricci soliton, then the manifold is an Einstein manifold.

§2. Preliminaries

Let M be a (2n+1)-dimensional connected differentiable manifold endowed with an almost contact metric structure (ϕ, ξ, η, g), where ϕ, ξ, η are tensor fields on M of types $(1,1)$, $(1,0)$, $(0,1)$ respectively, such that [5]

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1. \tag{2.1}$$

Then also

$$\phi\xi = 0, \quad \eta \circ \phi = 0, \quad \eta(X) = g(X, \xi). \tag{2.2}$$

Let Φ be fundamental 2-form of M defined by

$$\Phi(X, Y) = g(X, \phi Y) \qquad X, Y \varepsilon T(M).$$

M is said to be quasi-Sasakian if the almost contact structure (ϕ, ξ, η) is normal and the fundamental 2-form Φ is closed ($d\Phi = 0$), which was first introduced by Blair [4]. The normality condition gives that the induced almost complex structure of $M \times \mathbb{R}$ is integrable or equivalently, the torsion tensor field $N = [\phi, \phi] + 2\xi \otimes d\eta$ vanishes identically on M. The rank of a quasi-Sasakian structure is always an odd integer [4] which is equal to 1 if the structure is cosymplectic and it is equal to $(2n+1)$ if the structure is Sasakian.

A Riemannian manifold M is said to be an $\eta-$Einstein manifold if it satisfies the condition

$$S(X, Y) = \lambda g(X, Y) + \delta\eta(X)\eta(Y), \tag{2.3}$$

where λ and δ are smooth functions on the manifold.

In [10] De and Sengupta prove the following:

Lemma 1. *A parallel symmetric $(0,2)$ tensor field in a 3-dimensional non-cosymplectic quasi-Sasakian manifold is a constant multiple of the associated metric tensor.*

§3. 3-dimensional quasi-Sasakian manifold

An almost contact metric manifold M is a 3-dimensional quasi-Sasakian manifold if and only if [20]

$$\nabla_X \xi = -\beta\phi X, \quad X \varepsilon T(M), \tag{3.1}$$

for a certain function β on M, such that $\xi\beta = 0$, ∇ being the operator of the covariant differentiation with respect to the Levi-Civita connection of M. Clearly, such a quasi-Sasakian manifold is cosymplectic if and only if $\beta = 0$. If $\beta = constant$, then the manifold reduces to a $\beta-$Sasakian manifold and if in particular $\beta = 1$, the manifold becomes a Sasakian manifold. Here we have shown that the assumption $\xi\beta = 0$ is not necessary.

As a consequence of (3.1), we have [20]

$$(\nabla_X \phi)(Y) = \beta(g(X,Y)\xi - \eta(Y)X), \quad X, Y \varepsilon T(M). \tag{3.2}$$

Because of (3.1) and (3.2), we find

$$\nabla_X(\nabla_Y \xi) = -(X\beta)\phi Y - \beta^2\{g(X,Y)\xi - \eta(Y)X\} - \beta\phi(\nabla_X Y)$$

which implies that

$$R(X,Y)\xi = -(X\beta)\phi Y + (Y\beta)\phi X + \beta^2\{\eta(Y)X - \eta(X)Y\}. \tag{3.3}$$

Thus we get from (3.3)

$$\begin{aligned} R(X,Y,Z,\xi) &= (X\beta)g(\phi Y, Z) - (Y\beta)g(\phi X, Z) \\ &- \beta^2\{\eta(Y)g(X,Z) - \eta(X)g(Y,Z)\}, \end{aligned} \tag{3.4}$$

where $R(X,Y,Z,W) = g(R(X,Y,Z),W)$.
Putting $X = \xi$, in (3.4) we obtain

$$R(\xi,Y,Z,\xi) = \beta^2\{g(Y,Z) - \eta(Y)\eta(Z)\} + g(\phi Y, Z)\xi\beta. \tag{3.5}$$

Interchanging Y and Z of (3.5) yields

$$R(\xi,Z,Y,\xi) = \beta^2\{g(Y,Z) - \eta(Y)\eta(Z)\} + g(\phi Z, Y)\xi\beta. \tag{3.6}$$

Since $R(\xi,Y,Z,\xi) = R(Z,\xi,\xi,Y) = R(\xi,Z,Y,\xi)$, from (3.5) and (3.6) we have

$$\{g(\phi Y, Z) - g(\phi Z, Y)\}\xi\beta = 0.$$

Therefore, we can easily verify that $\xi\beta = 0$. So we have the following:

Proposition 1. *In a 3-dimensional non-cosymplectic quasi-Sasakian manifold the structure function β satisfies the condition $\xi\beta = 0$.*

Let M be a three-dimensional quasi-Sasakian manifold. The Ricci tensor S of M is given by [21]

$$\begin{aligned} S(Y,Z) &= (\frac{r}{2}-\beta^2)g(Y,Z)+(3\beta^2-\frac{r}{2})\eta(Y)\eta(Z) \\ &- \eta(Y)d\beta(\phi Z)-\eta(Z)d\beta(\phi Y), \end{aligned} \tag{3.7}$$

where r is the scalar curvature of M.

From (3.7), the Ricci operator Q can be written as

$$QX=(\frac{r}{2}-\beta^2)X+(3\beta^2-\frac{r}{2})\eta(X)\xi-\eta(X)(\phi grad\,\beta)-d\beta(\phi X)\xi, \tag{3.8}$$

where the gradient of a function f is related to the exterior derivative df by the formula $df(X)=g(grad\,f,X)$.

From (3.7) it is clear that if $\beta = constant$ then M is an $\eta-$Einstein manifold. Conversely if we consider that M is an $\eta-$Einstein manifold, then from (3.7) and (2.3), we have

$$\eta(X)d\beta(\phi Y)+\eta(Y)d\beta(\phi X)=(-\lambda+\frac{r}{2}-\beta^2)g(X,Y)+(-\delta-\frac{r}{2}+3\beta^2)\eta(X)\eta(Y).$$

Taking $Y=\xi$ in the last equation we get

$$d\beta(\phi X)=(-\lambda-\delta+2\beta^2)\eta(X).$$

Now taking ϕX instead of X in the above equation and using Proposition 1 we obtain that β is a constant. Hence the Ricci tensor S of an $\eta-$Einstein quasi-Sasakian manifold is of the form

$$S(Y,Z)=(\frac{r}{2}-\beta^2)g(Y,Z)+(3\beta^2-\frac{r}{2})\eta(Y)\eta(Z).$$

Hence we can state the following:

Theorem 1. *A 3-dimensional non-cosymplectic quasi-Sasakian manifold is an $\eta-$Einstein manifold if and only if β is constant.*

We verify the above theorem by an example.

Example: We consider the three-dimensional manifold $M=\{(x,y,z)\varepsilon\mathbb{R}^3,$ $(x,y,z)\neq 0\}$, where (x,y,z) are standard co-ordinate of $\mathbb{R}^3$.

The vector fields

$$e_1=\frac{\partial}{\partial x}-y\frac{\partial}{\partial z},\quad e_2=\frac{\partial}{\partial y},\quad e_3=\frac{\partial}{\partial z}$$

are linearly independent at each point of M. Let g be the Riemannian metric defined by

$$g(e_1, e_3) = g(e_1, e_2) = g(e_2, e_3) = 0$$
$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1.$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z\varepsilon T(M)$.

Let ϕ be the $(1,1)$ tensor field defined by

$$\phi(e_1) = -e_2, \quad \phi(e_2) = e_1, \quad \phi(e_3) = 0.$$

Then using the linearity of ϕ and g, we have

$$\eta(e_3) = 1, \quad \phi^2 Z = -Z + \eta(Z)e_3,$$

$$g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W),$$

for any $Z, W\varepsilon T(M)$.

Thus for $e_3 = \xi$, the structure (ϕ, ξ, η, g) defines an almost contact metric structure on M.

Let ∇ be the Levi-Civita connection with respect to metric g. Then we have

$$[e_1, e_2] = e_3, \quad [e_1, e_3] = 0 \quad \textit{and} \quad [e_2, e_3] = 0.$$

The Riemannian connection ∇ of the metric g is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &- g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y]) \end{aligned} \tag{3.9}$$

which is known as Koszul's formula.

Using (3.9) we obtain $2g(\nabla_{e_1} e_3, X) = 2g(-\frac{1}{2}e_2, X)$ for all $X\varepsilon T(M)$.

Thus, $\nabla_{e_1} e_3 = -\frac{1}{2}e_2$.

(3.9) further yields

$$\nabla_{e_1} e_3 = -\frac{1}{2}e_2, \quad \nabla_{e_1} e_2 = \frac{1}{2}e_3, \quad \nabla_{e_1} e_1 = 0, \quad \nabla_{e_2} e_3 = \frac{1}{2}e_1, \quad \nabla_{e_2} e_2 = 0,$$

$$\nabla_{e_2} e_1 = -\frac{1}{2}e_3 \quad \nabla_{e_3} e_3 = 0, \quad \nabla_{e_3} e_2 = \frac{1}{2}e_1, \quad \nabla_{e_3} e_1 = -\frac{1}{2}e_2. \tag{3.10}$$

We see that the structure (ϕ, ξ, η, g) satisfies the formula $\nabla_X \xi = -\beta\phi X$ for $\beta = -\frac{1}{2}$. Hence the manifold is a three-dimensional quasi-Sasakian manifold with the constant structure function β.

It is known that

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{3.11}$$

With the help of the above results and using (3.11) it can be easily verified that

$$R(e_1, e_2)e_3 = 0, \quad R(e_2, e_3)e_3 = \frac{1}{4}e_2, \quad R(e_1, e_3)e_3 = \frac{1}{4}e_1,$$
$$R(e_1, e_2)e_2 = -\frac{3}{4}e_1, \quad R(e_2, e_3)e_2 = -\frac{1}{4}e_3, \quad R(e_1, e_3)e_2 = 0,$$
$$R(e_1, e_2)e_1 = \frac{3}{4}e_2, \quad R(e_2, e_3)e_1 = 0, \quad R(e_1, e_3)e_1 = -\frac{1}{4}e_3.$$

From the above expression of the curvature tensor we obtain

$$S(e_1, e_1) = -\frac{1}{2}, \quad S(e_2, e_2) = -\frac{1}{2} \quad \textit{and} \quad S(e_3, e_3) = \frac{1}{2}.$$

Therefore,

$$\begin{aligned} r &= S(e_1, e_1) + S(e_2, e_2) + S(e_3, e_3) \\ &= -\frac{1}{2}. \end{aligned}$$

Here we note that the scalar curvature r is a constant.

With the help of the above expressions of the Ricci tensor it can be easily verified that the manifold satisfies (2.3) for $\lambda = -\frac{1}{2}$ and $\delta = 1$. Hence the manifold is an η-Einstein manifold. Therefore Theorem 1 is verified.

§4. Ricci solitons and Gradient Ricci solitons

Suppose a 3-dimensional quasi-Sasakian manifold admits a Ricci soliton defined by (1.1). It is well known that $\nabla g = 0$. Since λ in the Ricci soliton equation (1.1) is a constant, so $\nabla \lambda g = 0$. Thus $\pounds_V g + 2S$ is parallel. Hence using Lemma 1 we can say that $\pounds_V g + 2S$ is a constant multiple of metric tensors g, that is, $\pounds_V g + 2S = ag$, where a is constant. Hence $\pounds_V g + 2S + 2\lambda g$ reduces to $(a + 2\lambda)g$. Using (1.1) we get $\lambda = -a/2$. So we have the following:

Proposition 2. *In a 3-dimensional non-cosymplectic quasi-Sasakian manifold, the Ricci soliton (g, V, λ) is shrinking or expanding according as a is positive or negative.*

Now in particular we investigate the case $V = \xi$. Then (1.1) reduces to

$$\pounds_\xi g + 2S + 2\lambda g = 0. \tag{4.1}$$

It is known that [4] in a 3-dimensional quasi-Sasakian manifold ξ is Killing, that is, $\pounds_\xi g = 0$. Then from (4.1) $\lambda = -S(\xi, \xi) = -2\beta^2$, provided β is constant. Also from (4.1) it follows that the manifold is an Einstein manifold. But it is known [25] that a 3-dimensional Einstein manifold is a manifold of constant curvature.

Thus we have

Corollary 1. *In a 3-dimensional non-cosymplectic quasi-Sasakian manifold, the Ricci soliton* (g, ξ, λ) *is shrinking provided* β *is constant and the manifold is of constant curvature.*

Let M be a 3-dimensional non-cosymplectic quasi-Sasakian manifold with constant structure function β and g a gradient Ricci soliton. Then the equation (1.2) can be written as

$$\nabla_Y Df = QY + \lambda Y \tag{4.2}$$

for all vector fields Y in M, where D denotes the gradient operator of g. From (4.2) it follows that

$$R(X,Y)Df = (\nabla_X Q)Y - (\nabla_Y Q)X, \quad X, Y \varepsilon TM. \tag{4.3}$$

Using (3.3) we have

$$g(R(\xi, Y)Df, \xi) = g(\beta^2(Df - (\xi f)\xi), Y). \tag{4.4}$$

Also in a 3-dimensional quasi-Sasakian manifold, it follows that

$$g((\nabla_\xi Q)Y - (\nabla_Y Q)\xi, \xi) = 0. \tag{4.5}$$

From (4.3), (4.4) and (4.5) we get

$$\beta^2(Df - (\xi f)\xi) = 0$$

that is,

$$Df = (\xi f)\xi, \tag{4.6}$$

since M is non-cosymplectic. Using (4.6) in (4.2) we obtain

$$S(X,Y) + \lambda g(X,Y) = -\beta(\xi f)g(\phi Y, X) + Y(\xi f)\eta(X). \tag{4.7}$$

Putting $X = \xi$ in (4.7) we get

$$Y(\xi f) = (2\beta^2 + \lambda)\eta(Y). \tag{4.8}$$

From (4.7) and (4.8) we get

$$S(X,Y) + \lambda g(X,Y) = -\beta(\xi f)g(\phi Y, X) + (2\beta^2 + \lambda)\eta(X)\eta(Y). \tag{4.9}$$

Using (4.9) in (4.2), we have

$$\nabla_Y Df = (2\beta^2 + \lambda)\eta(Y)\xi - \beta(\xi f)\phi Y. \tag{4.10}$$

Using (4.10) we compute $R(X,Y)Df$ and obtain

$$g(R(X,Y)(\xi f)\xi,\xi) = (2\beta^2 + \lambda)d\eta(X,Y). \tag{4.11}$$

Thus we get

$$2\beta^2 + \lambda = 0. \tag{4.12}$$

Therefore from equation (4.8) we have $Y(\xi f) = 0$ that is $\xi f = c$, where c is a constant. Thus the equation (4.6) gives $df = c\eta$. Its exterior derivative implies that $cd\eta = 0$, which implies $c = 0$. Hence f is a constant. Consequently (4.2) reduces to $S(X,Y) = 2\beta^2 g(X,Y)$. Hence M is Einstein. So we have the following:

Theorem 2. *If the metric g of a 3-dimensional non-cosymplectic quasi-Sasakian manifold with constant structure function β is a gradient Ricci soliton, then the manifold is an Einstein manifold.*

Since a 3-dimensional Einstein manifold is a manifold of constant curvature, hence we get the following:

Corollary 2. *If the metric g of a 3-dimensional non-cosymplectic quasi-Sasakian manifold with constant structure function β is a gradient Ricci soliton, then the manifold is a manifold of constant curvature.*

Also using the result of Perelman [23], we can state the following:

Corollary 3. *If the metric g of a 3-dimensional non-cosymplectic compact quasi-Sasakian manifold with constant structure function β is a Ricci soliton, then the manifold is an Einstein manifold.*

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